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ORTHOCOMPLEMENTED DIFFERENCE LATTICES IN ASSOCIATION WITH GENERALIZED RINGS

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To David Foulis from whom we learnt on the poetry of the word "perp"

(*Communicated by Sylvia Pulmannová*)

ABSTRACT. Orthocomplemented difference lattices (ODLs) are orthocomplemented lattices endowed with an additional operation of "abstract symmetric difference". In studying ODLs as universal algebras or instances of quantum logics, several results have been obtained (see the references at the end of this paper where the explicite link with orthomodularity is discussed, too). Since the ODLs are "nearly Boolean", a natural question arises whether there are "nearly Boolean rings" associated with ODLs. In this paper we find such an association — we introduce some difference ring-like algebras (the DRAs) that allow for a natural one-to-one correspondence with the ODLs. The DRAs are defined by only a few rather plausible axioms. The axioms guarantee, among others, that a DRA is a group and that the association with ODLs agrees, for the subrings of DRAs, with the famous Stone (Boolean ring) correspondence.

1. Introduction

The algebraic properties of ODLs have been analysed in the papers [12]–[16]. Let us first recall (see [12]) that an ODL is an algebra $L = (X, \wedge, \vee, ^{\perp}, 0, 1, \triangle)$, where $\triangle\colon X^2 \to X$ is a binary operation and $(X, \wedge, \vee, ^{\perp}, 0, 1)$ is an orthocomplemented lattice. The presence of the "symmetric difference" $\triangle$ makes ODLs fairly close to Boolean algebras — for instance, $(X, \wedge, \vee, ^{\perp}, 0, 1)$ is an orthomodular lattice and $(X, \triangle)$ is a 2-group. It therefore seems hopeful to search for a correspondence à la Stone (Boolen ring) theorem. Indeed, in this search we managed to introduce certain ring-like algebras — called *difference ring-like algebras* (DRAs, see Def. 2.4) — and found them in a one-to-one correspondence with ODLs.

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It should be noted that several versions of ring-like correspondences have been obtained for mere orthocomplemented lattices ([2]–[6]). In that very general case the ring-like algebras used are necessarily more axiomatically involved since the modelling of an analogue of symmetric difference is then based on a term function in orthocomplemented lattices. As a result, the ring-like algebras in this very general case are groups exactly when one deals with a Boolean algebra. Thus, our investigation meets with the efforts previously made on Boolean algebras only.

2. Notions and results

Let us start by recalling the notions of the lattice side of our investigation.

Definition 2.1. Let $L = (X, \wedge, \vee, ^{\perp}, 0, 1)$, where $(X, \wedge, \vee, 0, 1)$ is a bounded lattice and $^{\perp}\colon X \to X$ is a unary operation. Then L is said to be an *orthocomplemented lattice* (abbr., an OCL) if the following formulas hold in L:

(O_1) $x \wedge x^{\perp} = 0$, $x \vee x^{\perp} = 1$,

(O_2) $x^{\perp\perp} = x$,

(O_3) $x \le y \implies y^{\perp} \le x^{\perp}$.

Basic properties of OCLs can be found in the pioneering work [8], or in the monographs [9] and [17]. Out of these properties, the de-Morgan identities $(x \wedge y)^{\perp} = x^{\perp} \vee y^{\perp}$, $(x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp}$ will be frequently used.

Definition 2.2. Let $L = (X, \wedge, \vee, ^{\perp}, 0, 1, \triangle)$, where $(X, \wedge, \vee, ^{\perp}, 0, 1)$ is an OCL and $\triangle\colon X^2 \to X$ is a binary operation. Then L is said to be an *orthocomplemented difference lattice* (abbr., an ODL) if the following identities hold in L:

(D_1) $x \triangle (y \triangle z) = (x \triangle y) \triangle z$,

(D_2) $x \triangle 1 = x^{\perp}$, $1 \triangle x = x^{\perp}$,

(D_3) $x \triangle y \le x \vee y$.

The ODLs have been systematically studied in [12]–[16] where, among others, several (non-Boolean or even non-set-representable) examples of ODLs can be found. Let us list and verify the properties of ODLs that we shall use in the sequel.

Proposition 2.3. *Let L be an ODL and let us suppose that $x, y \in L$. Then the following statements hold:*

(1) $x \triangle 0 = x$, $0 \triangle x = x$,

(2) $x \triangle x = 0$,

(3) $x \triangle y = y \triangle x$,

(4) $x^{\perp} \triangle y^{\perp} = x \triangle y$.

P r o o f. Let us first observe that the property (D_2) of the Def. 2.2 yields $1 \triangle 1 = 1^{\perp} = 0$. Let us go on in checking the properties (1)–(3).

(1) $x \triangle 0 = x \triangle (1 \triangle 1) = (x \triangle 1) \triangle 1 = x^{\perp} \triangle 1 = (x^{\perp})^{\perp} = x$. Further, $0 \triangle x = (1 \triangle 1) \triangle x = 1 \triangle (1 \triangle x) = 1 \triangle x^{\perp} = (x^{\perp})^{\perp} = x$.

(2) Let us first show that $x^{\perp} \triangle x^{\perp} = x \triangle x$. We consecutively obtain $x^{\perp} \triangle x^{\perp} = (x \triangle 1) \triangle (1 \triangle x) = (x \triangle (1 \triangle 1)) \triangle x = (x \triangle 0) \triangle x = x \triangle x$. Moreover, we have $x \triangle x \leq x$ as well as $x \triangle x = x^{\perp} \triangle x^{\perp} \leq x^{\perp}$. This implies that $x \triangle x \leq x \wedge x^{\perp} = 0$.

(3) $x \triangle y = (x \triangle y) \triangle 0 = (x \triangle y) \triangle [(y \triangle x) \triangle (y \triangle x)] = x \triangle (y \triangle y) \triangle x \triangle (y \triangle x) = x \triangle 0 \triangle x \triangle (y \triangle x) = x \triangle x \triangle (y \triangle x) = 0 \triangle (y \triangle x) = y \triangle x$.

(4) $x^{\perp} \triangle y^{\perp} = (1 \triangle x) \triangle (1 \triangle y) = (1 \triangle 1) \triangle (x \triangle y) = 0 \triangle (x \triangle y) = x \triangle y$ by applying the equalities (1)–(3). $\square$

Our intention is to find a ring-like counterpart of ODLs. The next definition is crucial in this effort.

Definition 2.4. An algebra $R = (X, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ is called a *difference ring-like algebra* (DRA) if
$(X, +, 0)$ is an Abelian group of characteristic 2 (i.e., $x + x = 0$ for all $x \in X$),
$(X, \cdot, 1)$ is a commutative monoid that is idempotent (i.e., $x \cdot x = x$ for all $x \in X$),
and if for all $x, y, z \in X$ the following identities hold:

(R_1) $x(1 + x) = 0$,

(R_2) $(1 + x)(1 + xy) = 1 + x$,

(R_3) $(x + y)(1 + xy) = x + y$.

Let us first see how DRAs can be obtained from ODLs. As a by-product, the consideration that follows would allow us to construct examples of DRAs that are not rings.

Proposition 2.5. *Let* $L = (X, \wedge, \vee, ^{\perp}, 0, 1, \triangle)$ *be an ODL. Let* $\mathbf{R}(L) = (X, +, \cdot, 0, 1)$ *be the algebra in which the operations* $+$ *and* $\cdot$ *are defined as follows* $(x, y \in X)$*:*

$$x + y = x \triangle y,$$
$$x \cdot y = x \wedge y.$$

Then the algebra $\mathbf{R}(L)$ *is a DRA.*

Proof. The property (D_1) in Def. 2.2 and the properties (1)–(3) in Prop. 2.3 imply that $(X, +, 0)$ is an Abelian group of characteristic 2. The properties of OCLs immediately give us that $(X, \cdot, 1)$ is a commutative idempotent monoid. It remains to verify the conditions (R_1)–(R_3) of Def. 2.4. Let $x, y \in X$.
(R_1) $x(1 + x) = x \wedge (1 \triangle x) = x \wedge x^{\perp} = 0$.
(R_2) $(1 + x)(1 + xy) = (1 \triangle x) \wedge (1 \triangle (x \wedge y)) = x^{\perp} \wedge (x \wedge y)^{\perp} = x^{\perp} \wedge (x^{\perp} \vee y^{\perp}) = x^{\perp} = 1 \triangle x = 1 + x$.
(R_3) $(x + y)(1 + xy) = (x \triangle y) \wedge (1 \triangle (x \wedge y)) = (x \triangle y) \wedge (x \wedge y)^{\perp} = (x \triangle y) \wedge (x^{\perp} \vee y^{\perp}) = x \triangle y$. The last equality follows from the equality (4), Prop. 2.3 and from the condition (D_3). Finally, $x \triangle y = x + y$. $\square$

In the next step we indicate how the DRAs induce ODLs. We shall need the following observation.

Proposition 2.6. *Let R be a DRA. Let us suppose that $x \in R$. Then $0 \cdot x = 0$.*

P r o o f. From the equation (R_3) we obtain, by putting $y = x$, $(x+x)(1+xx) = x+x$. Because $x+x = 0$ and $xx = x$, the previous identity gives us $0 \cdot (1+x) = 0$. If we write $1+x$ instead of x in the last equation, we obtain $0 \cdot (1+(1+x)) = 0$. Moreover, $1+(1+x) = (1+1)+x = 0+x = x$ and this completes the proof. □

Proposition 2.7. *Let $R = (X, +, \cdot, 0, 1)$ be a DRA. Let $\mathbf{L}(R) = (X, \wedge, \vee, ^\perp, 0, 1, \triangle)$ be the algebra in which the operations $\wedge, \vee, ^\perp$ and $\triangle$ are defined as follows ($x, y \in X$):*

$$\begin{aligned} x \wedge y &:= x \cdot y, \\ x \vee y &:= 1 + (1+x)(1+y), \\ x^\perp &:= 1 + x, \\ x \triangle y &:= x + y. \end{aligned}$$

Then the algebra $\mathbf{L}(R)$ is an ODL.

P r o o f. Since the operation $\cdot$ is associative, commutative and idempotent, the algebra $(X, \wedge)$ is a semilattice. It means that we can define an ordering $\leq$ on X by putting $x \leq y$ when $x \wedge y = x$. In other words, we write $x \leq y$ when $x \cdot y = x$. Since $0 \cdot x = 0$ (Prop. 2.6) and $x \cdot 1 = x$, we see that $(X, \wedge)$ is a semilattice with a least element, 0, and a greatest element, 1.

Suppose that $x \in X$. Then $x^{\perp\perp} = 1 + (1+x) = (1+1) + x = 0 + x = x$. Further, suppose that $x, y \in X$ with $x \leq y$ (i.e., $x \cdot y = x$). The condition (R_2) implies that $(1+y)(1+yx) = 1+y$. Since $yx = xy = x$, we see that $(1+y)(1+x) = 1+y$ and therefore $1+y \leq 1+x$. Thus, $y^\perp \leq x^\perp$. Since $x^{\perp\perp} = x$, we obtain the equivalence $x \leq y \iff y^\perp \leq x^\perp$. Making use of this equivalence we infer that the element $x \vee y = 1 + (1+x)(1+y) = (x^\perp \wedge y^\perp)^\perp$ is the least upper bound of x and y. We see that the algebra $(X, \wedge, \vee, 0, 1)$ is a bounded lattice.

In order to show that $(X, \wedge, \vee, ^\perp, 0, 1)$ is an OCL, it is sufficient to check the condition (O_1) of Def. 2.1. If $x \in X$ then $x \wedge x^\perp = x \cdot (1+x) = 0$ in view of the condition (R_1). Further, $x \vee x^\perp = (x^\perp \wedge x^{\perp\perp})^\perp = (x^\perp \wedge x)^\perp = 0^\perp = 1 + 0 = 1$ which means that $(X, \wedge, \vee, ^\perp, 0, 1, \triangle)$ is an OCL.

Finally, let us check the conditions (D_1)–(D_3) of Def. 2.2. The condition (D_1) is nothing but the associativity of the operation $+$. The condition (D_2) follows from the definition of $^\perp$ and the commutativity of $+$. In order to prove the condition (D_3), let us write $1+x$ and $1+y$ instead of x and y in the condition (R_3). We obtain the equality $((1+x)+(1+y))(1+(1+x)(1+y)) = (1+x)+(1+y)$. Since $(1+x)+(1+y) = (1+1)+(x+y) = 0+(x+y) = x+y$, we see that $(x+y)(1+(1+x)(1+y)) = x+y$. As a result, $(x \triangle y) \wedge [1 \triangle ((1 \triangle x) \wedge (1 \triangle y))]$

$= x \triangle y$. Thus, $(x \triangle y) \wedge (x^\perp \wedge y^\perp)^\perp = x \triangle y$. We have obtained that $x \triangle y \leq (x^\perp \wedge y^\perp)^\perp = x \vee y$, which is the condition (D_3). This concludes the proof. $\square$

The previous results enable us to formulate the main theorem of this paper. Let us denote by $\mathcal{ODL}$ the variety of ODLs and by $\mathcal{DRA}$ the variety of DRAs. The theorem says that these varieties are equivalent. This means (see e.g. [10]) that there is a bijection ε of $\mathcal{ODL}$ onto $\mathcal{DRA}$ such that for any $L \in \mathcal{ODL}$, the algebras L and $\varepsilon(L)$ have the same underlying sets and for any $K, L \in \mathcal{ODL}$ and any mapping f of K into L, f is a homomorphism of K into L if and only if f is a homomorphism of $\varepsilon(K)$ into $\varepsilon(L)$. In this case the mapping ε is said to be an *equivalence* between $\mathcal{ODL}$ and $\mathcal{DRA}$.

Theorem 2.8. *The mapping* $\mathbf{R}$ *of* Prop. 2.5 *is an equivalence between* $\mathcal{ODL}$ *and* $\mathcal{DRA}$. *Moreover,* $\mathbf{R}$ *restricted to Boolean algebras is the Stone equivalence between Boolean algebras and Boolean rings.*

Proof. Considering the mapping $\mathbf{L}$ of Prop. 2.7, let us we verify that $\mathbf{R}(\mathbf{L}(R)) = R$ and $\mathbf{L}(\mathbf{R}(L)) = L$ for any $R \in \mathcal{DRA}$ and any $L \in \mathcal{ODL}$. Since the mapping $\mathbf{R}$ obviously preserves homomorphisms, the verification proves the theorem (the "moreover" statement of Thm. 2.8 clearly follows from the definiton of $\mathbf{R}$ and the uniqueness of $\triangle$ in Boolean algebras [12]).

Let us first prove that $\mathbf{R}(\mathbf{L}(R)) = R$. Let $R = (X, +, \cdot, 0, 1)$ and $\mathbf{L}(R) = (X, \wedge, \vee, ^\perp, 0, 1, \triangle)$. Suppose that $\mathbf{R}(\mathbf{L}(R)) = (X, \oplus, \odot, 0, 1)$. Then for all $x, y \in X$ we have

$$x \oplus y = x \triangle y = x + y, \qquad x \odot y = x \wedge y = x \cdot y$$

and the first identity is proved.

Secondly, let us prove $\mathbf{L}(\mathbf{R}(L)) = L$. Let $L = (X, \wedge, \vee, ^\perp, 0, 1, \triangle)$ and let $\mathbf{R}(L) = (X, +, \cdot, 0, 1)$. Suppose that $\mathbf{L}(\mathbf{R}(L)) = (X, \sqcap, \sqcup, ^\star, 0, 1, \Delta)$. Then for all $x, y \in X$ we have

$$x \sqcap y = x \cdot y = x \wedge y,$$

$$x \sqcup y = 1 + (1 + x)(1 + y) = 1 \triangle ((1 \triangle x) \wedge (1 \triangle y)) = (x^\perp \wedge y^\perp)^\perp = x \vee y,$$

$$x^\star = 1 + x = 1 \triangle x = x^\perp, \quad \text{and} \quad x \Delta y = x + y = x \triangle y.$$

The second identity is verified and the proof of Thm. 2.8 is thus complete. $\square$

It is wortwhile observing in concluding our paper that Thm. 2.8 allows one to characterize the rings among DRAs. Moreover, this characterization can be expressed in terms of two variables.

Proposition 2.9. *Let* R *be a DRA. Then* R *is a ring if and only if the following identity holds for any* $x, y \in R$: $\quad x = 1 + (1 + xy)(1 + x(1 + y))$.

Proof. It is easy to see that getting back to the corresponding ODL via Thm. 2.8, we restate the condition of Prop. 2.9 in the form of the equality $x =$

$[(x \wedge y)^{\perp} \wedge (x \wedge y^{\perp})^{\perp}]^{\perp}$ in the ODL. This means the equality $x = (x \wedge y) \vee (x \wedge y^{\perp})$ which is known to imply that the ODL is Boolean. $\square$

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