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COMPOSITION OF ℓ-STABLE VECTOR FUNCTIONS

Dušan Bednařík* — Karel Pastor**

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ABSTRACT. The class of functions that are ℓ-stable at some point is a generalization of the class of $C^{1,1}$ functions. In this paper we prove that the class of vector functions that are ℓ-stable at some point is closed under composition.

1. Introduction

The class of $C^{1,1}$ fuctions (i.e., functions with locally Lipschitz derivative) has been intensively studied since 70's of last century, because these functions appear in some problems of applied mathematics as for example the proximal point method, the penalty function method or variational inequalities.

It was shown in [BP1] that an unconstrained second-order optimality condition, which was at first stated for the class of $C^{1,1}$ functions [GGR], holds also for the functions that are ℓ-stable at some point.

The properties of the class of ℓ-stable real-valued functions were then studied also in [BP2, BP3, LX]. It seems to be useful to generalize the class of ℓ-stable functions for vector-valued functions, see [BP4, BP6, BP7, G, GG], where several vector optimization results were presented under weaker assumptions than it was done earlier [GL, GGR, BP5]. In this paper, we will continue the study of the class of ℓ-stable vector functions.

In Section 2 we recall basic concepts of ℓ-stability. Section 3 aims to prove that the class of ℓ-stable vector functions is closed under composition.

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2. ℓ-stability for vector functions

Throughout of this text we will work with functions $f\colon \mathbb{R}^m \to \mathbb{R}^n$ where $m, n \in \mathbb{N}$. If X is an Euclidean space endowed with the Euclidean norm, then $S_X = \{x \in X : \|x\| = 1\}$ denotes the unit sphere, $B_X = \{x \in X : \|x\| \le 1\}$, and $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product on X.

Further, if $Y \subset X$, then $\operatorname{int} Y$, $\overline{Y}$ denote the topological interior and topological closure of Y, respectively. $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ will denote the space of all continuous linear operators $L\colon \mathbb{R}^m \to \mathbb{R}^n$. $f'(x)$ denotes the Fréchet derivative of the function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ at $x \in \mathbb{R}^m$, i.e. $f'(x) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$.

By a cone $C \subset \mathbb{R}^n$ we will always mean a nonempty, closed, convex and pointed cone with $\operatorname{int} C \ne \emptyset$. For definitions, see e.g. [Ja, RW]. We denote the (positive) polar cone to C by C' and define it as follows: $C' := \{\xi \in \mathbb{R}^n : \langle \xi, y \rangle \ge 0,\ \ y \in C\}$. From now on we always put $\Gamma_C := C' \cap S_{\mathbb{R}^n}$. Under assumptions set out above, it is well known that:

a) C' is also nonempty, closed, convex and pointed cone with $\operatorname{int} C' \ne \emptyset$;

b) for each $y \in C \setminus \{0\}$, $\xi \in \operatorname{int} C'$, it holds $\langle \xi, y \rangle > 0$;

c) $C = C''$.

For a function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ we define the *lower and upper directional derivatives at $x \in \mathbb{R}^m$ in the direction $h \in \mathbb{R}^m$ with respect to $\xi \in \mathbb{R}^n$*, respectively, by

$$f^\ell(x; h)(\xi) = \liminf_{t \downarrow 0} \frac{\langle \xi, f(x + th) - f(x) \rangle}{t},$$

and

$$f^u(x; h)(\xi) = \limsup_{t \downarrow 0} \frac{\langle \xi, f(x + th) - f(x) \rangle}{t}.$$

Specially, if $n = 1$, by $f'(x; h)$ we mean the limit

$$\lim_{t \downarrow 0} \frac{f(x + th) - f(x)}{t}.$$

Recall that if a Fréchet differentiable at x function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ satisfies

$$\big(\forall h \in S_{\mathbb{R}^m}\big) \left(f'(x)h = \lim_{y \to x, t \downarrow 0} \frac{f(y + th) - f(y)}{t} \right),$$

and this convergence is uniform for $h \in S_{\mathbb{R}^m}$, then f is said to be *strictly differentiable at x*.

Definition 2.1. A function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is said to be *ℓ-stable at x* if there is a cone $C \subset \mathbb{R}^n$, a neighbourhood U of x and a constant $K > 0$ such that

$$\big(y \in U \ \ \&\ \ h \in S_{\mathbb{R}^m} \ \ \&\ \ \xi \in \Gamma_C\big) \implies |f^\ell(y; h)(\xi) - f^\ell(x, h)(\xi)| \le K\|y - x\|. \quad (1)$$

It is not difficult to observe that each function f which is of $C^{1,1}$-class near a point x is also ℓ-stable at x. The reverse implication is not true (see Example 3.1).

The class of the functions that are ℓ-stable at some point satisfies some properties of regularity.

Proposition 2.1. ([BP5, Proposition 2.2]) *Let a function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be ℓ-stable at $x \in \mathbb{R}^m$. Then f is strictly differentiable at x and consequently f is Lipschitz on a neighbourhood of x.*

Using [BP2, Corollary 3] and Proposition 2.1, we can state the following result.

Theorem 2.1. *Let a function $f\colon \mathbb{R}^m \to \mathbb{R}$ be ℓ-stable at $x \in \mathbb{R}^m$, $g\colon \mathbb{R} \to \mathbb{R}$ be a function which is ℓ-stable at $y = f(x)$. Then $g \circ f$ is ℓ-stable at x.*

We would like to generalize the previous result for vector-valued functions. We will use the characterization of ℓ-stability by means of u-stability for scalar-valued functions.

We say that $f\colon \mathbb{R}^m \to \mathbb{R}$ is *u-stable* at $x \in \mathbb{R}^m$ if there exist a neighbourhood V of x and $L > 0$ such that

$$\big(\forall y \in V\big)\big(\forall h \in S_{\mathbb{R}^m}\big)\big(|f^u(y;h) - f^u(x;h)| \le L\|y - x\|\big).$$

Following [BP2, Corollary 1] and Proposition 2.1, we obtain the following proposition.

Proposition 2.2. ([BP3, Corollary 1]) *A function $f\colon \mathbb{R}^m \to \mathbb{R}$ is ℓ-stable at $x \in \mathbb{R}^m$ if and only if f is u-stable at x.*

For the sake of completness we note that we can replace $\mathbb{R}^m$ by an arbitrary linear normed space in Theorem 2.1 and Proposition 2.2 (see [BP2]), but we have to add the assumption of continuity in this case.

3. Composition of ℓ-stable functions

At first, we will derive that the ℓ-stability of vector function is equivalent to ℓ-stability of all its components (Proposition 3.1). We will use the following sequence of lemmas.

Lemma 3.1. ([BP6, Lemma 1]) *Let C be a cone in $\mathbb{R}^n$. Then*

$$\mathbb{R}^n = \big\{\alpha_1 y_1 + \alpha_2 y_2 : \ \alpha_1, \alpha_2 \in \mathbb{R}, \ y_1, y_2 \in \Gamma_C\big\}.$$

Lemma 3.2. *Let $f\colon \mathbb{R}^m \to \mathbb{R}$ be ℓ-stable at $x \in \mathbb{R}^m$. Then there exist $K > 0$ and a neighbourhood $\mathcal{U}$ of x such that*

$$|(\alpha f)^\ell(y;h) - (\alpha f)^\ell(x;h)| \le K\|y-x\|, \tag{2}$$

whenever $\alpha \in \mathbb{R}$, $|\alpha| \le 1$, $y \in \mathcal{U}$, $h \in S_{\mathbb{R}^m}$.

P r o o f. Suppose that f is ℓ-stable at $x \in \mathbb{R}^m$. Then an easy calculus shows that $(-f)$ must be u-stable at x. Hence by Proposition 2.2 $(-f)$ is also ℓ-stable at x. This means that there are $K > 0$ and a neighbourhood $\mathcal{U}$ of x such that

$$|(\pm f)^\ell(y;h) - (\pm f)^\ell(x,h)| \le K\|y-x\|, \tag{3}$$

whenever $y \in \mathcal{U}$, $h \in S_{\mathbb{R}^m}$. Now if $0 \le \alpha \le 1$, then for every $y \in \mathcal{U}$, $h \in S_{\mathbb{R}^m}$ we have by (3)

$$\begin{aligned} |(\alpha f)^\ell(y;h) - (\alpha f)^\ell(x;h)| &= |\alpha f^\ell(y;h) - \alpha f^\ell(x;h)| \\ &\le |\alpha| K\|y-x\| \\ &\le K\|y-x\|. \end{aligned}$$

If $-1 \le \alpha \le 0$, then we have again by (3)

$$\begin{aligned} |(\alpha f)^\ell(y;h) - (\alpha f)^\ell(x;h)| &= |((-\alpha)(-f))^\ell(y;h) - ((-\alpha)(-f))^\ell(x;h)| \\ &= |\alpha||(-f)^\ell(y;h) - (-f)^\ell(x;h)| \\ &\le K\|y-x\|, \end{aligned}$$

for every $y \in \mathcal{U}$, and for every $h \in S_{\mathbb{R}^m}$. □

By a similar manner, we can prove the following lemma.

Lemma 3.3. *Let $f\colon \mathbb{R}^m \to \mathbb{R}$ be an ℓ-stable function at $x \in \mathbb{R}^m$. Then there exist $L > 0$ and a neighbourhood $\mathcal{V}$ of x such that*

$$|(\alpha f)^u(y;h) - (\alpha f)^u(x;h)| \le L\|y-x\|,$$

whenever $|\alpha| \le 1$, $y \in \mathcal{V}$ and $h \in S_{\mathbb{R}^m}$.

Lemma 3.4. *Let $f_i\colon \mathbb{R}^m \to \mathbb{R}$, $i = 1,\dots,n$ be ℓ-stable at $x \in \mathbb{R}^m$. For $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$ put*

$$g_\xi(y) = \sum_{i=1}^n \xi_i f_i(y), \qquad y \in \mathbb{R}^m.$$

Then there are $K > 0$ and a neighbourhood $\mathcal{U}$ of x such that

$$|g_\xi^\ell(y;h) - g_\xi^\ell(x;h)| \le K\|y-x\|,$$

whenever $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$, $|\xi_i| \le 1$, $i = 1,\dots,n$, $y \in \mathcal{U}$, $h \in S_{\mathbb{R}^m}$.

P r o o f. Due to Proposition 2.1 every function f_i, $i = 1, \ldots, n$, is strictly differentiable at x. Next, by lemmas 3.2 and 3.3 there are $K > 0$ and a neighbourhood $\mathcal{U}$ of x such that for every $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ for which $|\xi_i| \leq 1$, $i = 1, \ldots, n$, $y \in \mathcal{U}$, $h \in S_{\mathbb{R}^m}$, we have:

$$|(\xi_i f_i)^\ell(y; h) - \xi_i f_i'(x)h| \leq K\|y - x\|, \tag{4}$$

$$|(\xi_i f_i)^u(y; h) - \xi_i f_i'(x)h| \leq K\|y - x\|, \tag{5}$$

$i = 1, \ldots, n$. From (4), (5) it follows that for every $i = 1, \ldots, n$

$$-K\|y - x\| + \xi_i f_i'(x)h \leq (\xi_i f_i)^\ell(y; h), \tag{6}$$

$$(\xi_i f_i)^u(y; h) \leq K\|y - x\| + \xi_i f_i'(x)h, \tag{7}$$

whenever $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $|\xi_i| \leq 1$, $y \in \mathcal{U}$, $h \in S_{\mathbb{R}^m}$. Adding up inequalities in (6) for $i = 1, \ldots, n$, we arrive at

$$-nK\|y - x\| + \sum_{i=1}^n \xi_i f_i'(x)h = -nK\|y - x\| + g_\xi'(x)h \leq \sum_{i=1}^n (\xi_i f_i)^\ell(y; h).$$

Similarly, adding up inequalities in (7) for $i = 1, \ldots, n$, we arrive at

$$\sum_{i=1}^n (\xi_i f_i)^u(y; h) \leq nK\|y - x\| + \sum_{i=1}^n \xi_i f_i'(x)h = nK\|y - x\| + g_\xi'(x)h.$$

Thus, if $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $|\xi_i| \leq 1$, $y \in \mathcal{U}$, $h \in S_{\mathbb{R}^m}$, then it holds

$$\begin{aligned} -nK\|y - x\| + g_\xi'(x)h &\leq \sum_{i=1}^n (\xi_i f_i)^\ell(y; h) \\ &\leq g_\xi^\ell(y; h) \leq g_\xi^u(y; h) \\ &\leq \sum_{i=1}^n (\xi_i f_i)^u(y; h) \\ &\leq nK\|y - x\| + g_\xi'(x)h. \end{aligned}$$

This implies for every $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $|\xi_i| \leq 1$, $y \in \mathcal{U}$, $h \in S_{\mathbb{R}^m}$:

$$|g_\xi^\ell(y; h) - g_\xi'(x)h| \leq nK\|y - x\|.$$

□

Lemma 3.5. *Let $f_i\colon \mathbb{R}^m \to \mathbb{R}$, $i = 1, \ldots, n$, be ℓ-stable at $x \in \mathbb{R}^m$, and let $\alpha_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n$. Then the function $f\colon \mathbb{R}^m \to \mathbb{R}$,*

$$f(y) = \sum_{i=1}^n \alpha_i f_i(y), \quad y \in \mathbb{R}^m,$$

is ℓ-stable at x.

P r o o f. We put

$$M := \sum_{i=1}^{n} |\alpha_i|,$$

and

$$g(y) := \sum_{i=1}^{n} \frac{\alpha_i}{M} f_i(y), \qquad \text{for all} \quad y \in \mathbb{R}^m.$$

Since we can write

$$f(y) = M \sum_{i=1}^{n} \frac{\alpha_i}{M} f_i(y) = M g(y), \qquad \text{for all} \quad y \in \mathbb{R}^m,$$

and it holds

$$\left| \frac{\alpha_i}{M} \right| \leq 1, \qquad \text{for every} \quad i = 1, 2, \dots, n,$$

Lemma 3.4 implies the existence of neighbourhood $\mathcal{U}$ and $K > 0$ such that we have

$$|f^\ell(y; h) - f^\ell(x; h)| = M |g^\ell(y; h) - g^\ell(x; h)| \leq MK \|y - x\|,$$

for every $y \in \mathcal{U}$ and for every $h \in S_{\mathbb{R}^m}$. Therefore the function f is ℓ-stable at x. □

Proposition 3.1. *Let $f = (f_1, f_2, \dots, f_n) \colon \mathbb{R}^m \to \mathbb{R}^n$ be a function and $x \in \mathbb{R}^m$. Then f is ℓ-stable at x if and only if the function f_i is ℓ-stable at x for every $i \in \{1, 2, \dots, n\}$.*

P r o o f. At first, we suppose that every component f_i, $i \in \{1, 2, \dots, n\}$, is ℓ-stable at x. Let us consider an arbitrary $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \Gamma$ and set

$$g_\xi(y) = \langle \xi, f(y) \rangle = \sum_{i=1}^{n} \xi_i f_i(y).$$

Then for every $y \in \mathbb{R}^m$, and for every $h \in \mathbb{R}^m$ we have $f^\ell(y; h)(\xi) = g_\xi^\ell(y; h)$. $\xi \in \Gamma$ implies that $|\xi_i| \leq 1$ for every $i \in \{1, 2, \dots, n\}$. Due to Lemma 3.4 there exist a constant $K > 0$ and a neighbourhood $\mathcal{U}$ satisfying

$$|f^\ell(y; h)(\xi) - f^\ell(x; h)(\xi)| \leq K \|y - x\|,$$

for every $\xi = (\xi_1, \dots, \xi_n) \in \Gamma$, $y \in \mathcal{U}$, and for every $h \in S_{\mathbb{R}^m}$, but it means that f is ℓ-stable at x.

On the other hand, we suppose that f is ℓ-stable at x and take an arbitrary $i \in \{1, 2, \dots, n\}$. By Lemma 3.1 we can find $a_j^i \in \mathbb{R}$, $\xi_j^i \in \Gamma$, $j = 1, 2$, such that

$$e^i = (0, \dots, 0, \underbrace{1}_{i\text{-th}}, 0, \dots, 0) = \sum_{j=1}^{2} a_j^i \xi_j^i.$$

From the definition of ℓ-stability, for every $\xi \in \Gamma$ it holds that the function $y \mapsto \langle \xi, f(y) \rangle$ is ℓ-stable at x. Using Lemma 3.5, we have that the function

$$f_i(y) = \langle e^i, f(y) \rangle = \left\langle \sum_{j=1}^{2} a_j^i \xi_j^i, f(y) \right\rangle = \sum_{j=1}^{2} a_j^i \langle \xi_j^i, f(y) \rangle,$$

is ℓ-stable at x. □

Remark 3.1. Using Proposition 3.1 one can observe that a function which is ℓ-stable at x with respect to some cone must be also ℓ-stable at x with respect to another cone. Of course, the constant of ℓ-stability depends on the choice of a cone, but it seems that there exists a valid constant for all cone C [DP].

Theorem 3.1. *Let $m, n, p \in \mathbb{N}$, a function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be ℓ-stable at $\hat{x} \in \mathbb{R}^m$, and let a function $g\colon \mathbb{R}^n \to \mathbb{R}^p$ be ℓ-stable at $\hat{y} = f(\hat{x}) \in \mathbb{R}^n$. Then the composition $g \circ f$ is ℓ-stable at $\hat{x}$.*

Proof. It is easy to see that due to Proposition 3.1 we can assume without any loss of generality that $p = 1$.

STEP 1.

Using Proposition 2.1, we can find a neighbourhood U of $\hat{x}$ and a neighbourhood V of $\hat{y}$ such that

$$f(U) \subset V,$$

$f = (f_1, f_2, \ldots, f_n)$ is Lipschitz on U, and g is Lipschitz on V. Then the composition $g \circ f$ is Lipschitz on U. Further, it follows from Proposition 2.1 that the composition $g \circ f$ is differentiable at $\hat{x}$ and

$$(g \circ f)'(\hat{x})(h) = g'(f(\hat{x})) \cdot f'(\hat{x})(h),$$

for every $h \in \mathbb{R}^m$.

STEP 2.

Let us consider $x \in U$, $h \in \mathbb{R}^m$ and a sequence $\{\tau_n\}$ such that $\tau_n \downarrow 0$, and

$$(g \circ f)^{\ell}(x; h) = \lim_{n \to \infty} \frac{1}{\tau_n} \big[(g \circ f)(x + \tau_n h) - (g \circ f)(x) \big].$$

Proposition 3.1 implies that the function f_i is ℓ-stable at $\hat{x}$ for every $i \in \{1, 2, \ldots, n\}$. Then, due to Proposition 2.1 and Proposition 2.2, we can suppose that the function f_i is Lipschitz on the neighbourhood U of $\hat{x}$ for every $i \in \{1, 2, \ldots, n\}$, and that:

$$\big(\exists K > 0\big)\big(\forall i \in \{1, 2, \ldots, n\}\big)\big(\forall x \in U\big)\big(\forall h \in S_{\mathbb{R}^m}\big)$$
$$\big(|f_i^{\tau_n}(x; h) - f_i'(\hat{x})h| \le K\|x - \hat{x}\|\big)$$

where the existence of the limit

$$f_i^{\tau_n}(x; h) := \lim_{n \to \infty} \frac{f_i(x + \tau_n h) - f_i(x)}{\tau_n}$$

is garanted by the lipschitzianity of f_i for every $i \in \{1, 2, \dots, n\}$.

Further, we set

$$f^{\tau_n}(x; h) := \lim_{n\to\infty} \frac{1}{\tau_n}[f(x + \tau_n h) - f(x)] = (f_1^{\tau_n}(x; h), \dots, f_n^{\tau_n}(x; h)).$$

Then

$$f(x + \tau_n h) = f(x) + \tau_n f^{\tau_n}(x; h) + o(\tau_n),$$

where

$$\lim_{n\to\infty} \frac{o(\tau_n)}{\tau_n} = 0 \in \mathbb{R}^n.$$

Further we have

$$\begin{aligned} \|f^{\tau_n}(x; h) - f'(\hat{x})h\| &= \| (f_1^{\tau_n}(x; h), \dots, f_n^{\tau_n}(x; h)) - (f_1'(\hat{x})h, \dots, f_n'(\hat{x})h) \| \\ &\le \sum_{i=1}^{n} |f_i^{\tau_n}(x; h) - f_i'(\hat{x})h| \le nK\|x - \hat{x}\|, \qquad (8) \end{aligned}$$

for every $h \in S_{\mathbb{R}^m}$. Without loss of generality we can also suppose the existence of the limit

$$g^{\tau_n}(f(x); f^{\tau_n}(x; h)) := \lim_{n\to\infty} \frac{1}{\tau_n}[g(f(x) + \tau_n f^{\tau_n}(x; h)) - g(f(x))].$$

By L_g we denote the constant of lipschitzianity of g on V. Again without loss of generality we can suppose that for every $n \in \mathbb{N}$ it holds

$$f(x) + \tau_n f^{\tau_n}(x; h) + o(\tau_n) \in V$$

and

$$f(x) + \tau_n f^{\tau_n}(x; h) \in V.$$

Then

$$\begin{aligned} &\left| \frac{1}{\tau_n}[g(f(x) + \tau_n f^{\tau_n}(x; h) + o(\tau_n)) - g(f(x) + \tau_n f^{\tau_n}(x; h))] \right| \\ \le\ & \frac{1}{\tau_n} L_g \|o(\tau_n)\| \longrightarrow 0 \qquad \text{for} \quad n \longrightarrow \infty \end{aligned}$$

Hence

$$\begin{aligned} (g \circ f)^{\ell}(x; h) &= \lim_{n\to\infty} \frac{1}{\tau_n} \big[(g \circ f)(x + \tau_n h) - (g \circ f)(x)\big] \\ &= \lim_{n\to\infty} \frac{1}{\tau_n} \big[g(f(x) + \tau_n f^{\tau_n}(x; h) + o(\tau_n)) - g(f(x))\big] \\ &= \lim_{n\to\infty} \bigg\{ \frac{1}{\tau_n} \big[g(f(x) + \tau_n f^{\tau_n}(x; h)) - g(f(x))\big] \\ &\quad + \frac{1}{\tau_n} \big[g(f(x) + \tau_n f^{\tau_n}(x; h) + o(\tau_n)) - g(f(x) + \tau_n f^{\tau_n}(x; h))\big] \bigg\} \\ &= g^{\tau_n}(f(x); f^{\tau_n}(x; h)). \qquad (9) \end{aligned}$$

STEP 3.
Without any loss of generality we can suppose that ℓ-stability of g, and Proposition 2.2 imply the existence of a constant $M > 0$ such that for every $y \in V$, for every $k \in S_{\mathbb{R}^n}$, and for every sequence $\{v_n\}_{n=1}^{\infty}$ for which $\lim_{n\to\infty} v_n = 0$ and the limit

$$g^{v_n}(y;k) := \lim_{n\to\infty} \frac{1}{v_n}\big[g(y+v_n k) - g(y)\big] = (g_1^{v_n}(y;k), \dots, g_p^{v_n}(y;k)),$$

exists, we have

$$|g^{v_n}(y;k) - g'(\hat{y})k| \leq M\|y-\hat{y}\|, \tag{10}$$

Now there are two possibilities:

a) $f^{\tau_n}(x;h) \neq 0$.

Following formulas (8), (9) and (10), we have

$$\begin{aligned}
& |(g\circ f)^{\ell}(x;h) - (g\circ f)^{\ell}(\hat{x};h)| \\
= \; & |g^{\tau_n}(f(x); f^{\tau_n}(x;h)) - g'(f(\hat{x}))f'(\hat{x})h| \\
\leq \; & |g^{\tau_n}(f(x); f^{\tau_n}(x;h)) - g'(f(\hat{x}))f^{\tau_n}(x;h)| \\
+ \; & |g'(f(\hat{x}))f^{\tau_n}(x;h) - g'(f(\hat{x})f'(\hat{x})h| \\
\leq \; & \|f^{\tau_n}(x;h)\| \left| g^{\tau_n}\left(f(x); \frac{f^{\tau_n}(x;h)}{\|f^{\tau_n}(x;h)\|}\right) - g'(f(\hat{x}))\frac{f^{\tau_n}(x;h)}{\|f^{\tau_n}(x;h)\|}\right| \\
& + \|g'(f(\hat{x}))\|\,\|f^{\tau_n}(x;h) - f'(\hat{x})h\| \\
\leq \; & \|f^{\tau_n}(x;h)\|M\|f(x) - f(\hat{x})\| + \|g'(f(\hat{x})\|\,nK\|x-\hat{x}\| \\
\leq \; & L_f^2 M\|x-\hat{x}\| + \|g'(f(\hat{x})\|\,nK\|x-\hat{x}\| \\
= \; & (L_f^2 M + nK\,\|g'(f(\hat{x})\|)\|x-\hat{x}\|,
\end{aligned}$$

where $L_f > 0$ denotes Lipschitz constant of f on U. Setting

$$N := L_f^2 Mp + nK\,\|g'(f(\hat{x})\| > 0,$$

then it holds

$$\big|(g\circ f)^{\ell}(x;h) - (g\circ f)^{\ell}(\hat{x};h)\big| \leq N\|x-\hat{x}\|, \tag{11}$$

for every $x \in U$ and for every $h \in S_{\mathbb{R}^m}$.

b) $f^{\tau_n}(x;h) = 0$. Let us calculate:

$$\begin{aligned}
& |(g\circ f)^{\ell}(x;h) - (g\circ f)^{\ell}(\hat{x};h)| \\
= \; & |g^{\tau_n}(f(x); f^{\tau_n}(x;h)) - g'(f(\hat{x}))f'(\hat{x})h| \\
= \; & |g'(f(\hat{x}))f'(\hat{x})h| \\
= \; & |g'(f(\hat{x}))f'(\hat{x})h - g'(f(\hat{x}))f^{\tau_n}(x;h) + g'(f(\hat{x}))f^{\tau_n}(x;h)| \\
= \; & |g'(f(\hat{x})f'(\hat{x})h - g'(f(\hat{x}))f^{\tau_n}(x;h)| \\
\leq \; & \|g'(f(\hat{x}))\|\,\|f'(\hat{x})h - f^{\tau_n}(x;h)\| \leq \|g'(f(\hat{x}))\|\,nK\|x-\hat{x}\|.
\end{aligned}$$

Thus, the inequality (11) holds in both cases a) and b). It means that the composition $g \circ f$ is ℓ-stable at $\hat{x}$ □

As an application of the previous results we present the following example where we use some ideas from [BP1, Example 2].

Example 3.1. Consider a sequence $a_n = 1/n$, $n = 1, 2, \ldots$. Then

$$\lim_{n\to\infty} \frac{a_{n+1} + a_n^2}{a_{n+1} + a_n} = \frac{1}{2} > 0.$$

Let us define a function $\varphi\colon [0,\infty) \to R$ as follows:

$$\varphi(u) = \begin{cases} a_1 & \text{if } u > a_1,\\ \frac{a_n^2 - a_{n+1}}{a_n - a_{n+1}}(u - a_{n+1}) + a_{n+1} & \text{if } u \in (a_{n+1}, a_n],\\ 0 & \text{if } u = 0. \end{cases}$$

Next, we will define a function $f\colon R \to R$ via the Riemann integral:

$$f(x) := \int_0^{|x|} \varphi(u)\,\mathrm{d}u, \qquad x \in \mathbb{R}.$$

Since φ is a piecewise affine function, the integral exists. Because of $f'(a_n; 1) = a_n$ and $f'(a_n; -1) = -a_n^2$ for every $n > 1$, the function f is not differentiable on any neighbourhood of 0. Nevertheless, it is easy to show that $f'(0) = 0$ and that f is ℓ-stable at $x = 0$.

Further, let us consider the composition of $g\colon R \to R^2$,

$$g(x) = \big(\sin f(x), f(x)\big) \qquad \text{for all} \quad x \in \mathbb{R},$$

and $h\colon R^2 \to R^2$,

$$h(u,v) = \Big(v - \frac{1}{2}u, u + v\Big) \qquad \text{for all} \quad (u,v) \in \mathbb{R}^2,$$

i.e., the function $h \circ g\colon R \to R^2$,

$$(h \circ g)(x) = \Big(f(x) - \frac{1}{2}\sin f(x), f(x) + \sin f(x)\Big) \qquad \text{for all} \quad x \in \mathbb{R}.$$

Since the function f is ℓ-stable at 0 and the function $\sin x$ is continuously differentiable, Theorem 3.1 and Proposition 3.1 imply that $h \circ g$ is ℓ-stable at 0.

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* *Department of Mathematics*
Faculty of Science
Rokitanského 62
CZ–500 03 Hradec Králové
CZECH REPUBLIC
E-mail: dbednarik@seznam.cz

** *Department of Mathematical Analysis and Applications of Mathematics*
Faculty of Science
Palacký University
17. listopadu 12
CZ–771 46 Olomouc
CZECH REPUBLIC
E-mail: karel.pastor@upol.cz