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SCHUR LEMMA AND LIMIT THEOREMS IN LATTICE GROUPS WITH RESPECT TO FILTERS

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Dedicated to Prof. David J. Foulis on the occasion of his 80th birthday

(*Communicated by Sylvia Pulmannová*)

ABSTRACT. Some Schur, Nikodým, Brooks-Jewett and Vitali-Hahn-Saks-type theorems for (ℓ)-group-valued measures are proved in the setting of filter convergence. Finally we pose an open problem.

1. Introduction

The theory of filter convergence, introduced in [13], has since been widely considered in the literature (see also [12]). The ideal convergence was introduced in [14] and independently in [15].

The aim of this paper is to give some conditions under which limit theorems, for instance Schur, Nikodým convergence, Brooks-Jewett, Vitali-Hahn-Saks theorems, hold with respect to filter convergence and for (ℓ)-group-valued measures.

In general it is impossible to give an answer analogous to the classical case when we deal with filter convergence even for positive real-valued measures (see [6: Example 3.4]).

The Schur l^1 theorem for Banach space-valued sequences was deeply studied in [1], where some results of [10] were extended to the filter convergence and the authors studied some classes of filters for which the Schur theorem holds and some other families of filters for which it is not valid. The technique used in [1] is inspired by the properties and the structure of Banach spaces. Further recent

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studies and developments of Schur theorems and related topics adapted to the context of filter or ideal convergence can be found in [11].

In this paper we extend the Schur l^1 theorem to (ℓ)-group-valued double sequences with respect to filter convergence, giving also an equivalent version for σ-additive (ℓ)-group-valued measures. We use some techniques of [1] and some other techniques inspired by [9]. As applications we present also some Nikodým convergence and Vitali-Hahn-Saks-type theorems for both σ-additive and finitely additive (ℓ)-group-valued measures, extending earlier results proved in [5]. Finally, we pose an open problem.

2. Preliminaries

Definitions 2.1.

(a) A Dedekind complete (ℓ)-group R is *super Dedekind complete* iff every nonempty subset $R_1 \subset R$ bounded from above contains a countable subset having the same supremum as R_1.

(b) Let R be an (ℓ)-group. We say that a sequence $(p_n)_n$ of positive elements of R is an (O)*-sequence* iff it is decreasing and $\bigwedge_n p_n = 0$.

(c) A bounded double sequence $(a_{t,r})_{t,r}$ in R is called (D)*-sequence* or *regulator* iff for all $t \in \mathbb{N}$ the sequence $(a_{t,r})_r$ is an (O)-sequence.

(d) An (ℓ)-group R is said to be *weakly σ-distributive* iff for every (D)-sequence $(a_{t,r})_{t,r}$ we have:

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \Big) = 0.$$

We now recall the following result, which will be useful in the sequel.

Lemma 2.1 (**Fremlin Lemma**). (see [5]) *Let R be any Dedekind complete (ℓ)-group, $(a_{t,r}^{(n)})_{t,r}$, $n \in \mathbb{N}$, be a sequence of regulators in R. Then for every $u \in R$, $u \geq 0$ there exists a (D)-sequence $(a_{t,r})_{t,r}$ in R such that:*

$$u \bigwedge \Big[\bigvee_{q} \Big(\sum_{n=1}^{q} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}^{(n)} \Big) \Big) \Big] \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \qquad \text{for all} \quad \varphi \in \mathbb{N}^{\mathbb{N}}.$$

We always assume that R is a Dedekind complete (ℓ)-group. We now recall the following:

Definitions 2.2. (see also [1])

(a) A *filter* $\mathcal{F}$ of $\mathbb{N}$ is a nonempty collection of subsets of $\mathbb{N}$ with $\emptyset \notin \mathcal{F}$, such that $A \cap B \in \mathcal{F}$ whenever A, $B \in \mathcal{F}$, and with the property that for each $A \in \mathcal{F}$ and $B \supset A$ we get $B \in \mathcal{F}$.

(b) Given a filter $\mathcal{F}$ of $\mathbb{N}$, we call *dual ideal* associated with $\mathcal{F}$ the collection $\{\mathbb{N} \setminus F : F \in \mathcal{F}\}$. The filter $\mathcal{F}_{\text{cofin}}$ is the filter of all subsets of $\mathbb{N}$ whose complement is finite, while its dual ideal $\mathcal{I}_{\text{fin}}$ is the ideal of all finite subsets of $\mathbb{N}$.

(c) A filter is said to be *free* iff it contains $\mathcal{F}_{\text{cofin}}$. In what follows, we always suppose that the involved filters are free.

(d) A filter $\mathcal{F}$ of $\mathbb{N}$ is called an *ultrafilter* iff for each $A \subset \mathbb{N}$ either A or $\mathbb{N} \setminus A$ belongs to $\mathcal{F}$.

(e) A subset of $\mathbb{N}$ is said to be *stationary* with respect to a filter $\mathcal{F}$ (or $\mathcal{F}$*-stationary*) iff it has nonempty intersection with every element of $\mathcal{F}$. We denote by $\mathcal{F}^*$ the collection of all $\mathcal{F}$-stationary sets.

(f) If $I \subset \mathbb{N}$ is an $\mathcal{F}$-stationary set, we call the *trace* of $\mathcal{F}$ on I the family of sets $\{A \cap I : A \in \mathcal{F}\}$ and we denote by $\mathcal{F}(I)$ the filter on $\mathbb{N}$ generated by the trace of $\mathcal{F}$ on I. Note that $\mathcal{F}(I) \supset \mathcal{F}$.

(g) We say that a filter $\mathcal{F}$ of $\mathbb{N}$ is *diagonal* iff for every decreasing sequence $(A_n)_n$ in $\mathcal{F}$ and for each $I \in \mathcal{F}^*$ there exists a set $J \subset I$, $J \in \mathcal{F}^*$ such that the set $J \setminus A_n$ is finite for all $n \in \mathbb{N}$.

Remark 1. Observe that the definition of diagonal filter can be formulated equivalently even without requiring that the involved sequence $(A_n)_n$ is decreasing. Indeed, if $(A_n)_n$ is any sequence in $\mathcal{F}$ and $A_1^* := A_1$, $A_n^* := \bigcap_{l=1}^{n} A_l$, $n \in \mathbb{N}$, then the sequence $(A_n^*)_n$ is in $\mathcal{F}$ and decreasing, and so for every $I \in \mathcal{F}^*$ there is a set $J \in \mathcal{F}^*$, $J \subset I$ such that $J \setminus A_n^*$ is finite for any $n \in \mathbb{N}$, and then a fortiori $J \setminus A_n \subset J \setminus A_n^*$ is finite too for all $n \in \mathbb{N}$.

Definitions 2.3.

(a) Given an infinite set $I \subset \mathbb{N}$, a *block* of I is a countable partition $\{D_k : k \in \mathbb{N}\}$ of I into nonempty finite subsets.

(b) A filter $\mathcal{F}$ of $\mathbb{N}$ is said to be *block-respecting* iff for every $I \in \mathcal{F}^*$ and for each block $\{D_k : k \in \mathbb{N}\}$ of I there exists a set $J \in \mathcal{F}^*$, $J \subset I$ such that $\text{card}(J \cap D_k) = 1$ for all $k \in \mathbb{N}$, where card denotes the number of elements of the set into brackets.

We recall the following results.

Proposition 2.1. (see [1: Remark 2.4]) *If $\mathcal{F}$ is a block-respecting filter, then $\mathcal{F}(J)$ is block-respecting too for every $J \in \mathcal{F}^*$.*

Proposition 2.2. (see [1: Remark 2.3]) *A filter $\mathcal{F}$ of $\mathbb{N}$ is block-respecting if and only if for every $I \in \mathcal{F}^*$ and for any block $\{D_k : k \in \mathbb{N}\}$ of I there exists a set $J \in \mathcal{F}^*$, $J \subset I$ such that card$(J \cap D_k) \le 1$ for all $k \in \mathbb{N}$.*

We now define the concepts of (O)- and (D)-convergence with respect to a filter in (ℓ)-groups (see also [6]).

Definitions 2.4.

(a) Let $\mathcal{F}$ be a filter of $\mathbb{N}$, we say that a sequence $(x_n)_n$ in R $(O\mathcal{F})$*-converges to* $x \in R$ iff there exists an (O)-sequence $(\sigma_p)_p$ such that $\{n \in \mathbb{N} : |x_n - x| \le \sigma_p\} \in \mathcal{F}$ for any $p \in \mathbb{N}$.

(b) A sequence $(x_n)_n$ in R $(D\mathcal{F})$*-converges to* $x \in R$ iff there is a (D)-sequence $(\alpha_{t,r})_{t,r}$ with the property that $\Big\{n \in \mathbb{N} : |x_n - x| \le \bigvee_{t=1}^{\infty} \alpha_{t,\varphi(t)}\Big\} \in \mathcal{F}$ for all $\varphi \in \mathbb{N}^{\mathbb{N}}$.

(c) Observe that, when $R = \mathbb{R}$, the $(O\mathcal{F})$- and $(D\mathcal{F})$-convergence coincide, and we denote them simply with $(\mathcal{F})$*-convergence*. Moreover, when $\mathcal{F} = \mathcal{F}_{\text{cofin}}$, Def. 2.4 (a) and (b) coincide with the classical ones given in [5].

(d) Let Λ be any arbitrary nonempty set and $\mathcal{F}$ be a filter of $\mathbb{N}$. A family $(\beta_{i,n})_{i\in\Lambda,n\in\mathbb{N}}$ of elements of R is said to be $(RO\mathcal{F})$*-convergent* to a family $(\beta_i)_{i\in\Lambda}$ (with respect to $i \in \Lambda$) iff there exists an (O)-sequence $(\sigma_p)_p$ with the property that for each $p \in \mathbb{N}$ and $i \in \Lambda$ we get $\{n \in \mathbb{N} : |\beta_{i,n} - \beta_i| \le \sigma_p\} \in \mathcal{F}$. The family $(\beta_{i,n})_{i\in\Lambda,n\in\mathbb{N}}$ is said to be $(RD\mathcal{F})$*-convergent* to a family $(\beta_i)_{i\in\Lambda}$ iff there is a regulator $(\alpha_{t,r})_{t,r}$ such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $i \in \Lambda$ we get

$$\Big\{n \in \mathbb{N} : |\beta_{i,n} - \beta_i| \le \bigvee_{t=1}^{\infty} \alpha_{t,\varphi(t)}\Big\} \in \mathcal{F}.$$

(e) We say that the family $(\beta_{i,n})_{i\in\Lambda,n\in\mathbb{N}}$ $(D\mathcal{F})$*-converges uniformly with respect to* $n \in \mathbb{N}$, or shortly $(U\mathcal{F})$*-converges* to $(\beta_i)_{i\in\Lambda}$, iff there is a (D)-sequence $(\alpha_{t,r})_{t,r}$ such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ we get

$$\Big\{n \in \mathbb{N} : \bigvee_{i\in\Lambda} |\beta_{i,n} - \beta_i| \le \bigvee_{t=1}^{\infty} \alpha_{t,\varphi(t)}\Big\} \in \mathcal{F}.$$

In a similar way we can give the concept of uniform filter convergence with respect to order convergence.

(f) A family $(\beta_{i,n})_{i\in\Lambda,n\in\mathbb{N}}$ is (RD)-, (RO)-, (U)*-convergent* iff it is $(RD\mathcal{F}_{\text{cofin}})$-, $(RO\mathcal{F}_{\text{cofin}})$-, $(U\mathcal{F}_{\text{cofin}})$-convergent respectively (see also [3, 9]).

The following result is an extension of [3: Theorem 3.4].

Theorem 2.3. *Let $\mathcal{F}$ be any free filter of $\mathbb{N}$. Then, in every Dedekind complete (ℓ)-group R, $(RO\mathcal{F})$-convergence implies $(RD\mathcal{F})$-convergence. Moreover, if R is a super Dedekind complete and weakly σ-distributive (ℓ)-group, then $(RD\mathcal{F})$-convergence implies $(RO\mathcal{F})$-convergence.*

Proof. Let $(\beta_{i,n})_{i\in\Lambda,n\in\mathbb{N}}$ be a family, $(RO\mathcal{F})$-convergent to $(\beta_i)_{i\in\Lambda}$, and let $(\sigma_p)_p$ be an (O)-sequence, satisfying the definition of $(RO\mathcal{F})$-convergence. For every $t, r \in \mathbb{N}$ define $\alpha_{t,r} := \sigma_{t+r}$. It is easy to check that the double sequence $(\alpha_{t,r})_{t,r}$ is a regulator.

Choose now arbitrarily $\varphi \in \mathbb{N}^{\mathbb{N}}$. By hypothesis, in correspondence with $p = 1+\varphi(1)$ and for all $i \in \Lambda$ we get: $\{n \in \mathbb{N} : |\beta_{i,n} - \beta_i| \le \sigma_{1+\varphi(1)} =: \alpha_{1,\varphi(1)}\} \in \mathcal{F}$, and a fortiori

$$\Big\{n \in \mathbb{N} : \ |\beta_{i,n} - \beta_i| \le \bigvee_{t=1}^{\infty} \alpha_{t,\varphi(t)}\Big\} \in \mathcal{F}.$$

This concludes the first part of the proof (Note that this part holds for every Dedekind complete (ℓ)-group, not necessarily super Dedekind complete or weakly σ-distributive).

We now turn to the second part. Let $(\beta_{i,n})_{i\in\Lambda,n\in\mathbb{N}}$ be $(RD\mathcal{F})$-convergent to $(\beta_i)_{i\in\Lambda}$, and $(\alpha_{t,r})_{t,r}$ be a regulator, satisfying the condition of $(RD\mathcal{F})$-convergence. Since R is super Dedekind complete and weakly σ-distributive, by [3: Theorem 3.1] there exists an (O)-sequence $(\sigma_p)_p$ with the property that to every $p \in \mathbb{N}$ there corresponds $\varphi_p \in \mathbb{N}^{\mathbb{N}}$ such that $\bigvee\limits_{t=1}^{\infty} \alpha_{t,\varphi_p(t)} \le \sigma_p$. From this, since by hypothesis

$$\Big\{n \in \mathbb{N} : \ |\beta_{i,n} - \beta_i| \le \bigvee_{t=1}^{\infty} \alpha_{t,\varphi_p(t)}\Big\} \in \mathcal{F}$$

for all $p \in \mathbb{N}$ and $i \in \Lambda$, then a fortiori $\{n \in \mathbb{N} : |\beta_{i,n} - \beta_i| \le \sigma_p\} \in \mathcal{F}$ for every $p \in \mathbb{N}$ and $i \in \Lambda$. This concludes the proof. $\square$

In [1] a Schur-type theorem was given for real-valued functions with respect to a filter convergence for diagonal and block-respecting filters. Here we extend this theorem to (ℓ)-group-valued functions.

This technical lemma is an extension of [1: Lemma 3.3].

Lemma 2.2. *Let $(a_{i,n})_{i,n}$ be a double sequence in R, $\mathcal{F}$ be a diagonal filter, suppose that $(RO\mathcal{F})\lim\limits_{i\in\mathbb{N}} a_{i,n} = 0$ (with respect to $n \in \mathbb{N}$) and let $(\sigma_p)_p$ be an associated (O)-sequence. Then for all $I \in \mathcal{F}^*$ there exists $J \in \mathcal{F}^*$, $J \subset I$ such that $(RO)\lim\limits_{i\in J} a_{i,n} = 0$ (with respect to the same (O)-sequence $(\sigma_p)_p$).*

Proof. By $(RO\mathcal{F})$-convergence we know that for all $n, p \in \mathbb{N}$ we get

$$A_{n,p} := \{i \in \mathbb{N} : \ |a_{i,n}| \le \sigma_p\} \in \mathcal{F}. \tag{2.1}$$

The family of the sets in (2.1) is obviously countable. Thus, since $\mathcal{F}$ is diagonal, in correspondence with the $A_{n,p}$'s and every $\mathcal{F}$-stationary subset $I \subset \mathbb{N}$ there is $J \in \mathcal{F}^*$, $J \subset I$, such that for every $n, p \in \mathbb{N}$ the set $J \setminus A_{n,p}$ is finite. We have:

$$B_{n,p} := \{i \in J : \ |a_{i,n}| \not\le \sigma_p\} \subset J \setminus A_{n,p},$$

and hence $B_{n,p}$ is finite too. Thus we get that to every n and $p \in \mathbb{N}$ there corresponds a positive integer $\overline{i}$ (without loss of generality belonging to J) such that $|a_{i,n}| \le \sigma_p$ whenever $i \ge \overline{i}$, $i \in J$. This ends the proof. $\square$

Remarks 1.

(a) Note that, in Lemma 2.2, the sequence $(\sigma_p)_p$ is independent of the choice of $I \in \mathcal{F}^*$.

(b) Moreover, observe that formulating the definition of diagonal filter in terms of sequences of sets which are not necessarily decreasing avoids us to proceed in terms of "neighborhoods of zero" according to the argument used in [1]. In general, (O)- and (D)-convergence are not topological: for example they coincide with almost everywhere convergence in the super Dedekind complete and weakly σ-distributive Riesz space $L^0(X, \mathcal{B}, \mu)$ of all measurable functions (up to sets of measure zero), where $(X, \mathcal{B}, \mu)$ is a measure space, with μ positive, σ-additive and σ-finite (see also [17]).

Arguing analogously as Lemma 2.2 it is possible to prove the following:

Lemma 2.3. *Let $(a_i)_i$ be a sequence in R, $\mathcal{F}$ be a diagonal filter, and assume that $(O\mathcal{F})\lim\limits_i a_i = 0$ with respect to an (O)-sequence $(\sigma_p)_p$. Then for every $I \in \mathcal{F}^*$ there exists $J \in \mathcal{F}^*$ such that $J \subset I$ and $(O)\lim\limits_{i\in J} a_i = 0$ with respect to the same (O)-sequence $(\sigma_p)_p$.*

Definition 2.1. The set $l^1(R)$ is the set of all sequences of the type $(a_j)_j$, with $a_j \in R$ for all $j \in \mathbb{N}$ and such that $\bigvee\limits_{q\in\mathbb{N}} \Big(\sum\limits_{j=1}^{q} |a_j|\Big) \in R$. As R is Dedekind complete, if $(a_j)_j$ belongs to $l^1(R)$, then $S := (O)\lim\limits_n \sum\limits_{j=1}^{n} a_j$ exists in R (see [9]). For every element $(a_j)_j$ in $l^1(R)$, we shall also write $S = (O)\lim\limits_n \sum\limits_{j=1}^{n} a_j = \sum\limits_{j=1}^{\infty} a_j$, and say that S is the *sum* of the sequence $(a_j)_j$.

Definitions 2.5.

(a) Let G be any infinite set and $\mathcal{E} \subset \mathcal{P}(G)$ be a lattice. We say that a set function $\mu\colon \mathcal{E} \to R$ is *bounded* or *order bounded* iff there is $w \in R$, $w \geq 0$, with $|\mu(A)| \leq w$ for all $A \in \mathcal{E}$. The set functions $\mu_j\colon \mathcal{E} \to R$, $j \in \mathbb{N}$, are *equibounded* iff there exists an element $u \in R$, $u \geq 0$, such that $|\mu_j(A)| \leq u$ for all $j \in \mathbb{N}$ and $A \in \mathcal{E}$.

(b) Given a finitely additive bounded set function $\mu\colon \mathcal{E} \to R$, we define μ^+, μ^-, $\|\mu\|\colon \mathcal{E} \to R$, by setting

$$\mu^+(A) := \bigvee_{B\in\mathcal{E}, B\subset A} \mu(B), \qquad \mu^-(A) := \bigvee_{B\in\mathcal{E}, B\subset A} (-\mu(B)),$$

$$\|\mu\|(A) := \mu^+(A) + \mu^-(A), \qquad A \in \mathcal{E}.$$

The quantities μ^+, μ^-, $\|\mu\|$ are called *positive part, negative part* and *variation* of μ respectively. Moreover, define the *semivariation* of μ with respect to $\mathcal{E}$ as follows:

$$v_{\mathcal{E}}(\mu)(A) = \bigvee_{B\in\mathcal{E}, B\subset A} |\mu(B)| \qquad \text{for all} \quad A \in \mathcal{E}.$$

We have clearly $v_{\mathcal{E}}(\mu)(A) \le \|\mu\|(A) \le 2v_{\mathcal{E}}(\mu)(A)$ for each $A \in \mathcal{E}$ (see also [9]).

We now give the concepts of strong boundedness, σ-additivity and absolute continuity for (ℓ)-group-valued measures.

Definition 2.2. Let G be any infinite set, $\mathcal{A} \subset \mathcal{P}(G)$ be an algebra and $\mu\colon \mathcal{A} \to R$ be any finitely additive measure. We say that μ is *strongly bounded* iff there exists a regulator $(a_{t,r})_{t,r}$ such that, for every disjoint sequence $(H_k)_k$ in $\mathcal{A}$ and all $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists an integer $\overline{k}$ with

$$|\mu(H_k)| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \tag{2.2}$$

holds, for all $k \ge \overline{k}$.

Remarks 2.

(a) Observe that, in the definition of strong boundedness, formula (2.2) can be equivalently replaced by

$$v_{\mathcal{A}}(\mu)(H_k) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

for all $k \ge \overline{k}$ (see [4: Remark 2.11]).

(b) We recall that every strongly bounded finitely additive measure is bounded too (see [4: Theorem 2.16]), while the converse implication is in general not true (see [4: Example 2.17], [5: Remark 2.8]).

This result extends [16: Lemma 2.2] to the setting of (ℓ)-groups.

Lemma 2.4. *Let $\mathcal{A} \subset \mathcal{P}(G)$ be a σ-algebra, and $\mu\colon \mathcal{A} \to R$ be a finitely additive strongly bounded measure (with respect to a common regulator $(a_{t,r})_{t,r}$).*

Then the regulator $(a_{t,r})_{t,r}$ is such that for each disjoint sequence $(E_k)_k$ in $\mathcal{A}$ and for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a finite set of positive integers Δ_φ such that

$$v_{\mathcal{A}}(\mu)\Big(\bigcup_{k\in\Delta} E_k\Big) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

whenever $\Delta \subset \mathbb{N}$ is a finite set, such that $\Delta \cap \Delta_\varphi = \emptyset$.

Proof. If the lemma is false, then there exists $\varphi \in \mathbb{N}^{\mathbb{N}}$ such that to every finite set $\Delta \subset \mathbb{N}$ there corresponds a finite set $\Delta' \subset \mathbb{N}$ with $\Delta \cap \Delta' = \emptyset$ and

$$v_{\mathcal{A}}(\mu)\Big(\bigcup_{k\in\Delta'} E_k\Big) \not\le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}.$$

From this, proceeding by induction, it follows that there exists a disjoint sequence $(\Delta_n)_n$ of finite subsets of $\mathbb{N}$, with

$$v_{\mathcal{A}}(\mu)\Big(\bigcup_{k\in\Delta_n} E_k\Big) \not\leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

for every $n \in \mathbb{N}$. Hence there exists a disjoint sequence $(T_n)_n$ in $\mathcal{A}$ such that $T_n \subset \bigcup\limits_{k\in\Delta_n} E_k$ and $|\mu(T_n)| \not\leq \bigvee\limits_{t=1}^{\infty} a_{t,\varphi(t)}$ for all $n \in \mathbb{N}$. This contradicts strong boundedness of μ, and the proof is finished. □

Definitions 2.6.

(a) Let G be any infinite set and $\mathcal{A} \subset \mathcal{P}(G)$ be a σ-algebra. A finitely additive set function $\mu\colon \mathcal{A} \to R$ is said to be *σ-additive* iff for every disjoint sequence $(E_s)_s$ in $\mathcal{A}$, $(D)\lim\limits_s v_{\mathcal{L}}(\mu)\Big(\bigcup\limits_{l=s}^{\infty} E_l\Big) = 0$, where $\mathcal{L}$ is the σ-algebra generated by the elements E_s, $s \in \mathbb{N}$, in the set $\bigcup\limits_{s=1}^{\infty} E_s$.

(b) Let $\lambda\colon \mathcal{A} \to [0,+\infty]$ be a finitely additive measure. A measure $\mu\colon \mathcal{A} \to R$ is *λ-absolutely continuous* iff for every decreasing sequence $(E_s)_s$ in $\mathcal{A}$ there exists a regulator $(a_{t,r})_{t,r}$ with the following property: in correspondence with every $\varphi \in \mathbb{N}^{\mathbb{N}}$, a positive real number δ can be found, with

$$v_{\mathcal{L}}(\mu)(E) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

whenever $E \in \{E_s :\ s \in \mathbb{N}\}$ and $\lambda(E) < \delta$ (see also [5]).

Remarks 3.

(a) In [9: Proposition 3.3] we proved also that, if $\mu\colon \mathcal{P}(\mathbb{N}) \to R$ is σ-additive, then for every finite and for any cofinite subset $A \subset \mathbb{N}$ we have

$$\|\mu\|(A) = \sum_{n\in A} |\mu(\{n\})|. \tag{2.3}$$

(b) Observe that in general our notions of σ-additivity and absolute continuity (with respect to $v_{\mathcal{L}}$) are weaker than the classical ones (with respect to $v_{\mathcal{A}}$), but they coincide when the measure takes values in a Banach lattice, or when $G = \mathbb{N}$ (see also [9]).

We now recall the following properties of σ-additive measures defined on $\mathcal{P}(\mathbb{N})$ (see [7: Propositions 2.5, 3.6]).

Proposition 2.4. *A set function $\nu\colon \mathcal{P}(\mathbb{N}) \to R$ is a σ-additive measure on $\mathcal{P}(\mathbb{N})$ if and only if*

$$\bigwedge_n v_{\mathcal{P}(\mathbb{N})}(\nu)(\{n, n+1, n+2, \dots\}) = 0.$$

Proposition 2.5. *Let $\mathcal{A} \subset \mathcal{P}(G)$ be a σ-algebra, $\mu\colon \mathcal{A} \to R$ be a finitely additive measure, $(H_n)_n$ be any decreasing sequence in $\mathcal{A}$, and set $B_n = H_n \setminus H_{n+1}$, $n \in \mathbb{N}$. For every $A \subset \mathbb{N}$ put $\nu(A) = \mu\Big(\bigcup_{n\in A} B_n\Big)$. Let $\mathcal{L}$ be the σ-algebra generated by the H_n's in H_1 and suppose that*

$$\bigwedge_n v_{\mathcal{L}}(\mu)(H_n) = 0.$$

Then ν is σ-additive on $\mathcal{P}(\mathbb{N})$.

Moreover we have

$$\begin{aligned} v_{\mathcal{P}(\mathbb{N})}(\nu)(\{n, n+1, n+2, \dots\}) &= \bigvee \big\{ |\nu(B)| :\ B \subset \{n, n+1, n+2 \dots\} \big\} \\ &= \bigvee \big\{ |\mu(C)| :\ C \in \mathcal{L} \text{ with } C \subset H_n \big\} \\ &= v_{\mathcal{L}}(\mu)(H_n) \qquad \text{for all} \quad n \in \mathbb{N}. \end{aligned} \tag{2.4}$$

3. The Schur Lemma

We now prove the following lemma, which extends [1: Theorem 2.6] to the case of (ℓ)-groups. Note that, if the involved filter is not block-respecting, then the Schur lemma does not hold even when $R = \mathbb{R}$ (see [1: Remark 3.4]). Here we denote by v the semivariation with respect to $\mathcal{P}(\mathbb{N})$. We begin with the following:

Lemma 3.1. *Let $\mathcal{F}$ be a block-respecting filter of $\mathbb{N}$, $\mu_j\colon \mathcal{P}(\mathbb{N}) \to R$, $j \in \mathbb{N}$, be a sequence of σ-additive equibounded set functions, and define $(\beta_{A,j})_{A\in\mathcal{P}(\mathbb{N}), j\in\mathbb{N}}$ by setting $\beta_{A,j} := \mu_j(A)$, $A \in \mathcal{P}(\mathbb{N})$, $j \in \mathbb{N}$. Suppose that:*

i) *$(D)\lim_j \mu_j(\{n\}) = 0$ for each $n \in \mathbb{N}$;*

ii) *the family $(\beta_{A,j})_{A\in\mathcal{P}(\mathbb{N}), j\in\mathbb{N}}$ $(RD\mathcal{F})$-converges to 0.*

Then $(D\mathcal{F})\lim_j \sum_{n=1}^{\infty} |\mu_j(\{n\})| = 0$.

Proof. Let $u := \bigvee_{A\in\mathcal{P}(\mathbb{N}), j\in\mathbb{N}} v(\mu_j)(A)$: such an element exists in R, thanks to equiboundedness of the μ_j's. For each $j \in \mathbb{N}$ let $(a^{(j)}_{t,r})_{t,r}$ be a (D)-sequence related with σ-additivity of μ_j. By virtue of Proposition 2.4, for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N}$ there is $\overline{n} \in \mathbb{N}$ (depending on φ and j) with

$$v(\mu_j)(A) \le \bigvee_{t=1}^{\infty} a^{(j)}_{t,\varphi(t+j)} \qquad \text{for all} \quad A \subset \{\overline{n}, \overline{n}+1, \overline{n}+2, \dots\}. \tag{3.1}$$

By virtue of the Fremlin Lemma 2.1 there exists a (D)-sequence $(a_{t,r})_{t,r}$ such that

$$u \bigwedge \Big[\bigvee_q \Big(\sum_{j=1}^{q} \Big(\bigvee_{t=1}^{\infty} a^{(j)}_{t,\varphi(t+j)} \Big) \Big) \Big] \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \tag{3.2}$$

for all $\varphi \in \mathbb{N}^{\mathbb{N}}$. From (3.1) and (3.2) it follows that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N}$ there exists $\overline{n} \in \mathbb{N}$ such that

$$v(\mu_j)(A) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \qquad \text{for all} \quad A \subset \{\overline{n}, \overline{n}+1, \overline{n}+2, \dots\}. \tag{3.3}$$

Let $(b_{t,r})_{t,r}$ satisfy the condition of $(RD\mathcal{F})$-convergence as in ii).

Since $(D) \lim\limits_j \mu_j(\{n\}) = 0$ for each $n \in \mathbb{N}$, then to every $n \in \mathbb{N}$ there corresponds a regulator $(d^{(n)}_{t,r})_{t,r}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\overline{j} \in \mathbb{N}$ with $|\mu_j(\{n\})| \le \bigvee\limits_{t=1}^{\infty} d^{(n)}_{t,\varphi(t+n)}$ whenever $j \ge \overline{j}$. Thanks to equiboundedness of the μ_j's, by virtue of the Fremlin Lemma 2.1, a regulator $(d_{t,r})_{t,r}$ can be found, with the property that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and for any positive integer q there exists $\overline{j} \in \mathbb{N}$ such that

$$\sum_{n=1}^{q} |\mu_j(\{n\})| \le \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \tag{3.4}$$

whenever $j \ge \overline{j}$. Set

$$c_{t,r} = 2(a_{t,r} + b_{t,r} + d_{t,r}), \qquad t, r \in \mathbb{N}. \tag{3.5}$$

We prove that the (D)-sequence $(c_{t,r})_{t,r}$ defined in (3.5) satisfies the condition of $(D\mathcal{F})$-convergence as in the thesis of the theorem.

Otherwise there exists $\varphi \in \mathbb{N}^{\mathbb{N}}$ with the property that the set

$$I^* := \Big\{ j \in \mathbb{N} : \sum_{n=1}^{\infty} |\mu_j(\{n\})| \le \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \Big\} = \Big\{ j \in \mathbb{N} : \|\mu_j\|(\mathbb{N}) \le \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \Big\} \tag{3.6}$$

does not belong to $\mathcal{F}$ (The equality in (3.6) follows from [9: Proposition 3.3]). From this it follows that every element F of $\mathcal{F}$ is not contained in I^*, that is F has nonempty intersection with $\mathbb{N} \setminus I^*$. This means that the set $\mathbb{N} \setminus I^*$ is $\mathcal{F}$-stationary. Let $I := \mathbb{N} \setminus I^*$. Note that I is an infinite set, because $\mathcal{F}$ is a free filter. We can find a disjoint sequence $(A_s)_{s \in I}$ of subsets of $\mathbb{N}$, such that

$$|\mu_s(A_s)| \not\le \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \qquad \text{for all} \quad s \in I. \tag{3.7}$$

We will construct a strictly increasing sequence $(n_h)_h$ in $\mathbb{N}$ such that $D_h := [n_{h-1}, n_h) \cap I \neq \emptyset$ for every $h \in \mathbb{N}$.

Let $n_0 := 1$. By σ-additivity of the μ_j's and Lemma 2.1, in correspondence with the above considered φ it is possible to find a natural number $m(n_0)$ with

$$v(\mu_1)\big(\{m(n_0)+1, m(n_0)+2, \dots\}\big) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}.$$

By (3.4), in correspondence with the finite number of indexes $1, 2, \dots, m(n_0)$ there exists $n_1 \in \mathbb{N}$, $n_1 > m(n_0)$, such that

$$\big|\mu_s\big(\{1\}\big)\big| + \dots + \big|\mu_s\big(\{m(n_0)\}\big)\big| \le \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \tag{3.8}$$

for all $s \ge n_1$. Proceeding analogously as above, it is possible to associate to n_1 a natural number $m(n_1) > n_1$ with

$$v(\mu_r)\big(\{m(n_1)+1, m(n_1)+2, \dots\}\big) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}, \qquad r = 1, \dots, n_1,$$

and to find an integer $n_2 > m(n_1)$ with

$$\big|\mu_s\big(\{1\}\big)\big| + \dots + \big|\mu_s\big(\{m(n_1)\}\big)\big| \le \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \tag{3.9}$$

for all $s \ge n_2$. By (3.7), (3.8) and (3.9) we get $[1{=}n_0, n_1) \cap I \ne \emptyset$ and $[n_1, n_2) \cap I \ne \emptyset$. Proceeding by induction, we get the existence of two strictly increasing sequences $(n_h)_h$ and $(m(n_h))_h$ in $\mathbb{N}$ such that for all $h \in \mathbb{N}$ we have:

$$1 = n_0 < m(n_0) < n_1 < m(n_1) < n_2 < \dots; \qquad [n_{h-1}, n_h) \cap I \ne \emptyset;$$

$$v(\mu_r)(\{m(n_h)+1, m(n_h)+2, \dots\}) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}, \qquad r = 1, \dots, n_h; \tag{3.10}$$

$$\big|\mu_s\big(\{1\}\big)\big| + \dots + \big|\mu_s\big(\{m(n_h)\}\big)\big| \le \bigvee_{t=1}^{\infty} d_{t,\varphi(t)}, \qquad s \ge n_{h+1}. \tag{3.11}$$

Since the filter $\mathcal{F}$ is block-respecting, there exists a set $J := \{j_1, j_2, \dots\} \in \mathcal{F}^*$, $J \subset I$, with $n_h \le j_h < n_{h+1}$ for every $h \in \mathbb{N}$. Of course, since $J \in \mathcal{F}^*$, then either $J_1 := \{j_1, j_3, j_5, \dots\}$ or $J_2 := \{j_2, j_4, j_6, \dots\}$ belongs to $\mathcal{F}^*$. Without loss of generality, let $J_1 \in \mathcal{F}^*$. We now proceed analogously as in [5: Lemma 3.5].

For every h, set $q_h := m(n_{2h})$. Since $n_h \le j_h < n_{h+1}$, we get in particular $j_{2h-1} < n_{2h}$. So, by (3.10) used with $2h$ and $r = j_{2h-1}$, we get

$$|\mu_{j_{2h-1}}(C)| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \tag{3.12}$$

whenever $C \subset \{q_h + 1, q_h + 2, \dots\}$.

Moreover, from (3.11) used with $2(h-1)$, we have

$$\big|\mu_s(\{1\})\big| + \cdots + \big|\mu_s(\{q_{h-1}\})\big| \leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \tag{3.13}$$

for all $s \geq n_{2h-1}$. Set now $T_h := A_{j_{2h-1}} \cap \{q_{h-1}+1, \ldots, q_h\}$, $A := \bigcup_{h=1}^{\infty} T_h$, $W_h := A \cap \{q_{h-1}+1, \ldots, q_h\}$, $h \in \mathbb{N}$. Since the A_s's are pairwise disjoint, for all $h \in \mathbb{N}$ we have $T_h = W_h$. Moreover we get:

$$\begin{aligned} \mu_{j_{2h-1}}(A) &= \mu_{j_{2h-1}}(A \cap \{1, \ldots, q_{h-1}\}) + \mu_{j_{2h-1}}(T_h) \\ &\quad + \mu_{j_{2h-1}}(A \cap \{q_h+1, q_h+2, \ldots\}); \\ \mu_{j_{2h-1}}(A_{j_{2h-1}}) &= \mu_{j_{2h-1}}(A_{j_{2h-1}} \cap \{1, \ldots, q_{h-1}\}) + \mu_{j_{2h-1}}(T_h) \\ &\quad + \mu_{j_{2h-1}}(A_{j_{2h-1}} \cap \{q_h+1, q_h+2, \ldots\}). \end{aligned} \tag{3.14}$$

From (3.12) we have:

$$\begin{aligned} \big|\mu_{j_{2h-1}}\big(A_{j_{2h-1}} \cap \{q_h+1, q_h+2, \ldots\}\big)\big| &\leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}, \\ \big|\mu_{j_{2h-1}}\big(A \cap \{q_h+1, q_h+2, \ldots\}\big)\big| &\leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}. \end{aligned} \tag{3.15}$$

From (3.13) we get:

$$\begin{aligned} \big|\mu_{j_{2h-1}}\big(A_{j_{2h-1}} \cap \{1, \ldots, q_{h-1}\}\big)\big| &\leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)}, \\ \big|\mu_{j_{2h-1}}\big(A \cap \{1, \ldots, q_{h-1}\}\big)\big| &\leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)}. \end{aligned} \tag{3.16}$$

From (3.14), (3.15) and (3.16) we obtain:

$$|\mu_{j_{2h-1}}(A) - \mu_{j_{2h-1}}(A_{j_{2h-1}})| \leq 2\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} + 2\bigvee_{t=1}^{\infty} d_{t,\varphi(t)}. \tag{3.17}$$

From (3.7) and (3.17) it follows that

$$|\mu_{j_{2h-1}}(A)| \not\leq 2\bigvee_{t=1}^{\infty} b_{t,\varphi(t)}$$

for all $h \in \mathbb{N}$, and thus $\Big\{l \in \mathbb{N} : \ |\mu_l(A)| \not\leq 2\bigvee_{t=1}^{\infty} b_{t,\varphi(t)}\Big\} \in \mathcal{F}^*$. Moreover, the sets

$$\Big\{l \in \mathbb{N} : \ |\mu_l(A)| \leq 2\bigvee_{t=1}^{\infty} b_{t,\varphi(t)}\Big\} \in \mathcal{F}$$

and

$$\Big\{l \in \mathbb{N} : \ |\mu_l(A)| \not\leq 2 \bigvee_{t=1}^{\infty} b_{t,\varphi(t)}\Big\}$$

have nonempty intersection, which is absurd. This ends the proof. □

The next step is to give our version of the Schur Lemma for filter convergence, which extends [1: Theorem 3.5] to the setting of (ℓ)-groups. From now on we suppose that R is super Dedekind complete and weakly σ-distributive.

Theorem 3.1. *Let $\mathcal{F}$ be a diagonal and block-respecting filter of $\mathbb{N}$, $\mu_j \colon \mathcal{P}(\mathbb{N}) \to R$, $j \in \mathbb{N}$, be a sequence of σ-additive equibounded set functions, and set $\beta_{A,j} := \mu_j(A)$, $A \in \mathcal{P}(\mathbb{N})$, $j \in \mathbb{N}$.*

If the family $(\beta_{A,j})_{A \in \mathcal{P}(\mathbb{N}), j \in \mathbb{N}}$ $(RD\mathcal{F})$-converges to 0, then

$$(D\mathcal{F}) \lim_j \sum_{n=1}^{\infty} |\mu_j(\{n\})| = 0.$$

P r o o f. Let $(a_{t,r})_{t,r}$ and $(b_{t,r})_{t,r}$ be as in the proof of Lemma 3.1. Observe that, since R is super Dedekind complete and weakly σ-distributive, from equivalence between $(RO\mathcal{F})$- and $(RD\mathcal{F})$-convergence (Theorem 2.3) and Lemma 2.2 it follows that there exists a regulator $(\alpha_{t,r})_{t,r}$ such that for every $I \in \mathcal{F}^*$ there is an $\mathcal{F}$-stationary set $J \subset I$ such that:

3.1.1) the double sequence $(\mu_j(\{n\}))_{n \in \mathbb{N}, j \in J}$ (RD)-converges to 0 with respect to $(\alpha_{t,r})_{t,r}$.

Set now $c_{t,r} := 2(a_{t,r} + b_{t,r} + \alpha_{t,r})$, $t, r \in \mathbb{N}$. We prove that the (D)-sequence $(c_{t,r})_{t,r}$ satisfies the thesis of the theorem. Otherwise, proceeding analogously as in the proof of Lemma 3.1, there are an $\mathcal{F}$-stationary set $I \subset \mathbb{N}$ and a function $\varphi \in \mathbb{N}^{\mathbb{N}}$ with the property that

$$\sum_{n=1}^{\infty} |\mu_j(\{n\})| \not\leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \tag{3.18}$$

whenever $j \in I$. In correspondence with I, let $J \in \mathcal{F}^*$, $J \subset I$, satisfy 3.1.1). As J is stationary and $\mathcal{F}$ is block-respecting, then the filter $\mathcal{F}(J)$ is block-respecting, too. Note that, since $\mathcal{F}(J)$ contains $\mathcal{F}$, it is easy to see that the family $(\beta_{A,j})_{A \in \mathcal{P}(\mathbb{N}), j \in J}$, $(RD\mathcal{F}(J))$-converges to 0 and the regulators related with $(D\mathcal{F}(J))$-convergence are the same as the ones involving $(D\mathcal{F})$-convergence. From (3.18) we get easily that the sequence $\Big(\sum\limits_{n=1}^{\infty} |\mu_j(\{n\})|\Big)_j$ cannot $(D\mathcal{F}(J))$-converge to 0 with respect to the (D)-sequence $(c_{t,r})_{t,r}$ (see also [1]). But by Lemma 3.1 applied with $\mathcal{F}(J)$ and the family $(\beta_{A,j})_{A \in \mathcal{P}(\mathbb{N}), j \in J}$ it follows that $(D\mathcal{F}(J)) \lim\limits_j \sum\limits_{n=1}^{\infty} |\mu_j(\{n\})| = 0$ with respect to the regulator $(c_{t,r})_{t,r}$. Thus we get a contradiction, and this ends the proof. □

A consequence of Theorem 3.1 is the following extension to the context of (ℓ)-groups of [10: Lemma 1].

Corollary 3.1.1. *Let $(a_{j,n})_{j,n}$ be any bounded double sequence in R, suppose that $\sum_{n=1}^{\infty} |a_{j,n}| \in R$ for all $j \in \mathbb{N}$, and set $\alpha_{A,j} := \sum_{n\in A} a_{j,n}$, $A \in \mathcal{P}(\mathbb{N})$, $j \in \mathbb{N}$.*

If the family $(\alpha_{A,j})_{A\in\mathcal{P}(\mathbb{N}),j\in\mathbb{N}}$ $(RD\mathcal{F})$-converges to 0, then

$$(D\mathcal{F})\lim_{j} \sum_{n=1}^{\infty} |a_{j,n}| = 0.$$

P r o o f. For every $A \subset \mathbb{N}$ and $j \in \mathbb{N}$, set $\mu_j(A) := \sum_{n\in A} a_{j,n}$. First of all observe that, by hypothesis, we have that $(a_{j,n})_n \in l^1(R)$ for all $j \in \mathbb{N}$. We now claim that

$$\|\mu_j\|(\mathbb{N}) = \sum_{n=1}^{\infty} |a_{j,n}| \qquad \text{for every} \quad j \in \mathbb{N}. \tag{3.19}$$

Indeed, by (2.3) we get

$$\sum_{k=1}^{n} |a_{j,k}| = \|\mu_j\|(\{1,\dots,n\}) \le \|\mu_j\|(\mathbb{N})$$

for any $j,n \in \mathbb{N}$, and hence $\sum_{n=1}^{\infty} |a_{j,n}| \le \|\mu_j\|(\mathbb{N})$ for all $j \in \mathbb{N}$.

We now turn to the converse inequality. For every $j \in \mathbb{N}$ and $A \subset \mathbb{N}$ we get

$$\sum_{n\in A} a_{j,n} \le \sum_{n\in A} a^+_{j,n} \le \sum_{n=1}^{\infty} a^+_{j,n}; \quad -\sum_{n\in A} a_{j,n} = \sum_{n\in A}(-a_{j,n}) \le \sum_{n\in A} a^-_{j,n} \le \sum_{n=1}^{\infty} a^-_{j,n}.$$

Note that all the involved quantities belong to R, since $(a_{j,n})_n \in l^1(R)$ for all j. Taking the supremum as A varies in $\mathcal{P}(\mathbb{N})$ we obtain:

$$\bigvee_{A\subset\mathbb{N}} \Big(\sum_{n\in A} a_{j,n}\Big) \le \sum_{n=1}^{\infty} a^+_{j,n}, \ \bigvee_{A\subset\mathbb{N}} \Big(-\sum_{n\in A} a_{j,n}\Big) \le \sum_{n=1}^{\infty} a^-_{j,n};$$

$$\|\mu_j\|(\mathbb{N}) = \bigvee_{A\subset\mathbb{N}} \Big(\sum_{n\in A} a_{j,n}\Big) + \bigvee_{A\subset\mathbb{N}} \Big(-\sum_{n\in A} a_{j,n}\Big) \le \sum_{n=1}^{\infty} a^+_{j,n} + \sum_{n=1}^{\infty} a^-_{j,n} = \sum_{n=1}^{\infty} |a_{j,n}|$$

for all $j \in \mathbb{N}$. Thus (3.19) is proved. From (3.19) and the fact that $(a_{j,n})_n \in l^1(R)$ for all j, arguing analogously as in [9: Proposition 3.3] it follows that the μ_j's are σ-additive. The assertion follows from this and Theorem 3.1. $\square$

Corollary 3.1.2. *Under the same notations and hypotheses as in* Theorem 3.1, *we get that the family*

$$(\mu_j(A))_{A\in\mathcal{P}(\mathbb{N}),j\in\mathbb{N}} \tag{3.20}$$

$(U\mathcal{F})$*-converges to* 0*. Furthermore for every* $I \in \mathcal{F}^*$ *there exists an* $\mathcal{F}$*-stationary set* $J \subset I$ *such that*

$$(D)\lim_n\Big[\bigvee_{j\in J} v(\mu_j)(\{n,n+1,\dots\})\Big] = 0. \tag{3.21}$$

Proof. First we prove $(U\mathcal{F})$-convergence to 0 of the family $(\mu_j(A))_{A\in\mathcal{P}(\mathbb{N}),j\in\mathbb{N}}$. By virtue of Theorem 3.1 and (3.19) it follows that

$$0 = (D\mathcal{F})\lim_j \sum_{n=1}^{\infty} |\mu_j(\{n\})| = (D\mathcal{F})\lim_j \|\mu_j\|(\mathbb{N}) = 0,$$

and thus the family in (3.20) $(U\mathcal{F})$-converges to 0. Similar conclusions hold even for the positive and negative variations and for the semivariations of the μ_j's.

From this, (3.19), Lemma 2.3 and equivalence between $(O\mathcal{F})$- and $(D\mathcal{F})$-convergence (see also Theorem 2.3) it follows that there exists an $\mathcal{F}$-stationary set $J \subset \mathbb{N}$ with the property that

$$\begin{aligned} 0 &= (O)\lim_{j\in J} \sum_{n=1}^{\infty} |\mu_j(\{n\})| = (O)\lim_{j\in J} \|\mu_j\|(\mathbb{N}) \\ &= (D)\lim_{j\in J} \sum_{n=1}^{\infty} |\mu_j(\{n\})| = (D)\lim_{j\in J} \|\mu_j\|(\mathbb{N}). \end{aligned} \tag{3.22}$$

Now, by virtue of (3.22), there exists a (D)-sequence $(h_{t,r})_{t,r}$ such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ a positive integer $\overline{j}$ can be associated, without loss of generality $\overline{j} \in J$, such that

$$|\mu_j(A)| \le \bigvee_{t=1}^{\infty} h_{t,\varphi(t)} \tag{3.23}$$

whenever $j \ge \overline{j}$, $j \in J$, and $A \subset \mathbb{N}$. Moreover, by virtue of σ-additivity of the μ_j's, $j \in \mathbb{N}$, their equiboundedness and Lemma 2.1, the (D)-sequence $(a_{t,r})_{t,r}$ in (3.3) is such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N}$ there is $\overline{n} = \overline{n}(\varphi, j) \in \mathbb{N}$ with

$$|\mu_j(A)| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \qquad \text{for all} \quad A \subset \big\{\overline{n}, \overline{n}+1, \overline{n}+2, \dots\big\}, \;\; A \subset \mathbb{N}. \tag{3.24}$$

Fix arbitrarily $\varphi \in \mathbb{N}^{\mathbb{N}}$, and let $\overline{j}$ as in (3.23). In correspondence with φ and $j = 1, \dots, \overline{j} - 1$, there exist $\overline{n}_1, \dots, \overline{n}_{\overline{j}-1}$ as in (3.24). Set $n^* := \max\{\overline{n}_1, \dots, \overline{n}_{\overline{j}-1}\}$, we have

$$|\mu_j(A)| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \qquad \text{for all} \quad A \subset \{n^*, n^*+1, n^*+2, \dots\}, \\ A \subset \mathbb{N}, \quad j = 1, \dots, \overline{j} - 1. \tag{3.25}$$

Moreover, for every $j \ge \overline{j}$, $j \in J$, and $A \subset \{n^*, n^*+1, n^*+2, \dots\}$, $A \subset \mathbb{N}$, we get

$$|\mu_j(A)| \le \bigvee_{t=1}^{\infty} h_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}. \tag{3.26}$$

Relation (3.21) follows from (3.25), (3.26) and Lemma 2.3. □

We now give a filter version of the Nikodým convergence theorem derived from a filter version of the Schur theorem. In [8] some versions of Vitali-Hahn-Saks and Nikodým theorems were proved with a different technique and with respect to a special class of ideals, requiring positivity of the involved measures (see also [2]).

Theorem 3.2. *Let $\mathcal{F}$ be a diagonal and block-respecting filter of $\mathbb{N}$, $\mathcal{A} \subset \mathcal{P}(G)$ be a σ-algebra, $\mu_j\colon \mathcal{A} \to R$, $j \in \mathbb{N}$, be a sequence of equibounded σ-additive measures. Assume that there exists a σ-additive measure $\mu_0\colon \mathcal{A} \to R$, such that the family $\mu_j(A)$, $A \in \mathcal{A}$, $j \in \mathbb{N}$, $(RO\mathcal{F})$-converges to $\mu_0(A)$, $A \in \mathcal{A}$.*

Then for each decreasing sequence $(H_n)_n$ in $\mathcal{A}$ with $\bigcap_{n=1}^{\infty} H_n = \emptyset$ and for every $\mathcal{F}$-stationary set $I \subset \mathbb{N}$ there exists an $\mathcal{F}$-stationary set $J \subset I$, such that

$$\bigwedge_n \Big[\bigvee_{j \in J} v_{\mathcal{L}}(\mu_j)(H_n) \Big] = 0, \tag{3.27}$$

where $\mathcal{L}$ is the σ-algebra generated by the H_n's in H_1.

P r o o f. Let $(H_n)_n$ be any decreasing sequence in $\mathcal{A}$ with $\bigcap_{n=1}^{\infty} H_n = \emptyset$, put $B_n = H_n \setminus H_{n+1}$ for all $n \in \mathbb{N}$, and $F = \bigcap_{n=1}^{\infty} H_n$.

As μ_j, $j = 0, 1, \dots$, is σ-additive, we get

$$\bigwedge_n v_{\mathcal{L}}(\mu_j)(H_n) := \bigwedge_n \bigvee \big\{ |\mu_j(C)| :\ C \in \mathcal{L} \ \text{with} \ C \subset H_n \big\} = 0. \tag{3.28}$$

For all $A \in \mathcal{P}(\mathbb{N})$ and $j = 0, 1, \dots$, set

$$\nu_j(A) = \mu_j\Big(\bigcup_{n \in A} B_n \Big). \tag{3.29}$$

By Proposition 2.5, the measures ν_j are σ-additive. The equiboundedness of the ν_j's and $(RO\mathcal{F})$-convergence of the family $\nu_j(A)$, $A \in \mathcal{P}(\mathbb{N})$, $j \in \mathbb{N}$, to $\nu_0(A)$,

$A \in \mathcal{P}(\mathbb{N})$, follow easily from the equiboundedness of the μ_j's and $(RO\mathcal{F})$-convergence of the family $\mu_j(A)$, $A \in \mathcal{A}$, $j \in \mathbb{N}$, to $\mu_0(A)$, $A \in \mathcal{A}$, respectively.

By applying Theorem 3.1 and taking into account (2.4), it follows that for every $\mathcal{F}$-stationary set $I \subset \mathbb{N}$ there exists an $\mathcal{F}$-stationary set $J \subset I$, satisfying (3.27). $\square$

4. The finitely additive case

We now prove a version of the Brooks-Jewett theorem, for finitely additive set functions, with respect to the filter convergence. We extend [1: Lemma 2.5] to the finitely additive case and to the context of (ℓ)-groups.

Theorem 4.1. *Let $\mathcal{F}$ be a block-respecting filter of $\mathbb{N}$, and $\mu_j \colon \mathcal{P}(\mathbb{N}) \to R$, $j \in \mathbb{N}$, be a sequence of (not necessarily positive) equibounded measures, strongly bounded with respect to a common regulator. Set $\zeta_{A,j} := \sum_{k \in A} \mu_j(\{k\})$, $A \in \mathcal{P}(\mathbb{N})$, $j \in \mathbb{N}$, and suppose that:*

i) *$(D) \lim_j \mu_j(\{k\}) = 0$ for each $k \in \mathbb{N}$;*

ii) *the family $(\zeta_{A,j})_{A \in \mathcal{P}(\mathbb{N}), j \in \mathbb{N}}$, $(RD\mathcal{F})$-converges to zero.*

Then $(D\mathcal{F}) \lim_j \Big(\bigvee_{A \in \mathcal{P}(\mathbb{N})} \big| \sum_{k \in A} \mu_j(\{k\}) \big| \Big) = 0$.

P r o o f. Proceeding analogously as in the proof of Lemma 3.1, by virtue of Lemma 2.4 and the Fremlin Lemma 2.1 we have the existence of a (D)-sequence $(a_{t,r})_{t,r}$ related with strong boundedness of the μ_j's, with the property that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N}$ there is a natural number $m(j) > j$ with

$$\Big| \sum_{k \in E} \mu_r(\{k\}) \Big| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}, \qquad r = 1, \dots, j, \tag{4.1}$$

whenever E is a finite subset of $\{m(j)+1, m(j)+2, \dots\}$. Taking the (D)-limits, from (4.1) we get that the integer $m(j)$ is such that

$$\Big| \sum_{k \in E} \mu_r(\{k\}) \Big| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}, \qquad r = 1, \dots, j, \tag{4.2}$$

whenever $E \subset \{m(j)+1, m(j)+2, \dots\}$. Moreover, proceeding as in Lemma 3.1, we get the existence of a regulator $(d_{t,r})_{t,r}$ such that for any positive integer q there exists $\overline{j} \in \mathbb{N}$ with

$$\sum_{k=1}^{q} |\mu_j(\{k\})| \le \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \tag{4.3}$$

whenever $j \ge \overline{j}$.

Let $(b_{t,r})_{t,r}$ be a regulator, satisfying the condition ii) of $(RD\mathcal{F})$-convergence to 0 of the family $(\zeta_{A,j})_{A\in\mathcal{P}(\mathbb{N}),j\in\mathbb{N}}$. Set $c_{t,r} := 2(a_{t,r} + b_{t,r} + d_{t,r})$, $t, r \in \mathbb{N}$. We now prove that the (D)-sequence $(c_{t,r})_{t,r}$ satisfies the condition of convergence in the thesis of the theorem.

Otherwise, there exists $\varphi \in \mathbb{N}^{\mathbb{N}}$ such that the set

$$I^* := \Big\{ j \in \mathbb{N} : \bigvee_{A\in\mathcal{P}(\mathbb{N})} \big| \sum_{k\in A} \mu_j(\{k\}) \big| \le \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \Big\}$$

does not belong to $\mathcal{F}$. From this it follows that every element F of $\mathcal{F}$ is not contained in I^*, that is F has nonempty intersection with $\mathbb{N} \setminus I^*$. This means that the set $\mathbb{N} \setminus I^*$ is $\mathcal{F}$-stationary. Thus there exist an $\mathcal{F}$-stationary set $I \subset \mathbb{N}$ and a disjoint sequence $(A_s)_{s\in I}$ of subsets of $\mathbb{N}$, such that

$$\Big| \sum_{k\in A_s} \mu_s(\{k\}) \Big| \not\le \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \qquad \text{for all} \quad s \in I. \tag{4.4}$$

As in Lemma 3.1, we construct a strictly increasing sequence $(n_h)_h$ in $\mathbb{N}$ with $D_h := [n_{h-1}, n_h) \cap I \ne \emptyset$ for any $h \in \mathbb{N}$.

Let $n_0 := 1$. In correspondence with φ there is a natural number $m(n_0)$ such that

$$\Big| \sum_{k\in E} \mu_1(\{k\}) \Big| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

whenever $E \subset \{m(n_0) + 1, m(n_0) + 2, \dots\}$.

By (4.3), we can associate to $1, 2, \dots, m(n_0)$ a natural number $n_1 > m(n_0)$, with

$$\big|\mu_s\big(\{1\}\big)\big| + \cdots + \big|\mu_s\big(\{m(n_0)\}\big)\big| \le \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \qquad \text{for all} \quad s \ge n_1.$$

Furthermore, in correspondence with n_1 there is an $m(n_1) \in \mathbb{N}$, $m(n_1) > n_1$, with

$$\Big| \sum_{k\in E} \mu_r(\{k\}) \Big| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}, \qquad r = 1, \dots, n_1$$

whenever $E \subset \{m(n_1)+1, m(n_1)+2, \dots\}$, and there exists an integer $n_2 > m(n_1)$ with

$$\big|\mu_s\big(\{1\}\big)\big| + \cdots + \big|\mu_s\big(\{m(n_1)\}\big)\big| \le \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \qquad \text{for all} \quad s \ge n_2.$$

Moreover, $[1 = n_0, n_1) \cap I \ne \emptyset$ and $[n_1, n_2) \cap I \ne \emptyset$. By induction, it is possible to determine two strictly increasing sequences $(n_h)_h$ and $(m(n_h))_h$ in $\mathbb{N}$ with

$$1 = n_0 < m(n_0) < n_1 < m(n_1) < n_2 < \dots; \qquad [n_{h-1}, n_h) \cap I \ne \emptyset$$

for all $h \in \mathbb{N}$;

$$\Big|\sum_{k\in E} \mu_r(\{k\})\Big| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}, \qquad r = 1,\dots,n_h \tag{4.5}$$

whenever $E \subset \{m(n_h)+1, m(n_h)+2,\dots\}$;

$$\big|\mu_s\big(\{1\}\big)\big| + \dots + \big|\mu_s\big(\{m(n_h)\}\big)\big| \le \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \qquad \text{for all} \quad s \ge n_{h+1}. \tag{4.6}$$

Since the filter $\mathcal{F}$ is block-respecting, there is a set $J := \{j_1, j_2, \dots\} \in \mathcal{F}^*$, $J \subset I$, with $n_h \le j_h < n_{h+1}$ for any h. Of course, since $J \in \mathcal{F}^*$, then either $J_1 := \{j_1, j_3, j_5, \dots\}$ or $J_2 := \{j_2, j_4, j_6, \dots\}$ belongs to $\mathcal{F}^*$. Without loss of generality, we can suppose that $J_1 \in \mathcal{F}^*$.

For every h, set $q_h := m(n_{2h})$. Since $n_h \le j_h < n_{h+1}$, we have $j_{2h-1} < n_{2h}$. So, by (4.5) used with $2h$ and $r = j_{2h-1}$, we get

$$\Big|\sum_{k\in E} \mu_{j_{2h-1}}(\{k\})\Big| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \tag{4.7}$$

whenever $E \subset \{q_h+1, q_h+2, \dots\}$.

From (4.6) used with $2(h-1)$, we obtain

$$\big|\mu_s\big(\{1\}\big)\big| + \dots + \big|\mu_s\big(q_{h-1})\}\big)\big| \le \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \qquad \text{for each} \quad s \ge n_{2h-1}. \tag{4.8}$$

As $j_{2h-1} \ge n_{2h-1}$, from (4.8) we get

$$|\mu_{j_{2h-1}}(\{1\})| + \dots + |\mu_{j_{2h-1}}(\{q_{h-1}\})| \le \bigvee_{t=1}^{\infty} d_{t,\varphi(t)}. \tag{4.9}$$

Set now $T_h := A_{j_{2h-1}} \cap \{q_{h-1}+1, \dots, q_h\}$, $A := \bigcup_{h=1}^{\infty} T_h$, $W_h := A \cap \{q_{h-1}+1, \dots, q_h\}$, $h \in \mathbb{N}$. As the A_s's are disjoint, then $T_h = W_h$ for every h. Moreover we get:

$$\sum_{k\in A_{j_{2h-1}}} \mu_{j_{2h-1}}(\{k\}) = \sum_{k\in A_{j_{2h-1}}\cap\{1,\dots,q_{h-1}\}} \mu_{j_{2h-1}}(\{k\}) + \mu_{j_{2h-1}}(T_h) + \sum_{k\in A_{j_{2h-1}}, k\ge q_h+1} \mu_{j_{2h-1}}(\{k\}) \tag{4.10}$$

and

$$\sum_{k\in A} \mu_{j_{2h-1}}(\{k\}) = \sum_{k\in A\cap\{1,\dots,q_{h-1}\}} \mu_{j_{2h-1}}(\{k\}) + \mu_{j_{2h-1}}(T_h) + \sum_{k\in A, k\ge q_h+1} \mu_{j_{2h-1}}(\{k\}). \tag{4.11}$$

From (4.7) we obtain:

$$\Big|\sum_{k\in A_{j_{2h-1}},k\geq q_h+1}\mu_{j_{2h-1}}(\{k\})\Big|\leq\bigvee_{t=1}^{\infty}a_{t,\varphi(t)},\tag{4.12}$$

$$\Big|\sum_{k\in A,k\geq q_h+1}\mu_{j_{2h-1}}(\{k\})\Big|\leq\bigvee_{t=1}^{\infty}a_{t,\varphi(t)}.$$

From (4.9) we get:

$$\Big|\sum_{k\in A_{j_{2h-1}}\cap\{1,\dots,q_{h-1}\}}\mu_{j_{2h-1}}(\{k\})\Big|\leq\bigvee_{t=1}^{\infty}d_{t,\varphi(t)},\tag{4.13}$$

$$\Big|\sum_{k\in A\cap\{1,\dots,q_{h-1}\}}\mu_{j_{2h-1}}(\{k\})\Big|\leq\bigvee_{t=1}^{\infty}d_{t,\varphi(t)}.$$

From (4.10), (4.11), (4.12) and (4.13) we have:

$$\Big|\sum_{k\in A_{j_{2h-1}}}\mu_{j_{2h-1}}(\{k\})-\sum_{k\in A}\mu_{j_{2h-1}}(\{k\})\Big|\leq 2\bigvee_{t=1}^{\infty}a_{t,\varphi(t)}+2\bigvee_{t=1}^{\infty}d_{t,\varphi(t)}.\tag{4.14}$$

From (4.4) and (4.14) it follows that

$$\Big|\sum_{k\in A}\mu_{j_{2h-1}}(\{k\})\Big|\not\leq 2\bigvee_{t=1}^{\infty}b_{t,\varphi(t)}$$

for all $h\in\mathbb{N}$. So, $\Big\{l\in\mathbb{N}:\ \big|\sum\limits_{k\in A}\mu_l(\{k\})\big|\not\leq 2\bigvee\limits_{t=1}^{\infty}b_{t,\varphi(t)}\Big\}\in\mathcal{F}^*$. Moreover, the sets

$$\Big\{l\in\mathbb{N}:\ \big|\sum_{k\in A}\mu_l(\{k\})\big|\leq 2\bigvee_{t=1}^{\infty}b_{t,\varphi(t)}\Big\}\in\mathcal{F}$$

and

$$\Big\{l\in\mathbb{N}:\ \big|\sum_{k\in A}\mu_l(\{k\})\big|\not\leq 2\bigvee_{t=1}^{\infty}b_{t,\varphi(t)}\Big\}$$

have nonempty intersection. This is absurd and finishes the proof. □

Analogously as in Theorem 3.1 it is possible to prove the following

Theorem 4.2. *Let $\mathcal{F}$ be a diagonal and block-respecting filter of $\mathbb{N}$, $\mu_j\colon\mathcal{P}(\mathbb{N})\to R$, $j\in\mathbb{N}$, be a sequence of strongly bounded and equibounded set functions, and set $\zeta_{A,j}:=\sum\limits_{k\in A}\mu_j(\{k\})$, $A\in\mathcal{P}(\mathbb{N})$, $j\in\mathbb{N}$.*

If the family $(\zeta_{A,j})_{A\in\mathcal{P}(\mathbb{N}),j\in\mathbb{N}}$ $(RD\mathcal{F})$-converges to 0, then

$$(D\mathcal{F})\lim_j\Big(\bigvee_{A\in\mathcal{P}(\mathbb{N})}\Big|\sum_{k\in A}\mu_j(\{k\})\Big|\Big)=0.$$

We now prove the following filter version of the Vitali-Hahn-Saks theorem in the finitely additive setting.

Theorem 4.3. *Let $\mathcal{F}$, $\mathcal{A}$ be as in* Theorem 3.2, $\lambda\colon \mathcal{A} \to [0,+\infty]$ *be a finitely additive measure, $\mu_j\colon \mathcal{A} \to R$, $j \in \mathbb{N}$, be a sequence of finitely additive, equibounded and λ-absolutely continuous measures. Suppose that there exists a finitely additive and λ-absolutely continuous measure $\mu_0\colon \mathcal{A} \to R$, such that the family $\mu_j(A)$, $A \in \mathcal{A}$, $j \in \mathbb{N}$, $(RO\mathcal{F})$-converges to $\mu_0(A)$, $A \in \mathcal{A}$.*

Then for each decreasing sequence $(H_n)_n$ in $\mathcal{A}$ with $\lim\limits_n \lambda(H_n) = 0$ and for every $\mathcal{F}$-stationary set $I \subset \mathbb{N}$ there exists an $\mathcal{F}$-stationary set $J \subset I$, such that

$$\bigwedge_n \Big[\bigvee_{j\in J} v_{\mathcal{L}}(\mu_j)(H_n)\Big] = 0, \tag{4.15}$$

where $\mathcal{L}$ is the σ-algebra generated by the H_n's in H_1.

Proof. Let $(H_n)_n$ be any decreasing sequence in $\mathcal{A}$ with $\lim\limits_n \lambda(H_n) = 0$, set $B_n = H_n \setminus H_{n+1}$ for every $n \in \mathbb{N}$, and let $F = \bigcap\limits_{n=1}^{\infty} H_n$.

Let $j = 0, 1, 2, \dots$ Since μ_j is λ-absolutely continuous, then we get (3.28). For all $A \in \mathcal{P}(\mathbb{N})$ and $j = 0, 1, 2, \dots$ let $\nu_j(A)$ be as in (3.29). By Proposition 2.5, the ν_j's are σ-additive. The equiboundedness of the ν_j's and $(RO\mathcal{F})$-convergence of the family $\nu_j(A)$, $A \in \mathcal{P}(\mathbb{N})$, $j \in \mathbb{N}$, to $\nu_0(A)$, $A \in \mathcal{P}(\mathbb{N})$, are easy consequences of equiboundedness of the μ_j's and $(RO\mathcal{F})$-convergence of the family $\mu_j(A)$, $A \in \mathcal{A}$, $j \in \mathbb{N}$, to $\mu_0(A)$, $A \in \mathcal{A}$, respectively.

By Theorem 3.1 and (2.4), it follows that for every $\mathcal{F}$-stationary set $I \subset \mathbb{N}$ there exists an $\mathcal{F}$-stationary set $J \subset I$, satisfying (4.15). $\square$

Open problem. Find versions of limit theorems for measures with respect to filter convergence, in which the limit measure is not necessarily requested to be σ-additive or strongly bounded.

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