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WEAK RELATIVELY UNIFORM CONVERGENCES ON MV-ALGEBRAS

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ABSTRACT. Weak relatively uniform convergences (wru-convergences, for short) in lattice ordered groups have been investigated in previous authors' papers. In the present article, the analogous notion for MV-algebras is studied. The system $s(A)$ of all wru-convergences on an MV-algebra A is considered; this system is partially ordered in a natural way. Assuming that the MV-algebra A is divisible, we prove that $s(A)$ is a Brouwerian lattice and that there exists an isomorphism of $s(A)$ into the system $s(G)$ of all wru-convergences on the lattice ordered group G corresponding to the MV-algebra A. Under the assumption that the MV-algebra A is archimedean and divisible, we investigate atoms and dual atoms in the system $s(A)$.

The notion of relatively uniform convergence (ru-convergence, for short) has been studied in archimedean vector lattices (cf. [17], [21]) and later in archimedean lattice ordered groups (cf. [2], [8], [9], [16], [18]). The notion of a regulator of a convergent sequence is essential in this theory. (For definitions, cf. Section 1 below.) Distinct convergent sequences have, in general, distinct regulators. Each positive element of the structure under consideration can serve as a regulator.

A different standpoint is applied in [5]; here, there are studied archimedean lattice ordered groups with a fixed regulator.

The notion of ru-convergence in archimedean lattice ordered groups was generalized in [7] in two directions. First, the lattice ordered group G under consideration was assumed to be abelian (this is a weaker condition than the assumption of the archimedean property). Secondly, it was assumed that the regulators form a set $M \neq \emptyset$ of archimedean elements of G such that M is closed with respect to the operation $+$. This type of convergence was called a weak relatively

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uniform convergence (wru-convergence, for short) generated by the set M of regulators and it was denoted by $\beta(M)$. The system $s(G)$ of all wru-convergences on G is partially ordered in a natural way. In [7] there is proved that $s(G)$ is a Brouwerian lattice.

Let A be an MV-algebra. In view of the well-known result of Mundici [19], there exists an abelian lattice ordered group G with a strong unit u such that, under the notation as in [4], we have $A = \Gamma(G, u)$.

In [3], there is introduced the notion of an MV-convergence as a convergence on an MV-algebra which makes the MV-operations continuous. In an analogical way, a convergence on a unital lattice ordered group (G, u), called lu-convergence, is defined. Connections between MV-convergences on the MV-algebra A and lu-convergences on the unital lattice ordered group (G, u) are dealt with, where $A = \Gamma(G, u)$ (cf. also Section 3 below).

A will be said to be archimedean if G is archimedean. In [4], a different terminology for MV-algebras is applied: instead of "archimedean" the term "semisimple" is used. A is archimedean if and only if A is semisimple (cf. [10]).

In [6], the notion of convergence with a fixed regulator on an archimedean MV-algebra A has been introduced and studied. In the definition of this type of convergence on A, the operations of the lattice ordered group G have been used.

The present paper can be considered as a sequel to the article [6]. First, a new definition of convergence with a fixed regulator on an MV-algebra A is given. In this definition, merely the operations in A are applied and the archimedean property of A is not assumed to be valid. The definition used in the present paper is equivalent with that from [6] in the case when the MV-algebra A is archimedean.

Our main interest consists in studying the notion of wru-convergences on an MV-algebra A; the definition is analogous as in the case of lattice ordered groups (in this definition, merely the operations from A are used). After deducing the basic properties of wru-convergences on A, we consider the system $s(A)$ of all wru-convergences on A; this system is partially ordered in an analogous way to $s(G)$. We prove that $s(A)$ is a Brouwerian lattice and that there exists an isomorphism of $s(A)$ into $s(G)$. Under the assumption that the MV-algebra A is archimedean and divisible, we investigate atoms and dual atoms of the lattice $s(A)$.

1. wru-convergence in abelian lattice ordered groups

The standard terminology and notation for lattice ordered groups will be used (cf., e.g. [1], [11]). All lattice ordered groups dealt with in the present paper are assumed to be abelian.

Let G be a lattice ordered group. In this section, we recall the notions of b-uniform convergence, wru-convergence and some relevant results.

An element $0 \leq b \in G$ is called *archimedean* if for each $0 < x \in G$ there exists $n \in N$ such that $nx \nleq b$. If each element $0 \leq b \in G$ is archimedean then G is said to be archimedean. The set of all archimedean elements of G will be denoted by $\mathcal{A}(G)$.

Lemma 1.1. (cf. [7]) *Let $b_1, b_2 \in \mathcal{A}(G)$. Then $b_1 + b_2 \in \mathcal{A}(G)$.*

Apparently, if $b \in \mathcal{A}(G)$ and $b' \in G$, $0 \leq b' \leq b$, then also $b' \in \mathcal{A}(G)$.

Definition 1.2. (cf. [7]) Let (x_n) be a sequence in G, $x \in G$ and $b \in \mathcal{A}(G)$. We say that (x_n) b-uniformly converges to x in G, written $x_n \stackrel{b}{\to}_\beta x$, if for each $k \in N$ there exists $n_0(b,k) \in N$ such that

$$k|x_n - x| \leq b$$

for each $n \in N$, $n \geq n_0(b,k)$.

The element b is referred to as a *regulator of convergence*.

In the whole section, M is assumed to be a nonvoid subset of $\mathcal{A}(G)$ closed with respect to the addition.

Definition 1.3. (cf. [7]) Let (x_n) be a sequence in G and $x \in G$. We say that the sequence (x_n) *$\beta(M)$-converges* to x, in symbols, $x_n \to_{\beta(M)} x$, if there exists $b \in M$ such that $x_n \stackrel{b}{\to}_\beta x$.

We denote this type of convergence as *wru-convergence* on G with the set M of regulators, or shortly, as *$\beta(M)$-convergence*.

If G is archimedean and if $M = G^+$, then $\beta(M)$-convergence coincides with ru-convergence (for definition of ru-convergence cf. [2], [18], [20]).

If the role of G is to be emphasized, then we write $\beta(G, M)$ instead of $\beta(M)$.

Next, we will apply the basic properties of $\beta(M)$-convergence presented in [7].

The symbol $\mathcal{S}(G)$ will denote the system of all nonempty subsets of $\mathcal{A}(G)$ closed with respect to the addition and $s(G)$ will be the system of all convergences $\beta(M)$ where M runs over the system $\mathcal{S}(G)$. For $M_1, M_2 \in \mathcal{S}(G)$ we put $\beta(M_1) \leq \beta(M_2)$ if for each sequence (x_n) in G and $x \in G$, the relation $x_n \to_{\beta(M_1)} x$ implies $x_n \to_{\beta(M_2)} x$. Then $s(G)$ turns out to be a partially ordered set.

When dealing with sequences in G, sometimes it is useful to consider a set $\emptyset \neq M \subseteq \mathcal{A}(G)$ which needs not be closed under the addition. This is a motivation to introduce the following definition.

Definition 1.4. (cf. [7]) Let M be a nonempty subset of $\mathcal{A}(G)$, (x_n) a sequence in G and $x \in G$. We say that the sequence (x_n) $\beta_0(M)$-converges to x, in symbols, $x_n \to_{\beta_0(M)} x$, if there is $b = b_1 + \dots + b_m$ with $b_i \in M$ $(i = 1, 2, \dots, m)$ such that $x_n \overset{b}{\to}_\beta x$.

If $M \in \mathcal{S}(G)$ then $\beta_0(M) = \beta(M)$.

Let M_1, M_2 be nonempty subsets of $\mathcal{A}(G)$. Apparently, if $M_1 \subseteq M_2$ then $\beta_0(M_1) \leqq \beta_0(M_2)$, but not conversely.

Given $\emptyset \neq M \subseteq \mathcal{A}(G)$, denote by $\widetilde{M}$ the set of all elements $b \in \mathcal{A}(G)$ such that for each sequence (x_n) in G and $x \in G$, the relation $x_n \overset{b}{\to}_\beta x$ implies $x_n \to_{\beta_0(M)} x$.

In 1.5–1.8 we assume that G is a divisible lattice ordered group.

Lemma 1.5. (cf. [7: Lemma 6.4]) *Let $\emptyset \neq M \subseteq \mathcal{A}(G)$. Then $\beta_0(M) = \beta_0(\widetilde{M})$.*

Lemma 1.6. (cf. [7: Lemma 6.5]) *Let M_1 and M_2 be nonempty subsets of $\mathcal{A}(G)$. Then $\beta_0(M_1) \leqq \beta_0(M_2)$ if and only if $\widetilde{M_1} \subseteq \widetilde{M_2}$.*

Lemma 1.7. (cf. [7: Lemma 6.3]) *If M is a nonempty subset of $\mathcal{A}(G)$, then $\widetilde{M} \in \mathcal{S}(G)$.*

Theorem 1.8. (cf. [7: Theorems 6.6, 6.7]) *The set $s(G)$ is a complete Brouwerian lattice. If I is a nonempty set and $M_i \in \mathcal{S}(G)$ for each $i \in I$, then*

$$\bigwedge_{i \in I} \beta(M_i) = \beta\Big(\bigcap_{i \in I} \widetilde{M_i}\Big),$$

$$\bigvee_{i \in I} \beta(M_i) = \beta\Big(\bigcup_{i \in I} M_i\Big)^{\sim}.$$

The equations $\beta_0(M) = \beta_0(\widetilde{M}) = \beta(\widetilde{M})$ holding on account of Lemma 1.5 for each $\emptyset \neq M \subseteq \mathcal{A}(G)$ and the relation $\beta_0(M) = \beta(M)$ that is valid for each $M \in \mathcal{S}(G)$ yield that $s(G)$ can be viewed as the system $s_0(G)$ of all convergences $\beta_0(M)$ where M runs over the system of all nonempty subsets of $\mathcal{A}(G)$.

2. wru-convergence in MV-algebras

An MV-algebra is a system $A = (A, \oplus, *, \neg, 0, 1)$ where A is a nonempty set, $\oplus, *$ are binary operations, $\neg$ is a unary operation and $0, 1$ are nulary operations on A satisfying the conditions (m_1)–(m_9) from [12]. For MV-algebras, a formally different but equivalent system of axioms has been applied in [4].

Theorem 2.1. (cf. [12]) *Let A be an MV-algebra. For each $a, b \in A$, put*

$$a \vee b = (a * \neg b) \oplus b, \qquad a \wedge b = \neg(\neg a \vee \neg b).$$

Then $(A, \vee, \wedge)$ is a distributive lattice with the least element 0 and the greatest element 1.

Let A' be a nonempty subset of A closed under the operations $\oplus, *, \neg, 0, 1$ in A. Then $A' = (A', \oplus, *, \neg, 0, 1)$ is called a *subalgebra* of A.

An isomorphism of MV-algebras is defined in a usual way.

The following two theorems are due to Mundici [19].

Theorem 2.2. *Let G be an abelian lattice ordered group with a strong unit u. Let A be the interval $[0, u]$ of G. For each a, b in A we put*

$$a \oplus b = (a + b) \wedge u, \quad \neg a = u - a, \quad 1 = u, \quad a * b = \neg(\neg a \oplus \neg b).$$

Then $A = (A, \oplus, *, \neg, 0, 1)$ is an MV-algebra.

If A is as in 2.2, we will write $A = \Gamma(G, u)$.

Theorem 2.3. *Let A be an MV-algebra. Then there exists an abelian lattice ordered group G with a strong unit u such that $A = \Gamma(G, u)$.*

Let us remark that if A and G are as in 2.2, then the partial order on A inherited from G is the same as the partial order on A defined by means of the lattice $(A, \vee, \wedge)$ in 2.1.

In what follows, unless otherwise stated, we assume that $A = \Gamma(G, u)$.

Definition 1.2 of b-uniform convergence in lattice ordered groups has been applied in [6] to archimedean MV-algebras assuming that (x_n) is a sequence in A, $x \in A$ and $b \in A$. However, such a definition was not given in MV-algebra operations; in fact, we used the operations concerning the lattice ordered group G (cf. Theorem 2.3).

In the present paper, we introduce a new definition of b-uniform convergence in A using merely the MV-algebra operations. Further, we prove that if (x_n) is a sequence in A, $x \in A$ and $b \in A$, then the following conditions are equivalent:

(i) (x_n) b-uniformly converges to x in A in the new definition.

(ii) (x_n) b-uniformly converges to x in G in the Definition 1.2.

Assume that $a_1, a_2 \in A$, $a_1 \leq a_2$. Then, $0 \leq a_2 - a_1 \leq u$, so, $a_2 - a_1 \in A$.

Lemma 2.4. (cf. [13]) *Let $a_1, a_2 \in A$, $a_1 \leq a_2$. Then*

$$a_2 - a_1 = \neg(a_1 \oplus \neg a_2).$$

Let $a \in A$. We denote

$$a \oplus a \oplus \cdots \oplus a = n \cdot a \qquad (n \text{ times})$$

and as usual, we write

$$a + a + \cdots + a = na \qquad (n \text{ times}).$$

Recall that for $a_1, a_2, \dots, a_n \in A$, the relation $a_1 \oplus a_2 \oplus \cdots \oplus a_n = (a_1 + a_2 + \cdots + a_n) \wedge u$ is valid. Hence $n \cdot a = na \wedge u$ for each $n \in N$.

An element $b \in A$ is called *archimedean* in A if b is archimedean in G. Let $\mathcal{A}(A)$ be the set of all archimedean elements of A. Then $\mathcal{A}(A) = \mathcal{A}(G) \cap A$.

Let (x_n) be a sequence in G, $x_n \geq 0$ for each $n \in N$, and $b \in \mathcal{A}(G)$. Apparently, $x_n \xrightarrow{b}_{\beta} 0$ if and only if for each $k \in N$ there exists $n_0 \in N$ such that $kx_n \leq b - x_n$ whenever $n \in N$, $n \geq n_0$. This is a motivation to define the notion of b-uniform convergence in A as follows.

Definition 2.5. Let (a_n) be a sequence in A and $b \in \mathcal{A}(A)$. We say that the sequence (a_n) *b-uniformly converges* to 0 in A, in symbols $a_n \xrightarrow{b}_{\alpha} 0$ if for each $k \in N$ there exists $n_0(b, k) \in N$ such that the relation

$$k \cdot a_n \leq b - a_n$$

is valid for each $n \in N$, $n \geq n_0(b, k)$.

From the relation $b - a_n \geq 0$ we get $a_n \leq b$ for each $n \in N$, $n \geq n_0$. Hence, $b - a_n \in A$ for each $n \in N$, $n \geq n_0$.

Let $a_1, a_2 \in A$. Then $a_1 - a_2 \leq a_1 \leq u$ and $a_2 - a_1 \leq a_2 \leq u$. Hence, $|a_1 - a_2| \in A$. Therefore, if (a_n) is a sequence in A and $a \in A$, then $|a_n - a| \in A$ for each $n \in N$.

Definition 2.6. Let (a_n) be a sequence in A, $a \in A$ and $b \in \mathcal{A}(A)$. We say that the sequence (a_n) *b-uniformly converges* to a and we write $a_n \xrightarrow{b}_{\alpha} a$ if $|a_n - a| \xrightarrow{b}_{\alpha} 0$.

Let (a_n) and a be as in 2.6. Then the elements $p_n = a_n \vee a$ and $q_n = a_n \wedge a$ belong to A for each $n \in N$. We get $q_n \leq p_n$ and $|a_n - a| = p_n - q_n$. Thus, we can express the elements $|a_n - a|$ and $b - a_n$ by using Lemma 2.4. We conclude that the Definition 2.6 of b-uniform convergence in A is given in terms of the MV-algebra operations.

Let (a_n) and b be as in 2.5. If $a_n \xrightarrow{b}_{\alpha} 0$, then for each $k \in N$ there exists $n_0 \in N$ such that $k \cdot a_n \leq b$ for each $n \in N$, $n \geq n_0$. The converse does not hold in general.

Example 2.7. Let G be the set of all convergent sequences of reals. If the operation $+$ and the relation $\leq$ are performed componentwise, G turns out to be an abelian lattice ordered group and the constant sequence $u = (1, 1, \dots)$ is a strong unit of G. Consider the MV-algebra $A = \Gamma(G, u)$ and the sequence (a_n)

in A defined as follows: $a_n = (t_1, t_2, t_3, \dots)$ such that $t_i = 0$ if $i \le n$ and $t_i = 1$ otherwise. Let $b = (0, 1, 1, \dots)$. For each $k \in N$ and each $n \in N$, we have

$$k \cdot a_n = ka_n \wedge u = a_n,$$

so

$$k \cdot a_n \le b$$

and

$$b - a_n = (0, 1, 1, \dots) - (0, 0, \dots, 0, 1, 1, \dots) = (0, 1, \dots, 1, 0, 0, \dots).$$

Hence, $k \cdot a_n \not\le b - a_n$, so, $a_n \overset{b}{\nrightarrow}_\alpha 0$.

Theorem 2.8. *Let (a_n) be a sequence in A and $b \in \mathcal{A}(A)$. Then the following conditions are equivalent:*

(i) $a_n \overset{b}{\to}_\beta 0$,

(ii) $a_n \overset{b}{\to}_\alpha 0$.

Proof.

(i) $\Longrightarrow$ (ii): Let $a_n \overset{b}{\to}_\beta 0$. Then for each $k \in N$ there exists $n_0 \in N$ such that $k \cdot a_n \le ka_n \le b - a_n$ for each $n \in N$, $n \ge n_0$. Thus, (ii) is valid.

(ii) $\Longrightarrow$ (i): Suppose that $a_n \overset{b}{\to}_\alpha 0$. We first prove that for each $k \in N$, there is $n_0 \in N$ such that the relation

$$k \cdot a_n = ka_n \tag{1}$$

holds for each $n \in N$, $n \ge n_0$.

We proceed by induction. Apparently, the relation (1) is valid for $k = 1$. Assume that (1) holds for some $k \in N$. In view of (ii), there exists $n_0 \in N$ such that $k \cdot a_n \le b - a_n$ for each $n \in N$, $n \ge n_0$. Consequently, $a_n + k \cdot a_n \le b$, so, $a_n + k \cdot a_n = a_n \oplus k \cdot a_n$ for each $n \in N$, $n \ge n_0$. We have

$$(k+1) \cdot a_n = a_n \oplus k \cdot a_n = a_n + k \cdot a_n = a_n + ka_n = (k+1)a_n$$

for each $n \in N$, $n \ge n_0$, and the relation (1) holds.

Then, $ka_n = k \cdot a_n \le b - a_n$, for each $n \in N$, $n \ge n_0$. Hence, (i) is satisfied. □

Corollary 2.9. *Let (a_n) be a sequence in A, $a \in A$ and $b \in \mathcal{A}(A)$. Then the following conditions are equivalent:*

(i) $a_n \overset{b}{\to}_\beta a$,

(ii) $a_n \overset{b}{\to}_\alpha a$.

Let $b_1, b_2 \in \mathcal{A}(A)$. Then $b_1, b_2 \in \mathcal{A}(G)$. By Lemma 1.1, $b_1 + b_2 \in \mathcal{A}(G)$. We have $b_1 \oplus b_2 \le b_1 + b_2$. Thus, $b_1 \oplus b_2 \in \mathcal{A}(G)$. Hence, we have:

Lemma 2.10. *Let $b_1, b_2 \in \mathcal{A}(A)$. Then $b_1 \oplus b_2 \in \mathcal{A}(A)$.*

In the rest of this section, M will be assumed to be a nonempty subset of $\mathcal{A}(A)$ closed with respect to the operation $\oplus$.

Definition 2.11. Let (a_n) be a sequence in A and $a \in A$. We say that the sequence (a_n) $\alpha(M)$-converges to a in A, written $a_n \to_{\alpha(M)} a$, if $a_n \overset{b}{\to}_\alpha a$ for some $b \in M$.

To avoid misunderstanding, the convergence in A will be denoted also by $\alpha(A, M)$ rather than $\alpha(M)$.

If A is archimedean and $M = A$, then we say that a sequence (a_n) in A relatively uniformly converges (ru-converges, for short) to an element $a \in A$, if $a_n \to_{\alpha(M)} a$.

Theorem 2.12. *Let (a_n), (a'_n) be sequences in A and $a, a' \in A$. If $a_n \to_{\alpha(M)} a$ and $a'_n \to_{\alpha(M)} a'$, then*

(i) $a_n \oplus a'_n \to_{\alpha(M)} a \oplus a'$,

(ii) $a_n \vee a'_n \to_{\alpha(M)} a \vee a'$,

(iii) $a_n \wedge a'_n \to_{\alpha(M)} a \wedge a'$,

(iv) $k \cdot a_n \to_{\alpha(M)} k \cdot a$ *for each* $k \in N$,

(v) *if* $c, d \in A, c \leq a_n \leq d$ *for each* $n \in N$, *then* $c \leq a \leq d$.

Proof.

(i) We have to prove that $|a_n \oplus a'_n - (a \oplus a')| \to_{\alpha(M)} 0$. The hypothesis implies $|a_n - a| \to_{\alpha(M)} 0$ and $|a'_n - a'| \to_{\alpha(M)} 0$. Hence, there exist $b_1, b_2 \in M$ with $|a_n - a| \overset{b_1}{\to}_\alpha 0$ and $|a'_n - a'| \overset{b_2}{\to}_\alpha 0$. Let us put $c_n = |a_n - a|$ and $c'_n = |a'_n - a'|$. Then (c_n) and (c'_n) are sequences in A. Denoting $b = b_1 \oplus b_2$, we get $b_1 \leq b$, $b_2 \leq b$ and $b \in M$. Hence, $c_n \overset{b}{\to}_\alpha 0$ and $c'_n \overset{b}{\to}_\alpha 0$. By Theorem 2.8, $c_n \overset{b}{\to}_\beta 0$ and $c'_n \overset{b}{\to}_\beta 0$. It is easy to verify (cf. [5]) that $c_n + c'_n \overset{b}{\to}_\beta 0$. Thus, for each $k \in N$, there exists $n_0 \in N$ such that

$$k(c_n + c'_n) \leq b$$

whenever $n \in N$, $n \geq n_0$.

We have

$$\begin{aligned} k|a_n \oplus a'_n - (a \oplus a')| &= k|(a_n + a'_n) \wedge u - (a + a') \wedge u| \\ &\leq k|(a_n + a'_n) - (a + a')| \leq k(|a_n - a| + |a'_n - a'|) \\ &= k(c_n + c'_n) \leq b \end{aligned}$$

for each $n \in N$, $n \geq n_0$. Hence $|a_n \oplus a'_n - (a \oplus a')| \overset{b}{\to}_\beta 0$. Again, in view of Theorem 2.8, $|a_n \oplus a'_n - (a \oplus a')| \overset{b}{\to}_\alpha 0$. Therefore, $|a_n \oplus a'_n - (a \oplus a')| \to_{\alpha(M)} 0$.

(ii) The hypothesis yields that there are $b_1, b_2 \in M$ with $a_n \overset{b_1}{\to}_\alpha a$ and $a'_n \overset{b_2}{\to}_\alpha a'$. Let b be an element from M as in (i). Using the procedure from (i), we obtain $a_n \overset{b}{\to}_\beta a$ and $a'_n \overset{b}{\to}_\beta a'$. Then, $a_n \vee a'_n \overset{b}{\to}_\beta a \vee a'$ (for the proof, cf. [5]). The sequence $(a_n \vee a'_n)$ is in A and $a \vee a' \in A$. Corollary 2.9 yields $a_n \vee a'_n \overset{b}{\to}_\alpha a \vee a'$ and (ii) holds.

(iii) The proof is dual to that of (ii).

(iv) and (v) are easy to verify. □

3. The partially ordered set of wru-convergences on A

As before, let $A = \Gamma(G, u)$. Denote by $\mathcal{S}(A)$ the system of all nonempty subsets of $\mathcal{A}(A)$ that are closed under the operation $\oplus$ and by $s(A)$ the system of all convergences $\alpha(M)$ where M runs over the system $\mathcal{S}(A)$.

Let us proceed similarly as in Section 2.

Assuming that $M_1, M_2 \in \mathcal{S}(A)$, we define the binary relation $\leq$ on $s(A)$ by putting $\alpha(M_1) \leq \alpha(M_2)$ if for each sequence (a_n) in A and $a \in A$, the relation $a_n \to_{\alpha(M_1)} a$ implies $a_n \to_{\alpha(M_2)} a$. Then $\leq$ is a partial order on the set $s(A)$.

Analogously as we did in lattice ordered groups, in MV-algebras we will consider wru-convergence without the assumption that the set of regulators is closed with respect to the operation $\oplus$; i.e., we apply the following definition.

Definition 3.1. Let M be a nonempty subset of $\mathcal{A}(A)$, (a_n) a sequence in A and $a \in A$. We say that the sequence (a_n) $\alpha_0(M)$-converges to a, written $a_n \to_{\alpha_0(M)} a$, if there is $b = b_1 \oplus \cdots \oplus b_m$ with $b_i \in M$ $(i = 1, \ldots, m)$ such that $a_n \overset{b}{\to}_\alpha a$.

Especially, if $M \in \mathcal{S}(A)$ then $\alpha_0(M) = \alpha(M)$.

Let M_1 and M_2 be nonempty subsets of $\mathcal{A}(A)$. Evidently, if $M_1 \subseteq M_2$ then $\alpha_0(M_1) \leq \alpha_0(M_2)$, but not conversely. In fact, let $b \in \mathcal{A}(A)$, $0 < b < u$, $M_1 = \{b, 2 \cdot b\}$, $M_2 = \{b\}$. Then, $M_1 \nsubseteq M_2$ but $\alpha_0(M_1) \leq \alpha_0(M_2)$. The relation $\alpha_0(M_1) = \alpha_0(M_2)$ is valid.

Assume that $\emptyset \neq M \subseteq \mathcal{A}(A)$. Let us form the set $\overline{M}$ of all elements $b \in \mathcal{A}(A)$ such that for each sequence (a_n) in A and $a \in A$, the relation $a_n \overset{b}{\to}_\alpha a$ implies $a_n \to_{\alpha_0(M)} a$. Then, $M \subseteq \overline{M}$, and obviously, $M \subseteq \widetilde{M}$. Further, if $b \in \overline{M}$ and $b_1 \in A$, $b_1 \leq b$, then $b_1 \in \overline{M}$, whence $0 \in \overline{M}$.

Taking into account Corollary 2.9 and the fact that $b_1 \oplus \cdots \oplus b_m \leq b_1 + \cdots + b_m$ whenever $b_1, \ldots, b_m \in M$, we obtain

Lemma 3.2. *Let $\emptyset \neq M \subseteq \mathcal{A}(A)$, (a_n) a sequence in A and $a \in A$. If $a_n \to_{\alpha_0(M)} a$, then $a_n \to_{\beta_0(M)} a$.*

An open question remains whether the converse assertion is valid.

Let $\emptyset \neq M \subseteq \mathcal{A}(A)$. We remark that Theorem 2.12 is valid also for $\alpha_0(M)$-convergence. The proof of this assertion is similar to the proof of Theorem 2.12.

We will apply the notion of a divisible MV-algebra.

The MV-algebra A is called *divisible* (cf. [15]) if for each $b \in A$ with $b \neq 0$ and each $n \in N$ there exists $a \in A$ such that

(i$_1$) $n \cdot a = b$,

(ii$_2$) $a < 2 \cdot a < 3 \cdot a < \ldots < (n-1) \cdot a < b$.

Lemma 3.3. (cf. [15]) *A is divisible if and only if G is divisible.*

Remark that if A is assumed to satisfy only the condition (i$_1$) then G need not be divisible (cf. [15]).

In 3.4–3.10 we suppose that A is a divisible MV-algebra.

Proposition 3.4. *Let $\emptyset \neq M \subseteq \mathcal{A}(A)$. Then $\overline{M}$ is closed with respect to the operation $\oplus$.*

Proof. Let $b_1, b_2 \in \overline{M}$. Then $b = b_1 \oplus b_2 \in \mathcal{A}(A)$ on account of Lemma 2.10. Assume that (a_n) is a sequence in A, $a \in A$ and $a_n \overset{b}{\to}_\alpha a$. We have to show that $a_n \to_{\alpha_0(M)} a$. By Corollary 2.9, $a_n \overset{b}{\to}_\beta a$. Then $c_n = |a_n - a|$ is a sequence in A and $c_n \overset{b}{\to}_\beta 0$. Thus for each $k \in N$ there exists $n_0 \in N$ such that

$$kc_n \leq b$$

whenever $n \in N$, $n \geq n_0$. According to Lemma 3.3, G is divisible. Then

$$c_n \leq \frac{1}{k}b = \frac{1}{k}(b_1 \oplus b_2) \leq \frac{1}{k}(b_1 + b_2) = \frac{1}{k}b_1 + \frac{1}{k}b_2$$

for each $n \in N$, $n \geq n_0$.

Using Riesz decomposition property for G, we get

$$c_n = c_n^1 + c_n^2, \quad 0 \leq c_n^1 \leq \frac{1}{k}b_1, \quad 0 \leq c_n^2 \leq \frac{1}{k}b_2$$

for each $n \in N$, $N \geq n_0$. Then

$$kc_n^1 \leq b_1, \qquad kc_n^2 \leq b_2$$

for each $n \in N$, $n \geq n_0$, i.e., $c_n^1 \overset{b_1}{\to}_\beta 0$, $c_n^2 \overset{b_2}{\to}_\beta 0$. Because $0 \leq c_n^i \leq c_n$ for $i = 1, 2$ and for each $n \in N$, we obtain that (c_n^1) and (c_n^2) are sequences in A. By Theorem 2.8, $c_n^1 \overset{b_1}{\to}_\alpha 0$ and $c_n^2 \overset{b_2}{\to}_\alpha 0$. The hypothesis implies $c_n^1 \to_{\alpha_0(M)} 0$ and $c_n^2 \to_{\alpha_0(M)} 0$. Applying Theorem 2.12 for $\alpha_0(M)$-convergence, we get $c_n = c_n^1 + c_n^2 = c_n^1 \oplus c_n^2 \to_{\alpha_0(M)} 0$. Consequently, $a_n \to_{\alpha_0(M)} a$. □

The above proof is a slight modification of the proof of [7: Lemma 2.12].

It is easy to verify that the inclusion $M \subseteq \overline{M}$ and Proposition 3.4 imply

$$\alpha_0(M) = \alpha_0(\overline{M}). \tag{1}$$

Lemma 3.5. *Let M_1 and M_2 be nonempty subsets of $\mathcal{A}(A)$. Then $\alpha_0(M_1) \leq \alpha_0(M_2)$ if and only if $\overline{M}_1 \subseteq \overline{M}_2$.*

The proof is simple, it will be omitted.

Lemma 3.6. *Let $\emptyset \neq M \subseteq \mathcal{A}(A)$. Then $\overline{M} \subseteq \widetilde{M}$.*

Proof. Let $b \in \overline{M}, (x_n)$ a sequence in G and $x \in G$. Assume that $x_n \overset{b}{\to}_\beta x$. Our purpose is to prove that $x_n \to_{\beta_0(M)} x$. We have $|x_n - x| \overset{b}{\to}_\beta 0$. Then there exists $m \in N$ such that $y_n = |x_n - x| \leq b$ for each $n \in N$, $n \geq m$, so (y_{n+m}) is a sequence in A and $y_{n+m} \overset{b}{\to}_\beta 0$. By Corollary 2.9, $y_{n+m} \overset{b}{\to}_\alpha 0$. Then, in view of the assumption, $y_{n+m} \to_{\alpha_0(M)} 0$. By Lemma 3.2, $y_{n+m} \to_{\beta_0(M)} 0$. From $y_n \to_{\beta_0(M)} 0$, we infer that $x_n \to_{\beta_0(M)} x$. Thus, $b \in \widetilde{M}$, and the proof is finished. □

Lemma 3.7. *Let $\emptyset \neq M \subseteq \mathcal{A}(A)$. Then $\beta_0(\overline{M}) = \beta_0(M)$.*

Proof. The relation $M \subseteq \overline{M}$ yields $\beta_0(M) \leq \beta_0(\overline{M})$. Using Lemmas 3.6 and 1.5, we get $\beta_0(\overline{M}) \leq \beta_0(\widetilde{M}) = \beta_0(M)$. □

Lemma 3.8. *Let $\emptyset \neq M \subseteq \mathcal{A}(A)$. Then $\overline{M} = \widetilde{M} \cap A$.*

Proof. In view of Lemma 3.6, $\overline{M} \subseteq \widetilde{M} \cap A$. Conversely, let $b \in \widetilde{M} \cap A$. Then $b \in \mathcal{A}(A)$. In order to prove that $b \in \overline{M}$, assume that (a_n) is a sequence in A, $a \in A$ and $a_n \overset{b}{\to}_\alpha a$. By Corollary 2.9, $a_n \overset{b}{\to}_\beta a$. Let $k \in N$. Then, there exists $n_1 \in N$ such that

$$k|a_n - a| \leq b$$

whenever $n \in N$, $n \geq n_1$. Thus, $k|a_n - a| \in A$ for every $n \in N$, $n \geq n_1$.

From $b \in \widetilde{M}$ and $a_n \overset{b}{\to}_\beta a$, we infer that $a_n \to_{\beta_0(M)} a$. Then, there exist $n_2 \in N$ and $b_1, \dots, b_m \in M$ such that

$$k|a_n - a| \leq b_1 + \dots + b_m$$

for every $n \in N$, $n \geq n_2$.

If $n_0 = \max(n_1, n_2)$, then, for each $n \in N, n \geq n_0$, we get $k|a_n - a| = k|a_n - a| \wedge u \leq (b_1 + \dots + b_m) \wedge u = b_1 \oplus \dots \oplus b_m$. Putting $b' = b_1 \oplus \dots \oplus b_m$, we have $b' \in \mathcal{A}(A)$ and $a_n \overset{b'}{\to}_\beta a$. By Corollary 2.9, $a_n \overset{b'}{\to}_\alpha a$. Consequently, $a_n \to_{\alpha_0(M)} a$. Thus $b \in \overline{M}$. □

By (1), we get $\alpha_0(M) = \alpha_0(\overline{M}) = \alpha(\overline{M})$ for each $\emptyset \neq M \subseteq \mathcal{A}(A)$. The relation $\alpha_0(M) = \alpha(M)$ is fulfilled for each $M \in \mathcal{S}(A)$. Consequently, $s(A)$ is equal to the system $s_0(A)$ of all $\alpha_0(M)$ where M runs over all nonempty subsets of $\mathcal{A}(A)$.

Theorem 3.9. *There exists an isomorphism of the partially ordered set $s(A)$ into $s(G)$.*

Proof. Instead of $s(A)$ and $s(G)$ we can consider $s_0(A)$ and $s_0(G)$, respectively. Assume that $\emptyset \neq M \subseteq \mathcal{A}(A)$. Define a mapping $f\colon s_0(A) \to s_0(G)$ by putting $f(\alpha_0(M)) = \beta_0(M)$.

For proving that f is corretly defined, suppose that M_1 and M_2 are nonempty subsets of $\mathcal{A}(A)$ and $\alpha_0(M_1) = \alpha_0(M_2)$ is satisfied. With respect to (1), $\alpha_0(\overline{M}_1) = \alpha_0(\overline{M}_2)$. By Lemma 3.5, $\overline{M}_1 = \overline{M}_2$, so, $\beta_0(\overline{M}_1) = \beta_0(\overline{M}_2)$. Using Lemma 3.7, we get $\beta_0(M_1) = \beta_0(M_2)$.

If the same arguments are applied, we get that f preserves the partial order $\leq$ from $s(A)$.

Let $\beta_0(M_1) \leq \beta_0(M_2)$. According to Lemma 1.5, $\beta_0(\widetilde{M}_1) \leq \beta_0(\widetilde{M}_2)$. By Lemma 1.6, we have $\widetilde{M}_1 \subseteq \widetilde{M}_2$. With respect to Lemma 3.8, $\overline{M}_1 = \widetilde{M}_1 \cap A \subseteq \widetilde{M}_2 \cap A = \overline{M}_2$. Hence $\alpha_0(\overline{M}_1) \leq \alpha_0(\overline{M}_2)$ and by (1), $\alpha_0(M_1) \leq \alpha_0(M_2)$.

Therefore the mapping f is injective and the proof is complete. □

Let us return to the results of the paper [3] in Theorem 3.3. Essential part of Theorem 3.3 is the following assertion:

> *If $A = \Gamma(G, u)$ then there exists a one-to-one correspondence between the system of all MV-convergences on A and the system of all lu-convergences on G.*

It is evident that neither the above Theorem 3.9 is a corollary of [3: Theorem 3.3] nor [3: Theorem 3.3] is a corollary of Theorem 3.9.

Theorem 3.10. *The set $s(A)$ is a complete Brouwerian lattice. If I is a nonempty set and $M_i \in \mathcal{S}(A)$ for each $i \in I$, then*

$$\bigwedge_{i\in I} \alpha(M_i) = \alpha\Big(\bigcap_{i\in I} \overline{M}_i\Big), \qquad \bigvee_{i\in I} \alpha(M_i) = \alpha\Big(\overline{\bigcup_{i\in I} M_i}\Big). \tag{2}$$

Proof. According to Theorem 1.8, $s(G)$ is a complete Browerian lattice. Analogously as in [7], we can prove that also $s(A)$ is a complete lattice and that the relations (2) are satisfied.

The sets $\bigcap_{i\in I}\overline{M}_i$, $\overline{\bigcup_{i\in I} M_i}$ and all M_i belong to $\mathcal{S}(A)$. In view of (1), $\alpha_0\big(\overline{\bigcup_{i\in I} M_i}\big) = \alpha_0\big(\bigcup_{i\in I} M_i\big)$. Then the relation (2) can be written in the form

$$\bigwedge_{i\in I}\alpha_0(M_i) = \alpha_0\Big(\bigcap_{i\in I}\overline{M}_i\Big), \qquad \bigvee_{i\in I}\alpha_0(M_i) = \alpha_0\Big(\overline{\bigcup_{i\in I} M_i}\Big). \tag{3}$$

It remains to prove that the lattice $s(A)$ is Brouwerian. A slightly modified procedure from [7] will be applied.

We suppose that M and M_i are elements of $\mathcal{S}(A)$ for each $i\in I$. We have to prove the relation

$$\alpha(M)\wedge\Big(\bigvee_{i\in I}\alpha(M_i)\Big) = \bigvee_{i\in I}\big(\alpha(M)\wedge\alpha(M_i)\big).$$

According to (3), we get

$$\begin{aligned}\alpha(M)\wedge\Big(\bigvee_{i\in I}\alpha(M_i)\Big) &= \alpha_0(M)\wedge\Big(\bigvee_{i\in I}\alpha_0(M_i)\Big)\\ &= \alpha_0(M)\wedge\alpha_0\Big(\bigcup_{i\in I} M_i\Big) = \alpha_0\Big(\overline{M}\cap\Big(\overline{\bigcup_{i\in I} M_i}\Big)\Big)\end{aligned}$$

and

$$\begin{aligned}\bigvee_{i\in I}\big(\alpha(M)\wedge\alpha(M_i)\big) &= \bigvee_{i\in I}\big(\alpha_0(M)\wedge\alpha_0(M_i)\big)\\ &= \bigvee_{i\in I}\alpha_0\big(\overline{M}\cap\overline{M}_i\big) = \alpha_0\Big(\bigcup_{i\in I}\big(\overline{M}\cap\overline{M}_i\big)\Big).\end{aligned}$$

It is sufficient to verify the validity of the relation

$$\alpha_0\Big(\overline{M}\cap\overline{\bigcup_{i\in I} M_i}\Big) \le \alpha_0\Big(\bigcup_{i\in I}\big(\overline{M}\cap\overline{M}_i\big)\Big).$$

Assume that (a_n) is a sequence in A, $a\in A$ and $a_n \to_{\alpha_0(\overline{M}\cap\overline{\bigcup_{i\in I} M_i})} a$. Then $a_n\to_{\alpha_0(\overline{M})} a$ and $a_n\to_{\alpha_0(\overline{\bigcup_{i\in I} M_i})} a$. From $\alpha_0\big(\overline{\bigcup_{i\in I} M_i}\big) = \alpha_0\big(\bigcup_{i\in I} M_i\big) \le \alpha_0\big(\bigcup_{i\in I}\overline{M}_i\big)$ it follows that $a_n\to_{\alpha_0(\bigcup_{i\in I}\overline{M}_i)} a$. Therefore $a_n \overset{b}{\to}_\alpha a$ and $a_n \overset{b'}{\to}_\alpha a$ where $b = b_1\oplus\dots\oplus b_m$ for some $b_1,\dots,b_m\in\overline{M}$ and $b' = b'_1\oplus\dots\oplus b'_p$ for some $b'_1,\dots,b'_p\in\bigcup_{i\in I}\overline{M}_i$. By Corollary 2.9, $a_n\overset{b}{\to}_\beta a$ and $a_n\overset{b'}{\to}_\beta a$. Then for each $k\in N$, there exists $n_0\in N$ such that

$$k|a_n - a|\le b \qquad\text{and}\qquad k|a_n-a|\le b'$$

for each $n \in N$, $n \geq n_0$.

Consequently,

$$\begin{aligned} k|a_n - a| &\leq b \wedge b' = (b_1 \oplus \dots \oplus b_m) \wedge (b'_1 \oplus \dots \oplus b'_p) \\ &= (b_1 + \dots + b_m) \wedge u \wedge (b'_1 + \dots + b'_p) \wedge u \\ &= (b_1 + \dots + b_m) \wedge (b'_1 + \dots + b'_p) \wedge u \\ &\leq (b_1 \wedge b'_1 + \dots + b_1 \wedge b'_p + \dots + b_m \wedge b'_1 + \dots + b_m \wedge b'_p) \wedge u \\ &= (b_1 \wedge b'_1) \oplus \dots \oplus (b_1 \wedge b'_p) \oplus \dots \oplus (b_m \wedge b'_1) \oplus \dots \oplus (b_m \wedge b'_p). \end{aligned}$$

Putting $b_0 = (b_1 \wedge b'_1) \oplus \dots \oplus (b_1 \wedge b'_p) \oplus \dots \oplus (b_m \wedge b'_1) \oplus \dots \oplus (b_m \wedge b'_p)$, we obtain $a_n \overset{b_0}{\to}_\beta a$ and by Corollary 2.9, $a_n \overset{b_0}{\to}_\alpha a$.

We have $b_j \wedge b'_\ell \leq b_j, b'_\ell$ $(j = 1, \dots, m;\ \ell = 1, \dots, p)$, where $b_j \wedge b'_\ell \in \overline{M} \cap \big(\bigcup\limits_{i \in I} \overline{M}_i \big) = \bigcup\limits_{i \in I} \big(\overline{M} \cap \overline{M}_i \big)$ $(j = 1, \dots, m;\ \ell = 1, \dots, p)$. We deduce that $a_n \to_{\alpha_0 (\bigcup_{i \in I} (\overline{M} \cap \overline{M}_i))} a$, as desired. □

4. Atoms and dual atoms in $s(A)$

In view of Theorem 3.10, the lattice $s(A)$ has the least element and the greatest element; these will be denoted by α^0 and α^1, respectively.

The notion of atom of $s(A)$ is defined in the usual way. Analogously, an element α of $s(A)$ is defined to be a dual atom if $\alpha < \alpha^1$ and if there does not exist any element α' in $s(A)$ with $\alpha < \alpha' < \alpha^1$.

If $y \in \mathcal{A}(A)$ and $M = \{n \cdot y\}_{n \in \mathbb{N}}$, then instead of $\alpha(A, M)$ we will write simply $\alpha(A, y)$.

Our aim is to prove the following results.

Theorem 4.1. *Assume that the MV-algebra A is archimedean and divisible. Let $\alpha \in s(A)$. Then the following conditions are equivalent:*

(i) *α is an atom of $s(A)$;*

(ii) *there exists an element $y \in \mathcal{A}(A)$ with $y > 0$ such that $\alpha = \alpha(A, y)$ and the inverval $[0, y]$ of A is a chain.*

Theorem 4.2. *Let A be as in* Theorem 4.1. *Denote by $a(s(A))$ and $a'(s(A))$ the set of all atoms or the set of all dual atoms in $s(A)$, respectively. Then* $\operatorname{card} a(s(A)) \leq \operatorname{card} a'(s(A))$.

We need some lemmas.

Lemma 4.3. *Let α be an atom of $s(A)$. Then there is $y \in \mathcal{A}(A)$ such that $\alpha = \alpha(A, y)$.*

P r o o f. There exists $M \subseteq \mathcal{A}(A)$ such that $\alpha = \alpha(A, M)$. Since $\alpha > \alpha^0$, there exists a sequence (t_n) in A such that $0 < t_n$ for each $n \in \mathbb{N}$ and $t_n \to_\alpha 0$; having in mind this relation, we conclude that there exists $y \in M$ such that y is the corresponding regulator. Then we also have $t_n \to_{\alpha(A,y)} 0$, whence $\alpha^0 < \alpha(A, y)$. Since $y \in M$, we get $\alpha(A, y) \leqq \alpha$. From this and from the fact that α is an atom of $s(A)$, we conclude that $\alpha = \alpha(A, y)$. □

The following assertion is easy to verify.

Lemma 4.4. *Let M_1 and M_2 be nonempty subsets of $\mathcal{A}(A)$ such that they are closed with respect to the operation $\oplus$. Assume that $m_1 \wedge m_2 = 0$ for each $m_1 \in M_1$ and each $m_2 \in M_2$. Then $\alpha(A, M_1) \wedge \alpha(A, M_2) = \alpha^0$.*

In Lemmas 4.5 and 4.6, we assume that A is an MV-algebra which is archimedean and divisible. We also suppose that $A \neq \{0\}$.

Lemma 4.5. *The relation $\overline{\{0\}} = \{0\}$ is valid.*

P r o o f. Let $0 < b \in A$. We apply the fact that A is divisible; we put $a_n = \frac{1}{n}b$ for each $n \in N$. Let k be a positive integer. Then there exists $n_0 \in \mathbb{N}$ with $ka_n \leq b$ for each $n \in \mathbb{N}$, $n \geq n_0$. Hence $a_n \xrightarrow{b}_\beta 0$. By Theorem 2.8, $a_n \xrightarrow{b}_\alpha 0$. This shows that b does not belong to $\overline{\{0\}}$. □

Lemma 4.6. *Let α and y be as in* Lemma 4.3*. Then the interval $[0, y]$ of A is a chain.*

P r o o f. By way of contradiction, assume that the interval $[0, y]$ of A fails to be a chain. Then there are elements $q_1, q_2 \in [0, y]$ such that $q_i > 0$ for $i = 1, 2$ and $q_1 \wedge q_2 = 0$. In view of Lemma 4.5, we have $\alpha^0 < \alpha(A, q_i)$, and clearly $\alpha(A, q_i) \leq \alpha(A, y)$ for $i = 1, 2$. Since α is an atom in $s(A)$, we have $\alpha(A, q_i) = \alpha$ for $i = 1, 2$, hence $\alpha(A, q_1) = \alpha(A, q_2)$. In view of Lemma 4.4 and Lemma 4.5, we arrived at a contradiction. □

The following assertion is easy to verify.

Lemma 4.7. *Assume that A_1 is a linearly ordered MV-algebra. Let y_1 and y_2 be positive archimedean elements of A_1. Then $\alpha(A_1, y_1) = \alpha(A_1, y_2)$.*

Sketch of the proof. First, we show that $a_n \xrightarrow{y}_\alpha 0$ if and only if $a_n \to_{\alpha(A,y)} 0$ for each sequence (a_n) in A_1 and each element y of A_1. Further, we verify that if A_1 is linearly ordered then G is linearly ordered, too. Indeed, if G is not linearly ordered then there are $0 < x$, $y \in G$ with the property $x \wedge y = 0$. The relation $g \wedge u > 0$ is valid for each $0 < g \in G$. By using these results we obtain that A_1 fails to be linearly ordered.

According to [14], we have:

Proposition 4.8. *Assume that A is an archimedean MV-algebra. Let C be a convex chain in A, $\{0\} \subset C$. Then there exists a uniquely determined maximal convex chain C' in A with $C \subseteq C'$. The set C' is closed with respect to the operation $\oplus$. Moreover, C' is a direct factor of A; thus, there is an MV-algebra D with $A = C' \times D$.*

As an easy consequence of Lemma 4.7 and Proposition 4.8, we obtain:

Lemma 4.9. *Let A be an archimedean MV-algebra and let C be a convex chain in A with $0 \in C$. Assume that y_1 and y_2 are nonzero elements of C. Then $\alpha(A, y_1) = \alpha(A, y_2)$.*

Lemma 4.10. *Let A be an archimedean and divisible MV-algebra. Assume that C is a convex chain in A with $0 \in C$ and let $0 < y \in C$. Then the convergence $\alpha(A, y)$ is an atom in $s(A)$.*

Proof. By way of contradiction, assume that there exists $\alpha \in s(A)$ such that $\alpha^0 < \alpha < \alpha(A, y)$. Under the usual notation, let $\alpha = \alpha(A, M)$. Then there exists $y_1 \in M$ such that $\alpha^0 < \alpha(A, y_1)$. We obviously have $\alpha(A, y_1) \leq \alpha(A, M)$, thus

$$\alpha(A, y_1) < \alpha(A, y). \tag{1}$$

If $y_1 \geq y$, then $\alpha(A, y_1) \geq \alpha(A, y)$, contradicting (1). Clearly, $y_1 > 0$. If $y_1 < y$, then $y_1 \in C$ and then Lemma 4.9 yields $\alpha(A, y_1) = \alpha(A, y)$; in view of (1), we arrived at a contradiction.

Suppose that y and y_1 are incomparable. Put $y \wedge y_1 = p$ and $y' = y - p$, $y_1' = y_1 - p$. We have $0 < y'$ and $0 < y_1'$. Then y' and y_1' belong to A and

$$y' \wedge y_1' = 0. \tag{2}$$

According to Lemma 4.4 and in view of (2) we get

$$\alpha(A, y') \wedge \alpha(A, y_1') = \alpha^0.$$

Further, we have $0 < y' < y$, hence $y' \in C$ and so, Lemma 4.9 yields $\alpha(A, y') = \alpha(A, y)$. Further, we have

$$\alpha^0 < \alpha(A, y_1') \leq \alpha(A, y_1) \leq \alpha(A, M) = \alpha < \alpha(A, y).$$

Hence

$$\alpha(A, y') \wedge \alpha(A, y_1') = \alpha(A, y) \wedge \alpha(A, y_1') = \alpha(A, y_1') > \alpha^0;$$

again, we arrived at a contradiction. This completes the proof. □

In view of Lemma 4.6 and Lemma 4.10, Theorem 4.1 is valid.

Let us apply the assumptions and the notation as in Proposition 4.8. For each element a of A we put

$$a^\perp = \{x \in A : \ x \wedge a = 0\}.$$

From Proposition 4.8, we conclude that for each $c \in C'$ the relation

$$c^{\perp} = D$$

is valid.

Let A, C', D be as in Proposition 4.8, $A = C' \times D$. Let us denote by $u(C')$ and $u(D)$ the component of u in C' and in D, respectively. Then the lattice $[0, u]$ is the direct product of the lattices $[0, u(C')]$, and $[0, u(D)]$, $[0, u] = [0, u(C')] \times [0, u(D)]$ and both direct product decompositions of the MV-algebra A and of the lattice $[0, u]$ coincide. This is a consequence of the fact that the lattice operations $\vee$ and $\wedge$ are defined by means of the operations $+, *$ and $\neg$.

The mentioned connection between direct product decompositions is used in the relation (r) below and also in the implication $A = C' \times D \implies D = C'^{\perp}$. This is applied to obtain the above equation $c^{\perp} = D$ for each $c \in C'$.

Proposition 4.11. *Assume that A is an archimedean and divisible MV-algebra. Let C be a convex chain in A, $0 \in C$ and $0 < y \in C$. Then the convergence $\alpha(A, y^{\perp})$ is a dual atom of the lattice $s(A)$.*

Proof. The set $y^{\perp}$ is closed with respect to the operation $\oplus$; we can construct the convergence $\alpha(A, y^{\perp})$. We put $x_n = \frac{1}{n}y$ for each $n \in N$. Analogously, as in the proof of Lemma 4.5, we can verify that $x_n \to_{\alpha(A,y)} 0$ in A. In view of Lemma 4.4, the relation $x_n \to_{\alpha(A,y^{\perp})} 0$ fails to be valid, hence $\alpha(A, y)$ fails to be equal or less than $\alpha(A, y^{\perp})$. Therefore, we have $\alpha(A, y^{\perp}) < \alpha^1$.

Assume that $\alpha \in s(A)$, $\alpha(A, y^{\perp}) < \alpha$. Under the standard notation, let $\alpha = \alpha(A, M)$. Then, we also have $\alpha = \alpha(A, \overline{M})$. We get $y^{\perp} \subset \overline{M}$. Thus, there exists $y_1 \in \overline{M}$ such that y_1 does not belong to $y^{\perp}$.

Consider the direct product decomposition $A = C' \times D$ from Proposition 4.8.

Let $y_1(C')$ and $y_1(D)$ be the component of y_1 in C' or in D, respectively. Then

$$y_1 = y_1(C') \oplus y_1(D) = y_1(C') \vee y_1(D). \tag{r}$$

If $y_1(C') = 0$, then $y_1 = y_1(D) \in y^{\perp}$ and we arrived at a contradiction. Thus, $y_1(C') > 0$.

Our aim is to prove that $\alpha = \alpha^1$. Clearly, $\alpha^1 = \alpha(A, u)$. Let $(t_n)_{n \in N}$ be a sequence in A such that $t_n \to_{\alpha(A,u)} 0$. For any element a of A, we denote its components in C' and in D by a^1 and a^2, respectively. Then we have

$$t_n^1 \to_{\alpha(A,u^1)} 0, \qquad t_n^2 \to_{\alpha(A,u^2)} 0. \tag{$*$}$$

From the second relation of $(*)$, we infer that $t_n^2 \to_{\alpha(A,\overline{M})} 0$. Since $u^1 \in C'$ and $y \in C'$, from Lemma 4.9, we conclude that $\alpha(A, u^1) = \alpha(A, y)$, thus $t_n^1 \to_{\alpha(A,y)} 0$.

Because $\alpha(A, \overline{M}) \leqq \alpha$ and $\alpha(A, y) \leqq \alpha$, we get

$$t_n^1 \to_\alpha 0, \qquad t_n^2 \to_\alpha 0$$

and hence $t_n = t_n^1 \vee t_n^2 \to_\alpha 0$. Therefore, $\alpha = \alpha^1$, completing the proof. □

It is well-known that if an MV-algebra B possesses direct product decompositions $B = C_1 \times D_1$ and $B = C_2 \times D_2$, then

$$C_1 = C_2 \implies D_1 = D_2;$$

namely, we have $D_1 = C_1^\perp$.

Thus, summarizing, the situation is as follows. Let A be an MV-algebra which is archimedean and divisible. Let $\alpha \in a(s(A))$. Then we have a direct product decomposition $A = C_1' \times D_1$ with the properties as in Proposition 4.8. Hence C_1' is linearly ordered; according to Proposition 4.11, we get $\alpha(A, D_1) \in a'(s(A))$. Moreover, in view of the above remark, D_1 is uniquely determined. We put $\varphi(\alpha) = \alpha'$, where $\alpha' = \alpha(A, D_1)$. We obtain an injective mapping of the set $a(s(A))$ into $a'(s(A))$. Hence we obtain the relation $\operatorname{card} a(s(A)) \leq \operatorname{card} a'(s(A))$. Therefore, Theorem 4.2 is valid.

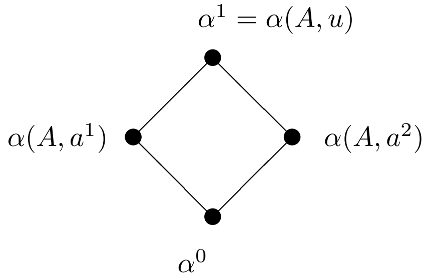
It is an open question whether the following assertion holds:

(+) Each dual atom of $s(A)$ can be obtained by the method described in Proposition 4.11.

Example 4.12. Let R be the additive group of all reals with the natural linear order and $B = \Gamma(R, 1)$. Assume that I is a nonempty set and that MV-algebra A is the direct product of MV-algebras A_i where $A_i = B$ for each $i \in I$ (for the direct product of MV-algebras cf. [4]). For any element $x \in A$ we denote its component in A_i by $x(i)$. Let u be the greatest element of A. Then $u(i) = 1$ for each $i \in I$. Given $i \in I$, denote by u^i the element of A with components $u^i(j) = 0$ for each $j \in J$, $j \neq i$, $u^i(i) = u(i)$ and we put $M_i = \{a \in A : a(i) = 0\}$. Particularly, the element $u_i \in A$ such that $u_i(j) = 1$ for each $j \in I$, $j \neq i$ and $u_i(i) = 0$ is included in M_i. In view of Theorem 4.1 and Proposition 4.11 (or directly, applying definitions of an atom and of a dual atom) we obtain that $\{\alpha(A, u^i) : i \in I\}$ and $\{\alpha(A, M_i) : i \in I\}$ are systems of all atoms and all dual atoms in $s(A)$, respectively. Evidently, $\{\alpha(A, M_i) : i \in I\} = \{\alpha(A, u_i) : i \in I\}$.

Especially, if $I = \{1, 2\}$, then two convergences $\alpha(A, u^1)$ and $\alpha(A, u^2)$ are all atoms and at the same time all dual atoms in $s(A)$. Hence the lattice $s(A)$ possesses the diagram on page 31.

In the previous example the supremum of all atoms in $s(A)$ satisfies the relation $\bigvee\limits_{i \in I} \alpha(A, u^i) = \alpha^1$. There arises a question if there exists an MV-algebra A and an element $\alpha \in s(A)$ covering the supremum of all atoms in $s(A)$.



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