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A NEW PROOF OF THE MONOTONE CONVERGENCE THEOREM OF LEBESGUE INTEGRAL ON σ-CLASS

Dinh Trung Hoa

Dedicated to Prof. David J. Foulis on the occasion of his 80th birthday

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. In this short note a new proof of the monotone convergence theorem of Lebesgue integral on σ-class is given.

1. Introduction

In the paper [1], J. Šipoš gave a proof of the monotone convergence theorem of Lebesgue integral on σ-class on language of quasi-integrals. In the other paper [2], S. Gudder and J. Zerbe have considered generalized measure and integration and proved the generalized monotone convergence theorem in the quantum logic framework which generalizes the traditional Hilbert space approach to quantum mechanics. In this work we give a combinatorial proof of the monotone convergence theorem of Lebesgue integral on σ-class. We first prove some reduced forms of the theorem and then using the limit transition to obtain the main theorem.

Let Ω be a non-empty set. The *σ-class* Δ of subsets of Ω is a collection of subsets of Ω which satisfies the following conditions

(1) $\Omega \in \Delta$,

(2) if $A \in \Delta$, then $A^c \in \Delta$,

(3) if $A_i \in \Delta \quad (i = 1, 2, \dots)$ are mutually disjoint, then $\bigcup_i A_i \in \Delta$,

where we denote the complement of the set A by A^c.

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Recall that a function $f\colon \Omega \to \mathbb{R}$ is *measurable* if $f^{-1}(A) \in \Delta$ for every $A \in \Delta$. Let μ be a measure on Δ. The triple (Ω, Δ, μ) is called to be a *generalized measurable space* (see [2], [3]). Let $f\colon \Omega \to \mathbb{R}$ be a measurable function, and $f^{-1}(B(\mathbb{R})) \subset \Delta$, the σ-algebra generated by f. Then $(\Omega, f^{-1}(B(\mathbb{R})), \mu)$ is a measure space, so $\int f \,\mathrm{d}\mu$ may be defined in the usual way on this space.

Note that the 3$^{\text{rd}}$ condition in definition of σ-class is an extension of the 3$^{\text{rd}}$ condition in definition of σ-algebra. Because of this extension many facts in the classical measure theory do not hold true in the case of σ-class (see [2], [3]).

2. Main results

Throughout the paper for sets A_i $(i \in \mathbb{N})$ the notation $A_1 + \dots + A_n + \dots$ or $\sum\limits_i A_i$ means the direct sum of these sets and they are mutually disjoint.

Lemma 1. *Let functions* $f = \sum\limits_{i=1}^{+\infty} \lambda_i \chi_{X_i}$, $g = \beta_1 \chi_{Y_1} + \beta_2 \chi_{Y_2}$ (*where* $\sum\limits_{i=1}^{+\infty} X_i = Y_1 + Y_2 = \Omega$) *satisfy the following condition*

$$f(\omega) \geq g(\omega) \geq 0 \qquad \text{for all} \quad \omega \in \Omega. \tag{1}$$

Then

$$\int f \,\mathrm{d}\mu \geq \int g \,\mathrm{d}\mu. \tag{2}$$

Proof. To prove the lemma, we consider four different cases.

The first case: either Y_1 or Y_2 is contained in some X_i $(i \in \mathbb{N})$. Without loss of generality we assume that $Y_1 \subset X_1$. Since $\sum\limits_{i=1}^{+\infty} X_i = Y_1 + Y_2 = \Omega$, it follows that $X_i \subset Y_2$ $(i > 2)$. On account of (1), we have

$$\begin{aligned}\int f \,\mathrm{d}\mu &= \lambda_1 \mu(X_1) + \sum_{i=2}^{+\infty} \lambda_i \mu(X_i) \\ &= \lambda_1 \mu(Y_1) + \lambda_1 \mu(X_1 \setminus Y_1) + \sum_{i=2}^{+\infty} \lambda_i \mu(X_i) \\ &\geq \beta_1 \mu(Y_1) + \beta_2 \Bigl(\mu(X_1 \setminus Y_1) + \sum_{i=2}^{+\infty} \mu(X_i) \Bigr) \\ &\geq \beta_1 \mu(Y_1) + \beta_2 \mu(Y_2) = \int g \,\mathrm{d}\mu.\end{aligned}$$

The second case: for any $i \in \mathbb{N}$ ($X_i \subset Y_1$ or $X_i \subset Y_2$). Denote

$$I_1 = \{i : \ X_i \subset Y_1\} \quad \text{and} \quad I_2 = \{i : \ X_i \subset Y_2\}.$$

We have

$$\begin{aligned}\int g \, \mathrm{d}\mu &= \beta_1 \mu(Y_1) + \beta_2 \mu(Y_2) \\ &= \beta_1 \sum_{i \in I_1} \mu(X_i) + \beta_2 \sum_{i \in I_2} \mu(X_i) \\ &\le \sum_{i \in I_1} \lambda_i \mu(X_i) + \sum_{i \in I_2} \lambda_i \mu(X_i) \\ &= \sum_{i=1}^{+\infty} \lambda_i \mu(X_i) = \int f \, \mathrm{d}\mu.\end{aligned}$$

The third case: for any $i \in \mathbb{N}$

$$\begin{cases} X_i \cap Y_1 \neq \emptyset, \\ X_i \cap Y_2 \neq \emptyset. \end{cases}$$

According to (1), we have

$$\lambda_i \ge \max\{\beta_1, \beta_2\} := \beta_0 \qquad (\text{for all} \quad i \in \mathbb{N}).$$

Consequently,

$$\begin{aligned}\int f \, \mathrm{d}\mu = \sum_{i=1}^{+\infty} \lambda_i \mu(X_i) &\ge \beta_0 \mu(\Omega) = \beta_0 (\mu(Y_1) + \mu(Y_2)) \\ &\ge \beta_1 \mu(Y_1) + \beta_2 \mu(Y_2) = \int g \, \mathrm{d}\mu.\end{aligned}$$

The fourth case: there exist some sets X_i such that $X_i \cap Y_1 \neq \emptyset$ and $X_i \cap Y_2 \neq \emptyset$, and the remaining sets are contained in either Y_1 or Y_2.

We denote

$$\begin{aligned} I_1 &= \{i : \ X_i \subset Y_1 \ \text{ or } \ X_i \subset Y_2\}, \\ I_2 &= \{i : \ X_i \cap Y_1 \neq \emptyset \ \text{ and } \ X_i \cap Y_2 \neq \emptyset\}. \end{aligned}$$

It is evident that $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = \mathbb{N}$. Let us consider the new sets

$$\Omega' = \Omega \setminus \bigcup_{i \in I_1} X_i = \bigcup_{i \in I_2} X_i,$$

$$Y_1' = Y_1 \setminus \bigcup_{i \in I_1} X_i, \quad Y_2' = Y_2 \setminus \bigcup_{i \in I_1} X_i.$$

Note that for all $i \in I_2$

$$\begin{cases} X_i \cap Y_1' \neq \emptyset, \\ X_i \cap Y_2' \neq \emptyset. \end{cases}$$

Obviously, Y_1', Y_2' and X_i $(i \in I_2)$ form a σ-class Δ' on Ω'. The restrictions of functions f and g on Ω' are integrable with respect to Δ' and again satisfy condition (1). Applying the above argument, we obtain

$$\int_{\Omega'} f \,\mathrm{d}\mu \geq \int_{\Omega'} g \,\mathrm{d}\mu,$$

i.e.,

$$\sum_{i\in I_2} \lambda_i \mu(X_i) \geq \beta_1 \mu\Big(Y_1 \setminus \bigcup_{i\in I_1} X_i\Big) + \beta_2 \mu\Big(Y_2 \setminus \bigcup_{i\in I_1} X_i\Big). \tag{3}$$

Denote

$$I_1^1 = \{i : \ X_i \subset Y_1\}, \quad \text{and} \quad I_1^2 = \{i : \ X_i \subset Y_2\}.$$

Therefore

$$\mu\Big(Y_1 \setminus \bigcup_{i\in I_1} X_i\Big) = \mu\Big(Y_1 \setminus \bigcup_{i\in I_1^1} X_i\Big) = \mu(Y_1) - \mu\Big(\bigcup_{i\in I_1} X_i\Big),$$

$$\mu\Big(Y_2 \setminus \bigcup_{i\in I_1} X_i\Big) = \mu\Big(Y_2 \setminus \bigcup_{i\in I_1^2} X_i\Big) = \mu(Y_2) - \mu\Big(\bigcup_{i\in I_2} X_i\Big).$$

From (3) we obtain

$$\begin{aligned}\sum_{i\in I_2} \lambda_i \mu(X_i) &\geq \beta_1\mu(Y_1) + \beta_2\mu(Y_2) - \beta_1 \sum_{i\in I_1^1} \mu(X_i) - \beta_2 \sum_{i\in I_1^2} \mu(X_i)\\ &\geq \beta_1\mu(Y_1) + \beta_2\mu(Y_2) - \sum_{i\in I_1^1} \lambda_i\mu(X_i) - \sum_{i\in I_1^2} \lambda_i\mu(X_i).\end{aligned}$$

Consequently,

$$\begin{aligned}\int f \,\mathrm{d}\mu &= \sum_{i\in I_2} \lambda_i\mu(X_i) + \sum_{i\in I_1^1} \lambda_i\mu(X_i) + \sum_{i\in I_1^2} \lambda_i\mu(X_i)\\ &\geq \beta_1\mu(Y_1) + \beta_2\mu(Y_2) = \int g \,\mathrm{d}\mu.\end{aligned}$$

□

Proposition 1. *Let functions* $f = \sum_{i=1}^{+\infty} \lambda_i \chi_{X_i}$, $g = \sum_{j=1}^{n} \beta_j \chi_{Y_j}$ *(where* $\sum_{i=1}^{+\infty} X_i = \sum_{j=1}^{n} Y_j = \Omega$*) satisfy the following condition*

$$f(\omega) \geq g(\omega) \geq 0 \qquad \textit{for all} \quad \omega \in \Omega. \tag{4}$$

Then

$$\int f \,\mathrm{d}\mu \geq \int g \,\mathrm{d}\mu. \tag{5}$$

For proof of this proposition we need the following lemma.

Lemma 2. *Let functions f, g be defined as in* Proposition 2*, in addition*

$$(\forall i)(\forall j)\big(X_i \subsetneq Y_j \ \ \& \ \ X_i \supsetneq Y_j\big). \tag{6}$$

Then

$$\int f \, \mathrm{d}\mu \geq \int g \, \mathrm{d}\mu.$$

P r o o f. The lemma is hence proved by using method of mathematical induction.

When $n = 2$, it reduces to Proposition 1.

We assume that the lemma holds true for all $k \leq n$. Now let $g = \sum\limits_{j=1}^{n+1} \beta_j \chi_{Y_j}$ satisfy (4) and (6). Without loss of generality we assume that

$$\beta_1 > \beta_2 > \cdots > \beta_n > \beta_{n+1}.$$

Consider the measurable with respect to σ-algebra Δ function

$$h = \sum_{j=1}^{n-1} \beta_j \chi_{Y_j} + \beta_n \chi_{Y_n \cup Y_{n+1}}.$$

Evidently,

$$h(\omega) \geq g(\omega).$$

Therefore,

$$\int h \, \mathrm{d}\mu \geq \int g \, \mathrm{d}\mu. \tag{7}$$

On the other hand, since

$$h(\omega) = g(\omega) \leq f(\omega) \qquad (\text{for all} \quad \omega \in Y_j, \quad j = \overline{1, n-1}\,)$$

and for all $\omega \in Y_n \cup Y_{n+1}$

$$h(\omega) = \beta_n \leq \beta_j \qquad (j = \overline{1, n-1}\,),$$

we obtain

$$h(\omega) \leq f(\omega) \qquad (\text{for all} \quad \omega \in \Omega).$$

By the inductive assumption it follows that

$$\int f \, \mathrm{d}\mu \geq \int h \, \mathrm{d}\mu. \tag{8}$$

Combining (7) and (8) we obtain

$$\int f \, \mathrm{d}\mu \geq \int g \, \mathrm{d}\mu.$$

□

P r o o f o f P r o p o s i t i o n 1. If the functions f, g satisfy conditions (4) and (6), then by Lemma 2, inequality (5) holds true. On the other hand, if there exist i_0, j_0 such that $X_{i_0} \subset Y_{j_0}$ or $Y_{j_0} \subset X_{i_0}$, then for the simplicity and without loss of generality we may assume that $X_{i_0} \subset Y_{j_0}$ (if the number of such sets is greater than 2, then we exclude them from further consideration. In the end we will have sets that are not subsets of the others.)

Evidently,

$$Y_{j_0} \setminus X_{i_0} \in \Delta.$$

Put

$$\Omega' = \Omega \setminus X_{i_0}.$$

On Ω' we give the structure of σ-class

$$\Delta' = \{\Delta \setminus X_{i_0}, Y_{j_0} \setminus X_{i_0}\}.$$

It is not difficult to show that Δ' is a σ-class on Ω' and functions f, g satisfy conditions (4) and (6) with respect to Ω'. By Lemma 2 we get

$$\int\limits_{\Omega'} f \, \mathrm{d}\mu \geq \int\limits_{\Omega'} g \, \mathrm{d}\mu,$$

i.e.

$$\sum_{i=1}^{i_0-1} \lambda_i \mu(X_i) + \sum_{i=i_0+1}^{+\infty} \lambda_i \mu(X_i) \geq \sum_{j=1}^{j_0} \beta_j \mu(Y_i) + \beta_{j_0} \mu(Y_{j_0} \setminus X_{i_0}) + \sum_{j=j_0+1}^{n} \beta_j \mu(Y_j).$$

Consequently,

$$\begin{aligned}\int\limits_{\Omega} f \, \mathrm{d}\mu = \int\limits_{\Omega'} f \, \mathrm{d}\mu + \lambda_{i_0} \mu(X_{i_0}) &\geq \int\limits_{\Omega'} g \, \mathrm{d}\mu + \lambda_{i_0} \mu(X_{i_0}) \\ &\geq \int\limits_{\Omega'} g \, \mathrm{d}\mu + \beta_{j_0} \mu(X_{j_0}) = \int\limits_{\Omega} g \, \mathrm{d}\mu.\end{aligned}$$

The Proposition 1 is proved. □

If the function g in the proposition is represented by the form $g = \sum\limits_{j=1}^{+\infty} \beta_j \chi_{Y_j}$ and f, g satisfy condition (4), then

$$\int\limits_{\Omega} g \, \mathrm{d}\mu = \sum_{j=1}^{+\infty} \beta_j \mu(Y_j) = \lim_{n \to +\infty} \sum_{j=1}^{n} \beta_j \mu(Y_j) \leq \int\limits_{\Omega} f \, \mathrm{d}\mu.$$

Theorem 1 (The monotone convergence theorem). *Let functions $f, g\colon \Omega \to \mathbb{R}$ be measurable on σ-class Δ and $f(\omega) \geq g(\omega)$ (for all $\omega \in \Omega$). Then*

$$\int f \,\mathrm{d}\mu \geq \int g \,\mathrm{d}\mu.$$

Proof. Consider functions

$$f_n = \sum_{k\in\mathbb{Z}} \frac{k+1}{2^n}\chi_{X_k}, \qquad X_k = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right),$$

$$g_n = \sum_{k\in\mathbb{Z}} \frac{k}{2^n}\chi_{Y_k}, \qquad Y_k = g^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right).$$

Evidently, f_n, g_n are integrable with respect to Δ, while the sequence $\{f_n\}$ monotonously decreases to f and the sequence $\{g_n\}$ monotonously increases to g. Consequently,

$$\int g \,\mathrm{d}\mu = \lim_n g_n \,\mathrm{d}\mu \leq \lim_n f_n \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu.$$

□

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Center of Research and Development
Duy Tan University
K7/25 Quang Trung
Da Nang City
VIETNAM
E-mail: trunghoa.math@gmail.com