

CONVERGENCE RATES IN THE COMPLETE MOMENT OF MOVING-AVERAGE PROCESSES

QING-PEI ZANG

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ABSTRACT. In this paper, we discuss precise asymptotics for a new kind of moment convergence of the moving-average process $X_k = \sum_{i=-\infty}^{\infty} a_{i+k}\varepsilon_i$, $k \geq 1$, where $\{\varepsilon_i : -\infty < i < \infty\}$ is a doubly infinite sequence of independent identically distributed random variables with mean zero and the finiteness of variance, $\{a_i : -\infty < i < \infty\}$ is an absolutely summable sequence of real numbers, i.e., $\sum_{i=-\infty}^{\infty} |a_i| < \infty$.

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1. Introduction and main results

We assume that $\{\varepsilon_i, \varepsilon_i : -\infty < i < \infty\}$ is a doubly infinite sequence of identically distributed variables. Let $\{a_i : -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers and $X_k = \sum_{i=-\infty}^{\infty} a_{i+k}\varepsilon_i$, $k \geq 1$, also set $S_n = \sum_{k=1}^n X_k$.

Many limiting results have been obtained for moving-average processes $\{X_k : k \geq 1\}$. For example, Burton and Dehling [1] have obtained a large deviation principle for $\{X_k : k \geq 1\}$ assuming $E \exp t\varepsilon_1 < \infty$ for all t , Ibragimov [5] has established the central limit theorem for $\{X_k : k \geq 1\}$, Li, et al. [6]

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derived convergence rates of moderate deviations and the precise asymptotics in the law of the iterated logarithm, Dong, et al. [3] also obtained moderate deviation principles for moving-average processes of real stationary sequences.

On the other hand, Gut and Spătaru [4] proved the precise asymptotics of i.i.d random variables. One of their results is as follows.

THEOREM 1.1. *Suppose that $\{Y_k : k \geq 1\}$ is a sequence of i.i.d random variables with $EY_1 = 0$ and $EY_1^2 = \sigma^2 < \infty$. Then*

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \sum_{n=1}^{\infty} \frac{1}{n \log n} P \left(\left| \sum_{k=1}^n Y_k \right| \geq \varepsilon \sqrt{n \log \log n} \right) = \sigma^2.$$

Chow [2] discussed the complete moment convergence of i.i.d random variables. He got the following result:

THEOREM 1.2. *Let $\{Y, Y_k : k \geq 1\}$ be a sequence of i.i.d random variables with $EY_1 = 0$. Suppose that $p \geq 1$, $\alpha > \frac{1}{2}$, $p\alpha > 1$, $E\{|Y|^p + |Y| \log(1 + |Y|)\} < \infty$. Then for any $\varepsilon > 0$, we have*

$$\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} E \left\{ \max_{j \leq n} \left| \sum_{k=1}^j Y_k \right| - \varepsilon n^\alpha \right\}_+ < \infty.$$

Recently, Liu and Lin [9] derived precise asymptotics for a new kind of complete moment convergence, one of their results is as follows:

THEOREM 1.3. *Suppose that $\{X, X_n : n \geq 1\}$ is a sequence of i.i.d. random variables, $0 < \delta \leq 1$,*

$$EX = 0, \quad EX^2 = \sigma^2 \quad \text{and} \quad EX^2 (\log^+ |X|)^\delta < \infty.$$

Then we have

$$\lim_{\varepsilon \searrow 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} ES_n^2 I \{|S_n| \geq \varepsilon n\} = 2\sigma^2.$$

Inspired by the above results, we will extend this kind of results to moving-average processes. Now we state our results as follows.

THEOREM 1.4. *Suppose that $\{\varepsilon, \varepsilon_n : n \geq 1\}$ is defined as above, and*

$$E\varepsilon = 0, \quad E\varepsilon^2 = \sigma^2, \quad E\varepsilon^3 < \infty. \quad (1.1)$$

If $\sum_{i=-\infty}^{\infty} a_i \neq 0$, we have

$$\lim_{\varepsilon \searrow 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} ES_n^2 I \{|S_n| \geq \varepsilon n\} = 2\tau^2, \quad (1.2)$$

where $\tau^2 = \sigma^2 \left(\sum_{i=-\infty}^{\infty} a_i \right)^2$.

THEOREM 1.5. Suppose that $\{\varepsilon, \varepsilon_n : n \geq 1\}$ is defined as above, and

$$E\varepsilon = 0, \quad E\varepsilon^2 = \sigma^2, \quad E\varepsilon^3 < \infty. \quad (1.3)$$

If $\sum_{i=-\infty}^{\infty} a_i \neq 0$, then, for $1 \leq p < 2$, $1 + \frac{p}{2} < r < 2$, $0 \leq q < \frac{2(r-p)}{2-p}$, we have

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{q}{p}} E|S_n|^q I \left\{ |S_n| \geq \varepsilon n^{\frac{1}{p}} \right\} \\ = \frac{2p}{2r-2p-2q+pq} E|Z|^{\frac{2(r-p)}{2-p}}, \end{aligned} \quad (1.4)$$

where Z has a normal distribution with mean 0 and variance $\tau^2 = \sigma^2 \left(\sum_{i=-\infty}^{\infty} a_i \right)^2$.

2. Some lemmas

First, we give some lemmas which will be used in the proofs. Lemmas 3.1 and 3.2 are from Burton and Dehling [1], Yang [10] respectively.

LEMMA 2.1. Let $\sum_{i=-\infty}^{\infty} a_i$ be an absolutely convergent series of real numbers with $a = \sum_{i=-\infty}^{\infty} a_i$ and $k \geq 1$. Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} \left| \sum_{j=i+1}^{i+n} a_j \right|^k = |a|^k.$$

LEMMA 2.2. Let $\{\varepsilon_i : -\infty < i < \infty\}$ be a sequence of random variables with $E\varepsilon_i = 0$, $0 < E\varepsilon_i^2 < \infty$, and $\{X_i : i \geq 1\}$ is defined as above, where $\{\varepsilon_i : -\infty < i < \infty\}$ is a sequence of real numbers with $\sum_{i=-\infty}^{\infty} |a_i| < \infty$. Then the moving-average process $\{X_k\}$ fulfills the CLT, that is,

$$\frac{S_n}{\tau\sqrt{n}} \xrightarrow{\mathcal{D}} N(0, 1) \quad \text{where} \quad \tau = \sigma \sum_{i=-\infty}^{\infty} a_i.$$

LEMMA 2.3 (Rosenthal inequality). ([8]) Assume $\{Y_i : i \geq 1\}$ is a sequence of independent random variables, $EY_i = 0$, $E|Y_i|^p < \infty$, for some $p \geq 2$ and every $i \in \mathbb{R}$. Then there exists $C = C(p)$, such that

$$E \left| \sum_{i \in \mathbb{R}} Y_i \right|^p \leq C \left\{ \sum_{i \in \mathbb{R}} E|Y_i|^p + \left(\sum_{i \in \mathbb{R}} |EY_i|^2 \right)^{\frac{p}{2}} \right\}.$$

Throughout the sequel, N represents standard normal variable and Z has a normal distribution with mean 0 and variance $\tau^2 = \sigma^2 \left(\sum_{i=-\infty}^{\infty} a_i \right)^2$. C will denote a positive constant although its value may change from one appearance to the next and let $[x]$ indicate the maximum integer not larger than x . Without loss of generality, we assume $\tau = 1$ in the sequel.

3. Proof of Theorem 1.4

In this section, we set $a(\varepsilon) = [\varepsilon^{-2}]$ for $0 < \varepsilon < 1$. Note that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} ES_n^2 I\{|S_n| \geq \varepsilon n\} = \varepsilon^2 \sum_{n=1}^{\infty} P(|S_n| \geq \varepsilon n) + \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\varepsilon n}^{\infty} 2xP(|S_n| \geq x) dx.$$

When taking $p = 1$ and $r = 2$ in Li [7], we have

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \sum_{n=1}^{\infty} P(|S_n| \geq \varepsilon n) = 1.$$

Thus, in order to prove (1.2), it suffices to show

$$\lim_{\varepsilon \searrow 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\varepsilon n}^{\infty} 2xP(|S_n| \geq x) dx = 2. \quad (3.1)$$

This will be proved by the following propositions.

PROPOSITION 3.1. *One has*

$$\lim_{\varepsilon \searrow 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\varepsilon n}^{\infty} 2xP(|N| \geq x/\sqrt{n}) dx = 2.$$

Proof. See the proof of [9, Proposition 3.1]. □

PROPOSITION 3.2. *One has*

$$\lim_{\varepsilon \searrow 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{a(\varepsilon)} \frac{1}{n^2} \left| \int_{\varepsilon n}^{\infty} 2xP(|S_n| \geq x) dx - \int_{\varepsilon n}^{\infty} 2xP(|N| \geq x/\sqrt{n}) dx \right| = 0. \quad (3.2)$$

Proof. Denote

$$\Delta_n = \sup_x \left| P\left(\frac{|S_n|}{\sqrt{n}} \geq x\right) - P(|N| \geq x) \right|,$$

it follows from Lemma 2 that $\Delta_n \rightarrow 0$ as $n \rightarrow \infty$. Obviously,

$$\begin{aligned}
 & \left| \sum_{n=1}^{a(\varepsilon)} \frac{1}{n^2} \left| \int_{\varepsilon n}^{\infty} 2x P(|S_n| \geq x) \, dx - \int_{\varepsilon n}^{\infty} 2x P(|N| \geq x/\sqrt{n}) \, dx \right| \right| \\
 &= \left| \sum_{n=1}^{a(\varepsilon)} \left| \int_0^{\infty} 2(x+\varepsilon) P(|S_n| \geq n(x+\varepsilon)) \, dx - \int_0^{\infty} 2(x+\varepsilon) P(|N| \geq \sqrt{n}(x+\varepsilon)) \, dx \right| \right| \\
 &\leq \sum_{n=1}^{a(\varepsilon)} \frac{1}{n} \int_0^{\infty} 2n(x+\varepsilon) |P(|S_n| \geq n(x+\varepsilon)) - P(|N| \geq \sqrt{n}(x+\varepsilon))| \, dx \\
 &\leq \sum_{n=1}^{a(\varepsilon)} \frac{1}{n} (\Delta_{n1} + \Delta_{n2} + \Delta_{n3}),
 \end{aligned}$$

where

$$\begin{aligned}
 \Delta_{n1} &= \int_0^{\frac{1}{\sqrt{n}\Delta_n^{1/4}}} 2n(x+\varepsilon) |P(|S_n| \geq n(x+\varepsilon)) - P(|N| \geq \sqrt{n}(x+\varepsilon))| \, dx; \\
 \Delta_{n2} &= \int_0^{\infty} 2n(x+\varepsilon) P(|S_n| \geq n(x+\varepsilon)) \, dx; \\
 \Delta_{n3} &= \int_0^{\infty} 2n(x+\varepsilon) P(|N| \geq \sqrt{n}(x+\varepsilon)) \, dx.
 \end{aligned}$$

Since $n \leq b(\varepsilon)$ implies $\varepsilon\sqrt{n} \leq 1$, we have

$$\begin{aligned}
 \Delta_{n1} &\leq \int_0^{\frac{1}{\sqrt{n}\Delta_n^{1/4}}} 2n(x+\varepsilon) \Delta_n \, dx \leq n\Delta_n \left(\frac{1}{\sqrt{n}\Delta_n^{1/4}} + \varepsilon \right)^2 \\
 &\leq \left(\Delta_n^{1/4} + \Delta_n^{1/2} \right)^2 \rightarrow 0, \quad \text{as } n \rightarrow \infty.
 \end{aligned} \tag{3.3}$$

Next, observe that

$$\sum_{k=1}^n X_k = \sum_{i=-\infty}^{\infty} \sum_{k=1}^n a_{k+i\varepsilon_i}.$$

Set $a_{ni} = \sum_{k=1}^n a_{k+i}$. Then

$$\sum_{k=1}^n X_k = \sum_{i=-\infty}^{\infty} a_{ni} \varepsilon_i =: \sum_{i=-\infty}^{\infty} Y_i.$$

From Lemma 1, we can assume, without loss of generality, that

$$\sum_{i=-\infty}^{\infty} |a_{ni}|^k \leq Cn, \quad n \geq 1, \quad k \geq 1 \quad \text{and} \quad \sum_{i=-\infty}^{\infty} |a_i| \leq 1.$$

And then, by Lemma 3 (rosenthal's inequality), we get

$$E|S_n|^3 \leq C \left(\sum_{i=-\infty}^{\infty} E|a_{ni} \varepsilon_i|^3 + \left(\sum_{i=-\infty}^{\infty} E|a_{ni} \varepsilon_i|^2 \right)^{\frac{3}{2}} \right) \leq Cn^{\frac{3}{2}}. \quad (3.4)$$

Thus, via Markov's inequality, we have

$$\begin{aligned} \Delta_{n2} &\leq C \int_{\frac{1}{\sqrt{n}\Delta_n^{1/4}}}^{\infty} n(x+\varepsilon) \frac{n^{\frac{3}{2}}}{n^3(x+\varepsilon)^3} dx \\ &\leq C \int_{\frac{1}{\sqrt{n}\Delta_n^{1/4}}}^{\infty} \frac{1}{\sqrt{n}(x+\varepsilon)^2} dx \\ &\leq C\Delta_n^{\frac{1}{4}} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.5)$$

Now, we estimate Δ_{n3} . By Markov's inequality, we have

$$\Delta_{n3} \leq Cn \int_{\frac{1}{\sqrt{n}\Delta_n^{1/4}}}^{\infty} (x+\varepsilon) \frac{1}{n^2(x+\varepsilon)^4} dx \leq C\Delta_n^{\frac{1}{2}} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.6)$$

From (3.3), (3.5), (3.6), the proof of this proposition is derived. $\square$

PROPOSITION 3.3. *One has*

$$\lim_{\varepsilon \searrow 0} \frac{1}{-\log \varepsilon} \sum_{n=a(\varepsilon)+1}^{\infty} \frac{1}{n^2} \left| \int_{\varepsilon n}^{\infty} 2xP(|S_n| \geq x) dx - \int_{\varepsilon n}^{\infty} 2xP(|N| \geq x/\sqrt{n}) dx \right| = 0. \quad (3.7)$$

P r o o f.

$$\begin{aligned}
 & \lim_{\varepsilon \searrow 0} \frac{1}{-\log \varepsilon} \sum_{n=a(\varepsilon)+1}^{\infty} \frac{1}{n^2} \left| \int_{\varepsilon n}^{\infty} 2xP(|S_n| \geq x) \, dx - \int_{\varepsilon n}^{\infty} 2xP(|N| \geq x/\sqrt{n}) \, dx \right| \\
 & \leq \lim_{\varepsilon \searrow 0} \frac{1}{-\log \varepsilon} \sum_{n=a(\varepsilon)+1}^{\infty} \int_0^{\infty} 2(x+\varepsilon)P(|S_n| \geq n(x+\varepsilon)) \, dx \\
 & \quad + \frac{1}{-\log \varepsilon} \sum_{n=b(\varepsilon)+1}^{\infty} \int_0^{\infty} 2(x+\varepsilon)P(|N| \geq \sqrt{n}(x+\varepsilon)) \, dx \\
 & =: I_1 + I_2.
 \end{aligned}$$

For I_1 , by (3.2) and Markov's inequality, we have

$$\begin{aligned}
 I_1 & \leq \lim_{\varepsilon \searrow 0} \frac{1}{-\log \varepsilon} \sum_{n=a(\varepsilon)+1}^{\infty} \frac{1}{n^2} \int_{\varepsilon n}^{\infty} \frac{n^{\frac{3}{2}}}{x^2} \, dx \\
 & \leq \frac{C}{-\log \varepsilon} \sum_{n=a(\varepsilon)+1}^{\infty} \frac{1}{\varepsilon n^{\frac{3}{2}}} \\
 & \leq \frac{C}{-\log \varepsilon} \frac{1}{\sqrt{a(\varepsilon)+1}} \rightarrow 0, \quad \text{as } \varepsilon \searrow 0.
 \end{aligned}$$

For I_2 , in view of Markov's inequality, we have

$$\begin{aligned}
 I_2 & \leq \lim_{\varepsilon \searrow 0} \frac{C}{-\log \varepsilon} \sum_{n=a(\varepsilon)+1}^{\infty} \frac{1}{n^2} \int_{\varepsilon n}^{\infty} \frac{n^2}{x^3} \, dx \\
 & \leq \frac{C}{-\log \varepsilon} \sum_{n=a(\varepsilon)+1}^{\infty} \frac{1}{\varepsilon^2 n^2} \\
 & \leq \frac{C}{-\varepsilon^2 \log \varepsilon} \frac{1}{a(\varepsilon)+1} \rightarrow 0, \quad \text{as } \varepsilon \searrow 0.
 \end{aligned}$$

Thus, we complete the proof of the proposition. $\square$

Now, Theorem 1.4 follows from the propositions.

4. Proof of Theorem 1.5

In this section, we set $b(\varepsilon) = [\varepsilon^{\frac{-2p}{2-p}}]$ for $0 < \varepsilon < 1$, and $1 \leq p < 2$. When taking $q = 0$, we have

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n \geq 1} n^{\frac{r}{p}-2} P(|S_n| \geq \varepsilon n^{\frac{1}{p}}) = \frac{p}{r-p} E|Z|^{\frac{2(r-p)}{2-p}}.$$

This is the result of Li [7]. Thus we only discuss the case $0 < p < 2$. Note that

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{q}{p}} E|S_n|^q I\left\{|S_n| \geq \varepsilon n^{\frac{1}{p}}\right\} \\ &= \varepsilon^q \sum_{n=1}^{\infty} n^{\frac{r}{p}-2} P\left(|S_n| \geq \varepsilon n^{\frac{1}{p}}\right) + \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{q}{p}} \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} qx^{q-1} P(|S_n| \geq x) dx. \end{aligned}$$

Via the result of Li [7]. Thus, in order to prove (1.4), it suffices to show

$$\begin{aligned} & \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{q}{p}} \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} qx^{q-1} P(|S_n| \geq x) dx \\ &= \frac{pq(2-p)}{(r-p)(2r-2p-2q+pq)} E|N|^{\frac{2(r-p)}{2-p}}. \end{aligned} \tag{4.1}$$

This will be proved by the following propositions.

PROPOSITION 4.1. *One has*

$$\begin{aligned} & \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{q}{p}} \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} qx^{q-1} P\left(|N| \geq \frac{x}{\sqrt{n}}\right) dx \\ &= \frac{2p}{2r-2p-2q+pq} E|Z|^{\frac{2(r-p)}{2-p}} E|N|^{\frac{2(r-p)}{2-p}}. \end{aligned}$$

Proof. Via the change of variation, we have

$$\begin{aligned} & \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{q}{p}} \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} qx^{q-1} P\left(|N| \geq \frac{x}{\sqrt{n}}\right) dx \\ &= \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{q}{p}+\frac{q}{2}} \int_{\varepsilon n^{\frac{1}{p}-\frac{1}{2}}}^{\infty} qx^{q-1} P(|N| \geq t) dt \end{aligned}$$

$$\begin{aligned}
 &= \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} \int_1^\infty x^{\frac{r}{p}-2-\frac{q}{p}+\frac{q}{2}} dx \int_{\varepsilon x^{\frac{1}{p}-\frac{1}{2}}}^\infty qx^{q-1} P(|N| \geq t) dt \\
 &= \frac{2pq}{2-p} \int_0^\infty y^{\frac{2(r-p)}{2-p}-q-1} dy \int_y^\infty t^{q-1} P(|N| \geq t) dt \\
 &= \frac{2p}{2-p} \frac{q}{\frac{2(r-p)}{2-p}-q} \int_0^\infty t^{\frac{2(r-p)}{2-p}-1} P(|N| \geq t) dt \\
 &= \frac{2p}{2r-2p-2q+pq} E|Z|^{\frac{2(r-p)}{2-p}} E|N|^{\frac{2(r-p)}{2-p}}.
 \end{aligned}$$

This completes the proof of Proposition 4.1. $\square$

PROPOSITION 4.2. *For $M > 1$, one has*

$$\begin{aligned}
 \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n=1}^{Mb(\varepsilon)} n^{\frac{r}{p}-2-\frac{q}{p}} \left| \int_{\varepsilon n^{\frac{1}{p}}}^\infty qx^{q-1} P(|S_n| \geq x) dx \right. \\
 \left. - \int_{\varepsilon n^{\frac{1}{p}}}^\infty qx^{q-1} P(|N| \geq x/\sqrt{n}) dx \right| = 0.
 \end{aligned}$$

Proof. It is easy to see that

$$\begin{aligned}
 &\sum_{n=1}^{Mb(\varepsilon)} n^{\frac{r}{p}-2-\frac{q}{p}} \left| \int_{\varepsilon n^{\frac{1}{p}}}^\infty qx^{q-1} P(|S_n| \geq x) dx - \int_{\varepsilon n^{\frac{1}{p}}}^\infty qx^{q-1} p(|N| \geq x/\sqrt{n}) dx \right| \\
 &\leq \sum_{n=1}^{Mb(\varepsilon)} n^{\frac{r}{p}-2} \int_0^\infty q(x+\varepsilon)^{q-1} \left| p(|S_n| \geq (x+\varepsilon)n^{\frac{1}{p}}) \right. \\
 &\quad \left. - p(|N| \geq (x+\varepsilon)n^{\frac{1}{p}-\frac{1}{2}}) \right| dx \\
 &\leq \sum_{n=1}^{Mb(\varepsilon)} n^{\frac{r}{p}-2-\frac{q(2-p)}{2p}} (\Delta'_{n1} + \Delta'_{n2}),
 \end{aligned}$$

where

$$\begin{aligned}\Delta'_{n1} &= n^{\frac{q(2-p)}{2p}} \int_0^{\frac{1}{n^{\frac{2-p}{2p}} \Delta_n^{1/2q}}} q(x+\varepsilon)^{q-1} \left| p \left(|S_n| \geq (x+\varepsilon)n^{\frac{1}{p}} \right) \right. \\ &\quad \left. - p \left(|N| \geq (x+\varepsilon)n^{\frac{1}{p}-\frac{1}{2}} \right) \right| dx, \\ \Delta'_{n2} &= n^{\frac{q(2-p)}{2p}} \int_{\frac{1}{n^{\frac{2-p}{2p}} \Delta_n^{1/2q}}}^{\infty} q(x+\varepsilon)^{q-1} \left| P \left(|S_n| \geq (x+\varepsilon)n^{\frac{1}{p}} \right) \right. \\ &\quad \left. - p \left(|N| \geq (x+\varepsilon)n^{\frac{1}{p}-\frac{1}{2}} \right) \right| dx.\end{aligned}$$

Since $n \leq Mb(\varepsilon)$ implies $\varepsilon n^{\frac{2-p}{2p}} \leq M^{\frac{2-p}{2p}}$, one can easily obtain that

$$\Delta'_{n1} \leq \Delta_n n^{\frac{q(2-p)}{2p}} \left(\frac{1}{n^{\frac{2-p}{2p}} \Delta_n^{1/2q}} + \varepsilon \right)^q \leq \left(\Delta_n^{\frac{1}{2q}} + M^{\frac{2-p}{2p}} \Delta_n^{\frac{1}{q}} \right)^p. \quad (4.2)$$

By Markov's inequality, we have

$$\Delta'_{n2} \leq C n^{\frac{q(2-p)}{2p}} \int_{\frac{1}{n^{\frac{2-p}{2p}} \Delta_n^{1/2q}}}^{\infty} \frac{1}{(x+\varepsilon)^{3-q} n^{\frac{2}{p}}} dx \leq C \Delta_n^{\frac{1}{q}-\frac{1}{2}}. \quad (4.3)$$

Denote $\Delta'_n = \Delta'_{n1} + \Delta'_{n2}$, it follows that

$$m^{-\frac{r}{p}+1+\frac{q(2-p)}{2p}} \sum_{n=1}^m n^{\frac{r}{p}-2-\frac{q(2-p)}{2p}} \Delta'_n \rightarrow 0, \quad \text{as } m \rightarrow \infty.$$

We have

$$\begin{aligned}\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n=1}^{Mb(\varepsilon)} n^{\frac{r}{p}-2-\frac{q}{p}} &\left| \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} q x^{q-1} P(|S_n| \geq x) dx \right. \\ &\left. - \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} q x^{q-1} P(|N| \geq x/\sqrt{n}) dx \right| \end{aligned}$$

$$\begin{aligned}
 &= \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n=1}^{Mb(\varepsilon)} n^{\frac{r}{p}-2-\frac{q(2-p)}{2p}} \Delta'_n \\
 &\leq \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} [Mb(\varepsilon)]^{\frac{r}{p}-1-\frac{q(2-p)}{2p}} [Mb(\varepsilon)]^{-\frac{r}{p}+1+\frac{q(2-p)}{2p}} \sum_{n=1}^{Mb(\varepsilon)} n^{\frac{r}{p}-2-\frac{q(2-p)}{2p}} \Delta'_n = 0.
 \end{aligned}$$

Then this proposition is proved. $\square$

PROPOSITION 4.3. *For $M > 1$, one has*

$$\begin{aligned}
 \lim_{M \rightarrow \infty} \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n > Mb(\varepsilon)} n^{\frac{r}{p}-2-\frac{q}{p}} \left| \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} qx^{q-1} P(|S_n| \geq x) dx \right. \\
 \left. - \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} qx^{q-1} P(|N| \geq x/\sqrt{n}) dx \right| = 0.
 \end{aligned}$$

Proof. Observe that

$$E|S_n|^2 \leq CE \left(\sum_{i=-\infty}^{\infty} a_{ni}^2 \varepsilon_i^2 \right) \leq Cn.$$

Thus, by Markov's inequality, we have

$$\begin{aligned}
 &\varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n > Mb(\varepsilon)} n^{\frac{r}{p}-2-\frac{q}{p}} \left| \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} qx^{q-1} P(|S_n| \geq x) dx \right. \\
 &\quad \left. - \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} qx^{q-1} P(|N| \geq x/\sqrt{n}) dx \right| \\
 &\leq C\varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n > Mb(\varepsilon)} n^{\frac{r}{p}-2-\frac{q}{p}} \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} \frac{n}{x^{3-q}} dx \\
 &\leq C\varepsilon^{\frac{2(r-p)}{2-p}-2} \sum_{n > Mb(\varepsilon)} n^{\frac{r}{p}-\frac{2}{p}-1} \\
 &\leq CM^{\frac{r-2}{p}} \rightarrow 0, \quad \text{as } M \rightarrow \infty.
 \end{aligned}$$

We complete the proof of this proposition. $\square$

Our main result now follows from the propositions via triangle inequality.

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School of Mathematical science

Huaiyin normal university

Huaian 223300

CHINA

E-mail: zqphunhu@yahoo.com.cn