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EXISTENCE AND UNIQUENESS OF MILD SOLUTION FOR FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS OF NEUTRAL TYPE WITH NONLOCAL CONDITIONS

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ABSTRACT. In this paper, we prove the existence and uniqueness of mild solution of a class of nonlinear fractional integrodifferential equations of neutral type with nonlocal conditions in a Banach space. New results are obtained by fixed point theorem.

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1. Introduction

Many problems in electromagnetics, viscoelasticity, acoustics, hydrology and other areas of application can be modeled by fractional differential equations. In recent years, fractional differential equations have been investigated by many researchers(see, e.g., [1, 2, 3, 4, 6, 7, 8, 9, 10, 16, 17, 18, 20] and references therein). Moreover, the fractional neutral functional differential equations have been studied extensively (see, e.g., [2, 17] and references therein).

An integrodifferential equation arises in modeling processes in many fields like engineering, physics, biology, finance, etc. For this reason the study of this type of equation in Banach space has received a significant amount of attention in the last few years, we refer to [4, 12, 21] and references therein.

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In this paper, we will study the existence and uniqueness of mild solution for the following fractional integrodifferential equations of neutral type with nonlocal conditions, which are more general than those in many previous publications.

$$\begin{aligned} {}^cD^q\left(u\left(t\right)-G\left(t,\psi\left(t\right)\right)\right) &= A\left(u\left(t\right)-G\left(t,\psi\left(t\right)\right)\right)+F\left(t,\,\phi\left(t\right)\right) \\ &\quad + \int_0^t k\left(t,s,u\left(s\right),\int_0^s \rho\left(s,\tau,u\left(\tau\right)\right)\,\mathrm{d}\tau\right)\mathrm{d}s, \\ t &\in [0,\,T] \\ u(0)+g(u) &= u_0, \end{aligned} \tag{1.1}$$

where $0 < q < 1$, $T > 0$ and

$$\begin{aligned} (t,\,\psi(t)) &= (t,\,u(t),\,u(\nu_1(t)),\ldots,u(\nu_m(t))), \\ (t,\,\phi(t)) &= (t,\,u(t),\,u(\sigma_1(t)),\ldots,u(\sigma_n(t))). \end{aligned}$$

The fractional derivative ${}^cD^q$ is understood here in the Caputo sense. A generates an analytic compact semigroup of uniformly bounded linear operators $S(\cdot)$ on a Banach space X, $G\colon [0,\,T]\times X_\alpha^{m+1}\to X_\alpha$, $F\colon [0,\,T]\times X_\alpha^{n+1}\to X$, $g\colon X_\alpha\to X_\alpha$ are given functions to be specified later, $k\colon \Delta\times X_\alpha\times X_\alpha\to X$, $\rho\colon \Delta\times X_\alpha\to X_\alpha$ are continuous functions, where $X_\alpha = D(A^\alpha)$, for some $\alpha\in(0,\,1)$, the domain of the fractional power of A, $\Delta = \big\{(t,\,s):\ 0\le s\le t\le T\big\}$. The $\nu_i(t)$ $(i=1,2,\ldots,m)$ and $\sigma_j(t)$ $(j=1,2,\ldots,n)$ are continuous scalar valued functions defined on $[0,\,T]$ such that $\nu_i(t)\le t$ $(i=1,2,\ldots,m)$ and $\sigma_j(t)\le t$ $(j=1,2,\ldots,n)$.

As in [11, 12, 13, 14, 22] and the related references given there, we pay attention to the nonlocal condition because in many cases a nonlocal condition $u(0)+g(u)=u_0$ is more realistic than the classical condition $u(0)=u_0$ in treating physical problems.

This paper is organized as follows. In Section 2, we recall some basic definitions and preliminary results. In Section 3 and Section 4, we give the existence and uniqueness theorems of mild solution of Eq. (1.1) and their proofs. In the last section, as an application controllability problem is studied.

2. Preliminaries

In this paper, we set $J=[0,\,T]$, a compact interval in $\mathbf{R}$. We denote by X a Banach space with norm $\|\cdot\|$, $C(J,\,X_\alpha)$ the Banach space of all continuous functions $J\to X_\alpha$ endowed with the supnorm given by

$$\|u\|_\infty := \sup_{t\in J}\|u\|_\alpha, \qquad \text{for} \quad u\in C(J,\,X_\alpha).$$

Moreover, we abbreviate $\|u\|_{L^1(J,\,\mathbf{R}^+)}$ with $\|u\|_{L^1}$, for any $u \in L^1(J, \mathbf{R}^+)$.

Let $A\colon D(A) \to X$ be the infinitesimal generator of an analytic compact semigroup of uniformly bounded linear operators $S(\cdot)$, i.e., there exists $M \geq 1$ such that $\|S(t)\| \leq M$ for $t \geq 0$, and without loss of generality, we assume $0 \in \rho(A)$. So we can define the fractional power A^α for $0 < \alpha < 1$, as a closed linear operator on its domain $D(A^\alpha)$ with inverse $A^{-\alpha}$, one has the following known result.

Lemma 2.1. ([19, 13])

(1) *$X_\alpha = D(A^\alpha)$ is a Banach space with the norm $\|x\|_\alpha := \|A^\alpha x\|$ for $x \in D(A^\alpha)$.*

(2) *$S(t)\colon X \to X_\alpha$ for each $t > 0$ and $\alpha > 0$.*

(3) *For every $x \in D(A^\alpha)$ and $t \geq 0$, $S(t)A^\alpha x = A^\alpha S(t)x$.*

(4) *For every $t > 0$, $A^\alpha S(t)$ is bounded on X and there exists $M_\alpha > 0$ such that*

$$\|A^\alpha S(t)\| \leq M_\alpha t^{-\alpha}. \tag{2.1}$$

(5) *The restriction $S_\alpha(t)$ of $S(t)$ to X_α is exactly the part of $S(t)$ in X_α. For $x \in X_\alpha$,*

$$\|S(t)x\|_\alpha = \|A^\alpha S(t)x\| = \|S(t)A^\alpha x\| \leq \|S(t)\|\|x\|_\alpha.$$

Lemma 2.2. ([13]) *$S_\alpha(\cdot)$ is an immediately compact semigroup in X_α, and hence it is immediately norm-continuous.*

Definition 2.3. ([20]) The fractional integral of order β with the lower limit zero for a function $f \in AC[0, \infty)$ is defined as

$$I^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\,\mathrm{d}s, \qquad t > 0, \quad 0 < \beta < 1,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.4. ([20]) Riemann-Liouville derivative of order β with the lower limit zero for a function $f \in AC[0, \infty)$ can be written as

$${}^L D^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{-\beta} f(s)\,\mathrm{d}s, \qquad t > 0, \quad 0 < \beta < 1.$$

Definition 2.5. ([20]) The Caputo derivative of order β for a function $f \in AC[0, \infty)$ can be written as

$${}^c D^\beta f(t) = {}^L D^\beta (f(t) - f(0)), \qquad t > 0, \quad 0 < \beta < 1.$$

Remark 2.6. (1) If $f(t) \in C^1[0, \infty)$, then

$$ {}^cD^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} f'(s)\,\mathrm{d}s = I^{1-\beta} f'(t), \qquad t > 0, \quad 0 < \beta < 1. $$

(2) The Caputo derivative of a constant is equal to zero.

Based on the work in [3, 4, 15], we set

$$ \begin{aligned} Q(t) &= \int_0^\infty \xi_q(\theta) S(t^q\theta)\,\mathrm{d}\theta, \\ R(t) &= q \int_0^\infty \theta t^{q-1} \xi_q(\theta) S(t^q\theta)\,\mathrm{d}\theta \end{aligned} $$

and ξ_q is a probability density function defined on $(0, \infty)$ such that

$$ \xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}} \varpi_q\left(\theta^{-\frac{1}{q}}\right) \geq 0, $$

where

$$ \varpi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \qquad \theta \in (0, \infty). $$

Remark 2.7. It is not difficult to verify that for $v \in [0, 1]$,

$$ \int_0^\infty \theta^v \xi_q(\theta)\,\mathrm{d}\theta = \int_0^\infty \theta^{-qv} \varpi_q(\theta)\,\mathrm{d}\theta = \frac{\Gamma(1+v)}{\Gamma(1+qv)}. $$

Then, we can see

$$ \begin{aligned} \|R(t)\| &\leq \frac{qM}{\Gamma(1+q)} t^{q-1}, \qquad t > 0, \\ \|R(t)\|_\alpha &\leq \frac{M_\alpha \Gamma(1-\alpha)}{\Gamma\left(q(1-\alpha)\right)} t^{q(1-\alpha)-1} := L_0 t^{q(1-\alpha)-1}, \qquad t > 0. \end{aligned} $$

We define the mild solution for Eq. (1.1) as follows:

Definition 2.8. A continuous function $u\colon J \to X$ satisfying the equation

$$ \begin{aligned} u(t) &= Q(t)(u_0 - g(u) - G(0, \psi(0))) + G(t, \psi(t)) \\ &\quad + \int_0^t R(t-s)\left[F(s, \phi(s)) + H(u)(s)\right]\mathrm{d}s \end{aligned} $$

for $t \in J$ is called a mild solution of equation (1.1), where

$$H(u)(t) \;=\; \int_0^t k\left(t,\, s,\, u(s),\, \int_0^s \rho(s,\, \tau,\, u(\tau))\, \mathrm{d}\tau\right) \mathrm{d}s.$$

The following theorem ([5]) will be used later.

Theorem 2.9 (**Nonlinear alternative for single-valued maps**). *Let E be a Banach space, V be a closed convex subset of E, U be an open subset of V and $0 \in U$. Suppose that $\mathfrak{F}\colon \overline{U} \to V$ is a continuous, compact map. Then either*

(i) *$\mathfrak{F}$ has a fixed point in $\overline{U}$, or*

(ii) *there exist $u \in \partial U$ (the boundary of U in V) and $\lambda \in (0,1)$ with $u = \lambda \mathfrak{F}(u)$.*

3. Existence of solution

We shall require the following assumptions:

(H1) For each $t \in J$, the function $F(t, \cdot, \cdot, \dots, \cdot)\colon X_\alpha^{n+1} \to X$ is continuous, for each $(x_0, x_1, \dots, x_n) \in X_\alpha^{n+1}$ the function $F(\cdot, x_0, x_1, \dots, x_n)\colon J \to X$ is strongly measurable, and there exist $\mu_i \in C(J, \mathbf{R}^+)$ and $\Omega_F\colon [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that

$$\|F(t,\, u(t),\, u(\sigma_1(t)), \dots, u(\sigma_n(t)))\| \le \sum_{i=1}^{n} \mu_i(t)\Omega_F(\|u\|_\alpha),$$
$$u \in C(J,\, X_\alpha).$$

(H2) The function $G\colon J \times X_\alpha^{m+1} \to X_\alpha$ is completely continuous and there exist $\eta_i \in C(J, \mathbf{R}^+)$ and $\Upsilon_G\colon [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that

$$\|G(t, u(t), u(\nu_1(t)), \dots, u(\nu_m(t)))\|_\alpha \le \sum_{i=1}^{m} \eta_i(t)\Upsilon_G(\|u\|_\alpha),$$
$$u \in C(J,\, X_\alpha).$$

(H3) There exist $\nu(t) \in L^1(J, \mathbf{R}^+)$ and $W_\rho\colon [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that

$$\left\|k\left(t, s, u, \int_0^s \rho(s,\, \tau,\, u)\, \mathrm{d}\tau\right)\right\| \le \nu(t)W_\rho(\|u\|_\alpha), \qquad u \in C(J,\, X_\alpha).$$

(H4) $g \in C(C(J,\, X_\alpha),\, X_\alpha)$ is completely continuous and there exists $b > 0$ such that

$$\|g(u)\|_\alpha \leq b\,(1 + \|u\|_\infty).$$

(H5) There exists $\overline{M} > 0$ such that

$$\frac{\overline{M}}{M\left(\|u_0\|_\alpha + b\,(1+\overline{M})\right) + C_4\Upsilon_G(\overline{M}) + \left(C_1\Omega_F(\overline{M}) + C_3W_\rho(\overline{M})\right)\frac{L_0T^{q(1-\alpha)}}{q(1-\alpha)}} > 1,$$

where

$$C_1 := \sum_{i=1}^{n} \sup_{t\in J} \mu_i(t),$$
$$C_2 := \sum_{i=1}^{m} \sup_{t\in J} \eta_i(t),$$
$$C_3 := \|\nu\|_{L^1}, and C_4 = C_2(M+1).$$

We define $\Psi\colon C(J,\, X_\alpha) \to C(J,\, X_\alpha)$ by

$$\begin{aligned}(\Psi u)(t) \quad := \quad & Q(t)(u_0 - g(u) - G(0,\psi(0))) + G(t,\, \psi(t)) \\ & + \int_0^t R(t-s)\,[F(s,\, \phi(s)) + H(u)(s)]\;\mathrm{d}s.\end{aligned}$$

We can see that if Ψ has a fixed point in $C(J,\, X_\alpha)$, then this fixed point is a mild solution of problem (1.1). Now we show that Ψ has a fixed point. The proof will be given in several steps.

Proposition 3.1. *The operator Ψ maps bounded sets of $C(J,\, X_\alpha)$ into bounded sets of $C(J,\, X_\alpha)$.*

Proof. Let $B_r = \{u \in C(J,\, X_\alpha) : \ \|u\|_\infty \leq r\}$. For any $u \in B_r$, and noting that

$$\|H(u)(t)\| \leq \|\nu\|_{L^1}W_\rho(\|u\|_\alpha),$$

we have

$$\begin{aligned}\|(\Psi u)(t)\|_\alpha \leq{} & \|Q(t)(u_0 - g(u) - G(0,\psi(0)))\|_\alpha + \|G(t,\, \psi(t))\|_\alpha \\ & + \int_0^t \|R(t-s)[F(s,\, \phi(s)) + H(u)(s)]\|_\alpha\,\mathrm{d}s \\ \leq{} & M\,(\|u_0\|_\alpha + b(1+\|u\|_\infty) + C_2\Upsilon_G(r)) + C_2\Upsilon_G(r) \\ & + L_0\int_0^t (t-s)^{q(1-\alpha)-1}\|F(s,\, \phi(s)) + H(u)(s)\|\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\le M' + L_0\left(C_1\Omega_F(r) + \|\nu\|_{L^1} W_\rho(r)\right) \int_0^t (t-s)^{q(1-\alpha)-1}\,\mathrm{d}s \\
&\le M' + \frac{L_0 T^{q(1-\alpha)}}{q(1-\alpha)}\left(C_1\Omega_F(r) + C_3 W_\rho(r)\right) := r_0,
\end{aligned}
$$

where $M' = M\left(\|u_0\|_\alpha + b(1+r) + C_2\Upsilon_G(r)\right) + C_2\Upsilon_G(r)$. Hence $\|(\Psi u)(t)\|_\alpha \le r_0$. This means that $\Psi B_r \subset B_{r_0}$. □

Proposition 3.2. *Ψ maps bounded sets of $C(J, X_\alpha)$ into equicontinuous sets of $C(J, X_\alpha)$.*

Proof. Let $t_2 > 0$, $t_2 < t_1$ and $\varepsilon > 0$ be small enough, then for $u \in B_r$,

$$
\begin{aligned}
&\|(\Psi u)(t_1) - (\Psi u)(t_2)\|_\alpha \\
&\le \|(Q(t_1) - Q(t_2))(u_0 - g(u) - G(0, \psi(0)))\|_\alpha \\
&\quad + \|G(t_1, \psi(t_1)) - G(t_2, \psi(t_2))\|_\alpha \\
&\quad + \int_0^{t_2} \|(R(t_1 - s) - R(t_2 - s))[F(s, \phi(s)) + H(u)(s)]\|_\alpha\,\mathrm{d}s \\
&\quad + \int_{t_2}^{t_1} \|R(t_1 - s)[F(s, \phi(s)) + H(u)(s)]\|_\alpha\,\mathrm{d}s \\
&= \mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3 + \mathfrak{I}_4.
\end{aligned}
$$

Noting that the continuity of $S(t)$ in the uniform operator topology for $t > 0$ by the compactness of $S(t)$, we can see

$$
\begin{aligned}
\mathfrak{I}_1 &= \|(Q(t_1) - Q(t_2))(u_0 - g(u) - G(0, \psi(0)))\|_\alpha \\
&\le \left(\|u_0\|_\alpha + b + br + C_2\Upsilon_G(r)\right) \int_0^\infty \xi_q(\theta)\|S(t_1^q\theta) - S(t_2^q\theta)\|\,\mathrm{d}\theta \\
&\longrightarrow 0, \quad \text{as} \quad t_2 \to t_1.
\end{aligned}
$$

The hypothesis of G ensures that $\mathfrak{I}_2$ tends to 0, as $t_2 \to t_1$.

For $\mathfrak{I}_3$, from (H1), (H2) and (2.1) we have

$$
\begin{aligned}
\mathfrak{I}_3 = q \int_0^{t_2} \Bigg\| [F(s, \phi(s)) + H(u)(s)] \int_0^\infty \theta\xi_q(\theta)[(t_1 - s)^{q-1} S\left((t_1 - s)^q\theta\right) \\
- (t_2 - s)^{q-1} S((t_2 - s)^q\theta)]\,\mathrm{d}\theta \Bigg\|_\alpha \mathrm{d}s
\end{aligned}
$$

$$\begin{aligned}
&\le q(C_1\Omega_F(r)+C_3W_\rho(r))\int_0^{t_2}\int_0^{\infty}\theta\xi_q(\theta)\big[|(t_1-s)^{q-1}-(t_2-s)^{q-1}|\\
&\quad\cdot\|S((t_1-s)^q\theta)\|_\alpha+(t_2-s)^{q-1}\|S((t_1-s)^q\theta)-S((t_2-s)^q\theta)\|_\alpha\big]\,\mathrm{d}\theta\,\mathrm{d}s\\
&\le (C_1\Omega_F(r)+C_3W_\rho(r))\Big[L_0\int_0^{t_2}\frac{|(t_1-s)^{q-1}-(t_2-s)^{q-1}|}{(t_1-s)^{q\alpha}}\,\mathrm{d}s\\
&\quad+q\int_0^{t_2-\varepsilon}(t_2-s)^{q-1}\int_0^{\infty}\theta\xi_q(\theta)\|S((t_1-s)^q\theta-\varepsilon^q\theta)-S((t_2-s)^q\theta-\varepsilon^q\theta)\|\\
&\quad\cdot\|A^\alpha S\left(\varepsilon^q\theta\right)\|\,\mathrm{d}\theta\,\mathrm{d}s+L_0\int_{t_2-\varepsilon}^{t_2}\left(\frac{(t_2-s)^{q-1}}{(t_1-s)^{q\alpha}}+\frac{(t_2-s)^{q-1}}{(t_2-s)^{q\alpha}}\right)\mathrm{d}s\Big]\\
&\le (C_1\Omega_F(r)+C_3W_\rho(r))\Big[L_0\int_0^{t_2}\frac{|(t_1-s)^{q-1}-(t_2-s)^{q-1}|}{(t_1-s)^{q\alpha}}\,\mathrm{d}s\\
&\quad+qM_\alpha\int_0^{t_2-\varepsilon}\frac{(t_2-s)^{q-1}}{\varepsilon^{q\alpha}}\int_0^{\infty}\theta^{1-\alpha}\xi_q(\theta)\|S((t_1-s)^q\theta-\varepsilon^q\theta)\\
&\quad-S((t_2-s)^q\theta-\varepsilon^q\theta)\|\,\mathrm{d}\theta\,\mathrm{d}s+L_0\int_{t_2-\varepsilon}^{t_2}\left(\frac{(t_2-s)^{q-1}}{(t_1-s)^{q\alpha}}+\frac{(t_2-s)^{q-1}}{(t_2-s)^{q\alpha}}\right)\mathrm{d}s\Big]\\
&=(C_1\Omega_F(r)+C_3W_\rho(r))(\mathfrak{I}_3'+\mathfrak{I}_3''+\mathfrak{I}_3''').
\end{aligned}$$

Obviously, $\mathfrak{I}_3'\to 0$, as $t_2\to t_1$ and the continuity of the function $t\to\|S(t)\|$ for $t\in(0,T)$ leads to $\lim\limits_{t_2\to t_1}\mathfrak{I}_3''=0$. As $t_2\to t_1$ and $\varepsilon\to 0$, $\mathfrak{I}_3'''\to 0$. For $\mathfrak{I}_4$, we have

$$\begin{aligned}
\mathfrak{I}_4&=\int_{t_2}^{t_1}\|R(t_1-s)[F(s,\,\phi(s))+H(u)(s)]\|_\alpha\,\mathrm{d}s\\
&\le q\int_{t_2}^{t_1}\|F(s,\,\phi(s))+H(u)(s)\|\int_0^{\infty}\theta\xi_q(\theta)(t_1-s)^{q-1}\|A^\alpha S((t_1-s)^q\theta)\|\,\mathrm{d}\theta\,\mathrm{d}s\\
&\le L_0(C_1\Omega_F(r)+C_3W_\rho(r))\int_{t_2}^{t_1}(t_1-s)^{q(1-\alpha)-1}\,\mathrm{d}s\to 0,\qquad\text{as}\quad t_2\to t_1.
\end{aligned}$$

Similarly, using the compactness of the set $g(B_r)$ we can prove that the functions Ψu, $u \in B_r$ are equicontinuous at $t = 0$.

Hence, Ψ maps bounded sets of $C(J, X_\alpha)$ into equicontinuous sets of $C(J, X_\alpha)$. □

Proposition 3.3. *The set* $\{(\Psi u)(t) :\ u \in B_r\}$ *is relatively compact in* X_α.

Proof. Obviously, $\{(\Psi u)(0) :\ u \in B_r\}$ is relatively compact in X_α.

Fix $t \in (0, T]$. For $0 < \eta < t$ and $u \in B_r$, set

$$\begin{aligned}(\Psi_\eta u)(t) &= Q(t)(u_0 - g(u) - G(0, \psi(0))) + G(t,\, \psi(t))\\ &\quad + \int_0^{t-\eta} R(t-s)\,[F(s,\, \phi(s)) + H(u)(s)]\, \mathrm{d}s.\end{aligned}$$

Since $S_\alpha(t)$ is compact for each $t \in (0, T]$ on X_α and from (H2), the sets $\{(\Psi_\eta u)(t) :\ u \in B_r\}$ are relatively compact in X_α. Furthermore,

$$\begin{aligned}&\|(\Psi u)(t) - (\Psi_\eta u)(t)\|_\alpha\\ \le\ & \int_{t-\eta}^{t} \|R(t-s)\,[F(s,\, \phi(s)) + H(u)(s)]\,\|_\alpha\, \mathrm{d}s\\ \le\ & q(C_1\Omega_F(r) + C_3 W_\rho(r)) \int_{t-\eta}^{t} (t-s)^{q-1} \int_0^\infty \theta \xi_q(\theta) \|A^\alpha S((t-s)^q\theta)\|\, \mathrm{d}\theta\, \mathrm{d}s\\ \le\ & L_0(C_1\Omega_F(r) + C_3 W_\rho(r)) \int_{t-\eta}^{t} (t-s)^{q(1-\alpha)-1}\, \mathrm{d}s\\ =\ & \frac{L_0(C_1\Omega_F(r) + C_3 W_\rho(r))}{q(1-\alpha)} \cdot \eta^{q(1-\alpha)},\end{aligned}$$

which implies that the set $\{(\Psi u)(t) :\ u \in B_r\}$ is relatively compact in X_α. □

Proposition 3.4. *The operator* Ψ *is a continuous mapping.*

Proof. Let $\{u_k\} \subset C(J,\, X_\alpha)$ with $u_k \to u$ in $C(J,\, X_\alpha)$. Then there is an integer p such that $\|u_k(t)\|_\alpha \le p$ for all k and $t \in J$, so $u_k \in B_p$ and $u \in B_p$. By (H1), as $k \to \infty$,

$$F(s,\, u_k(s),\, u_k(\sigma_1(s)),\, \ldots, u_k(\sigma_n(s))) \to F(s,\, u(s),\, u(\sigma_1(s)),\, \ldots, u(\sigma_n(s))),$$

and

$$\begin{aligned}&\|F(s,\, u_k(s),\, u_k(\sigma_1(s)),\, \ldots, u_k(\sigma_n(s))) - F(s,\, u(s),\, u(\sigma_1(s)),\, \ldots, u(\sigma_n(s)))\|\\ &\le 2C_1\Omega_F(p).\end{aligned}$$

By (H2), as $k \to \infty$,

$$G(t, u_k(t), u_k(\nu_1(t)), \ldots, u_k(\nu_m(t))) \to G(t, u(t), u(\nu_1(t)), \ldots, u(\nu_m(t))).$$

It follows from the hypotheses of k and ρ that

$$H(u_k)(s) \to H(u)(s), \quad k \to \infty.$$

Moreover, noting that g is completely continuous on $C(J, X_\alpha)$, we can obtain that $g(u_k) \to g(u)$ as $k \to \infty$.

Now, by means of the Lebesgue Dominated Convergence Theorem one proves that

$$\begin{aligned}
&\|(\Psi u_k)(t) - (\Psi u)(t)\|_\alpha \\
\le\; &\|Q(t)(g(u_k) - g(u))\|_\alpha \\
&+ \|Q(t)(G(0, u_k(0), u_k(\nu_1(0)), \ldots, u_k(\nu_m(0))) \\
&- G(0, u(0), u(\nu_1(0)), \ldots, u(\nu_m(0))))\|_\alpha \\
&+ \|G(t, u_k(t), u_k(\nu_1(t)), \ldots, u_k(\nu_m(t))) \\
&- G(t, u(t), u(\nu_1(t)), \ldots, u(\nu_m(t)))\|_\alpha \\
&+ \int_0^t \|R(t-s)[F(s, u_k(s), u_k(\sigma_1(s)), \ldots, u_k(\sigma_n(s))) \\
&- F(s, u(s), u(\sigma_1(s)), \ldots, u(\sigma_n(s))) + H(u_k)(s) - H(u)(s)]\|_\alpha \, \mathrm{d}s \\
\le\; &M(\|g(u_k) - g(u)\|_\alpha \\
&+ \|G(0, u_k(0), u_k(\nu_1(0)), \ldots, u_k(\nu_m(0))) \\
&- G(0, u(0), u(\nu_1(0)), \ldots, u(\nu_m(0)))\|_\alpha) \\
&+ \|G(t, u_k(t), u_k(\nu_1(t)), \ldots, u_k(\nu_m(t))) \\
&- G(t, u(t), u(\nu_1(t)), \ldots, u(\nu_m(t)))\|_\alpha \\
&+ L_0 \int_0^t (t-s)^{q(1-\alpha)-1} \|F(s, u_k(s), u_k(\sigma_1(s)), \ldots, u_k(\sigma_n(s))) \\
&- F(s, u(s), u(\sigma_1(s)), \ldots, u(\sigma_n(s))) + H(u_k)(s) - H(u)(s)\| \, \mathrm{d}s \to 0, \\
&\text{as} \quad k \to \infty.
\end{aligned}$$

Therefore, Ψ is continuous. □

Now, we prove that Ψ is completely continuous. Next, we can give the existence result of problem (1.1).

Theorem 3.5. *Assume that* (H1)–(H5) *hold. Then Eq.* (1.1) *has a mild solution for every* $u_0 \in X_\alpha$.

Proof. For $\lambda \in (0, 1)$ and any $t \in J$, let u satisfy $u(t) = \lambda(\Psi u)(t)$. Then from (H1)–(H4) we have

$$
\begin{aligned}
&\|u(t)\|_\alpha \\
&\le \|Q(t)(u_0 - g(u) - G(0, \psi(0)))\|_\alpha + \|G(t, \psi(t))\|_\alpha \\
&\quad + \int_0^t \|R(t-s)[F(s, \phi(s)) + H(u)(s)]\|_\alpha \, \mathrm{d}s \\
&\le M(\|u_0\|_\alpha + \|g(u)\|_\alpha + C_2 \Upsilon_G(\|u\|_\alpha)) + C_2 \Upsilon_G(\|u\|_\alpha) \\
&\quad + q \int_0^t (t-s)^{q-1} \|F(s, \phi(s)) + H(u)(s)\| \int_0^\infty \theta \xi_q(\theta) \|A^\alpha S((t-s)^q \theta)\| \, \mathrm{d}\theta \, \mathrm{d}s \\
&\le M(\|u_0\|_\alpha + b\,(1 + \|u\|_\infty) + C_2 \Upsilon_G(\|u\|_\infty)) + C_2 \Upsilon_G(\|u\|_\infty) \\
&\quad + L_0 \int_0^t (t-s)^{q(1-\alpha)-1} \|F(s, \phi(s)) + H(u)(s)\| \, \mathrm{d}s \\
&\le M(\|u_0\|_\alpha + b\,(1 + \|u\|_\infty)) + C_2 \Upsilon_G(\|u\|_\infty)(M+1) \\
&\quad + (C_1 \Omega_F(\|u\|_\infty) + C_3 W_\rho(\|u\|_\infty)) \frac{L_0 T^{q(1-\alpha)}}{q(1-\alpha)}.
\end{aligned}
$$

Therefore

$$
\frac{\|u\|_\infty}{\widetilde{M} + C_4 \Upsilon_G(\|u\|_\infty) + (C_1 \Omega_F(\|u\|_\infty) + C_3 W_\rho(\|u\|_\infty)) \, \frac{L_0 T^{q(1-\alpha)}}{q(1-\alpha)}} \le 1,
$$

where $\widetilde{M} := M\,(\|u_0\|_\alpha + b\,(1 + \|u\|_\infty))$.

Then, by (H5), there exists $\overline{M} > 0$ such that $\|u\|_\infty \ne \overline{M}$.

Let $\Pi = \{u \in C(J, X_\alpha) : \ \|u\|_\infty < \overline{M}\}$. Then $\Psi \colon \overline{\Pi} \to C(J, X_\alpha)$ is continuous and compact. It is obvious to see that there exists no $u \in \partial\Pi$ such that $u = \lambda \Psi u$, for $0 < \lambda < 1$. Consequently, by Theorem 2.9, the operator Ψ has a fixed point in $\overline{\Pi}$. This shows that Eq. (1.1) has a mild solution. □

Clearly, we have the following result from above proofs.

Theorem 3.6. *Assume that* (H1), (H3)–(H5) *hold. Then equation*

$$
\begin{aligned}
{}^cD^q u(t) &= Au(t) + F(t, u(t), u(\sigma_1(t)), u(\sigma_2(t)), \dots, u(\sigma_n(t))) \\
&\quad + \int_0^t k\left(t, s, u(s), \int_0^s \rho(s, \tau, u(\tau))\,\mathrm{d}\tau\right) \mathrm{d}s, \qquad t \in [0, T], \\
u(0) + g(u) &= u_0
\end{aligned}
\tag{3.1}
$$

has a mild solution for every $u_0 \in X_\alpha$.

4. Existence and uniqueness of solution

Now we assume that

(H1') The function $F\colon J \times X_\alpha^{n+1} \to X$ is continuous, and for any z_i, $\tilde{z}_i \in X_\alpha$ $(i = 0, 1, 2, \dots, n)$ there exists $L_1(t) \in L^1(J, \mathbf{R}^+)$ such that

$$\|F(t, z_0, z_1, z_2, \dots, z_n) - F(t, \tilde{z}_0, \tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_n)\| \le L_1(t) \sum_{i=0}^{n} \|z_i - \tilde{z}_i\|_\alpha.$$

(H2') The function $G\colon J \times X_\alpha^{m+1} \to X_\alpha$ is continuous, and for any y_i, $\tilde{y}_i \in X_\alpha$ $(i = 0, 1, 2, \dots, m)$ there exists a constant $\widetilde{C} > 0$ such that

$$\|G(t, y_0, y_1, y_2, \dots, y_m) - G(t, \tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_m)\|_\alpha \le \widetilde{C} \sum_{i=0}^{m} \|y_i - \tilde{y}_i\|_\alpha.$$

(H3') There exists $L_2(t) \in L^1(J, \mathbf{R}^+)$ such that

$$\|k(t, s, u_1, u_2) - k(t, s, v_1, v_2)\| \le L_2(t)(\|u_1 - v_1\|_\alpha + \|u_2 - v_2\|_\alpha),$$
$$u_1, u_2, v_1, v_2 \in X_\alpha.$$

(H4') There exists $L_3(t) \in L^1(J, \mathbf{R}^+)$ such that

$$\|\rho(t, s, w_1) - \rho(t, s, w_2)\|_\alpha \le L_3(t)\|w_1 - w_2\|_\alpha, \qquad w_1, w_2 \in X_\alpha.$$

(H5') $g \in C(C(J, X_\alpha), X_\alpha)$ and there exists $b_1 > 0$ such that

$$\|g(u) - g(v)\|_\alpha \le b_1\|u - v\|_\infty, \qquad u, v \in C(J, X_\alpha).$$

(H6') There exists γ with $0 < \gamma < 1$ such that

$$\omega(t) := Mb_1 + (M + 1)\widetilde{C}(m + 1) + L_0(n + 1)\Gamma(\delta) \cdot I^\delta L_1(t) + L_0 b_2 \frac{t^\delta}{\delta} \le \gamma,$$

where $\delta := q(1 - \alpha)$.

Theorem 4.1. *Assume that* (H1')–(H6') *hold, then Eq.* (1.1) *has a unique mild solution on* J.

Proof. Define the mapping $\tilde{\Psi} : C(J, X_\alpha) \to C(J, X_\alpha)$ by

$$\begin{aligned}(\tilde{\Psi}u)(t) &= Q(t)(u_0 - g(u) - G(0, \psi(0))) + G(t, \psi(t)) \\ &\quad + \int_0^t R(t-s)[F(s, \phi(s)) + H(u)(s)]\,\mathrm{d}s.\end{aligned}$$

Obviously, $\tilde{\Psi}$ is well defined on $C(J, X_\alpha)$. Taking $u, v \in C(J, X_\alpha)$, we have

$$\|H(u)(t) - H(v)(t)\| \le \|L_2\|_{L^1}(1 + \|L_3\|_{L^1})\|u - v\|_\infty := b_2\|u - v\|_\infty.$$

Furthermore, we have

$$\begin{aligned}
&\|(\tilde{\Psi}u)(t) - (\tilde{\Psi}v)(t)\|_\alpha \\
&\le \|Q(t)(g(u) - g(v))\|_\alpha \\
&\quad + \|Q(t)(G(0, u(0), u(\nu_1(0)), \dots, u(\nu_m(0))) \\
&\quad - G(0, v(0), v(\nu_1(0)), \dots, v(\nu_m(0))))\|_\alpha \\
&\quad + \|G(t, u(t), u(\nu_1(t)), \dots, u(\nu_m(t))) - G(t, v(t), v(\nu_1(t)), \dots, v(\nu_m(t)))\|_\alpha \\
&\quad + \int_0^t \|R(t-s)\{[F(s, u(s), u(\sigma_1(s)), \dots, u(\sigma_n(s))) \\
&\quad - F(s, v(s), v(\sigma_1(s)), \dots, v(\sigma_n(s)))] + [H(u)(s) - H(v)(s)]\}\|_\alpha\,\mathrm{d}s \\
&\le M\|g(u) - g(v)\|_\alpha + (M+1)\widetilde{C}(m+1)\|u - v\|_\infty \\
&\quad + L_0 \int_0^t (t-s)^{q(1-\alpha)-1}(\|F(s, u(s), u(\sigma_1(s)), \dots, u(\sigma_n(s))) \\
&\quad - F(s, v(s), v(\sigma_1(s)), \dots, v(\sigma_n(s)))\| + \|H(u)(s) - H(v)(s)\|)\,\mathrm{d}s \\
&\le Mb_1\|u - v\|_\infty + (M+1)\widetilde{C}(m+1)\|u - v\|_\infty \\
&\quad + L_0\Bigg[(n+1)\int_0^t (t-s)^{q(1-\alpha)-1}L_1(s)\,\mathrm{d}s \\
&\qquad\qquad + b_2\int_0^t (t-s)^{q(1-\alpha)-1}\,\mathrm{d}s\Bigg] \cdot \|u - v\|_\infty \\
&= \Bigg\{Mb_1 + (M+1)\widetilde{C}(m+1) \\
&\qquad + \left[L_0(n+1)\Gamma(\delta) \cdot I^\delta L_1(t) + L_0 b_2 \frac{t^\delta}{\delta}\right]\Bigg\} \cdot \|u - v\|_\infty \\
&= \omega(t)\|u - v\|_\infty.
\end{aligned}$$

Therefore, we obtain

$$\|(\tilde{\Psi}u)(t) - (\tilde{\Psi}v)(t)\|_\infty \leq \gamma\|u - v\|_\infty < \|u - v\|_\infty,$$

the result follows from the contraction mapping principle. □

5. Application

As an application of Theorem 3.5, we consider the system (1.1) with a control parameter as follows:

$$\begin{aligned}
{}^cD^q u(t) &= Au(t) + F(t, u(t), u(\sigma_1(t)), \ldots, u(\sigma_n(t))) \\
&\quad + \int_0^t k\left(t, s, u(s), \int_0^s \rho(s, \tau, u(\tau))\,\mathrm{d}\tau\right) \mathrm{d}s + (Bx)(t), \qquad t \in J, \\
u(0) + g(u) &= u_0,
\end{aligned} \tag{5.1}$$

where B is a bounded linear operator from a Banach space Y to X, and $x \in L^2(J, Y)$.

Definition 5.1. The system (5.1) is said to be controllable on J, if for every u_0, $u_1 \in X_\alpha$ there exists a control $x(t) \in L^2(J, Y)$ such that the mild solution $u(t)$ of (5.1) satisfies $u(T) = u_1$.

It is not difficult to see that the mild solution of (5.1) can be given by

$$u(t) := Q(t)(u_0 - g(u)) + \int_0^t R(t - s)\left[F(s, \phi(s)) + H(u)(s) + (Bx)(s)\right] \mathrm{d}s.$$

To obtain the controllability result for the system (5.1), we need the following hypotheses:

(H7) The linear operator $\Lambda\colon L^2(J, Y) \to X_\alpha$ defined by

$$\Lambda x = \int_0^T R(T - s)(Bx)(s)\,\mathrm{d}s$$

induces an invertible operator Λ^{-1} defined on $L^2(J, Y)/ker\Lambda$ and there exists $K > 0$ such that $\|B\Lambda^{-1}\| \leq K$.

(H8) There exists $M^* > 0$ such that

$$\frac{M^*}{M\left(\|u_0\|_\alpha + b\left(1 + M^*\right)\right) + \left(C_1\Omega_F(M^*) + C_3W_\rho(M^*)\right)\frac{L_0T^{q(1-\alpha)}}{q(1-\alpha)} + \frac{T^qMKN}{\Gamma(1+q)}} > 1,$$

where

$$N = \|u_1\|_\alpha + M\left(\|u_0\|_\alpha + b(1+M^*)\right) + \left(C_1\Omega_F(M^*) + C_3W_\rho(M^*)\right)\frac{L_0T^{q(1-\alpha)}}{q(1-\alpha)}.$$

Theorem 5.2. *If the hypotheses* (H1), (H3), (H4), (H7), *and* (H8) *hold, then the system* (5.1) *is controllable on* J.

P r o o f. Using (H7), for an arbitrary function $u(\cdot)$, we define the control

$$x(t) = \Lambda^{-1}\left\{u_1 - Q(T)(u_0 - g(u)) - \int_0^T R(T-s)[F(s,\phi(s)) + H(u)(s)]\,\mathrm{d}s\right\}(t).$$

Using this control, we define $\Phi\colon C(J, X_\alpha) \to C(J, X_\alpha)$ by

$$(\Phi u)(t) = Q(t)(u_0 - g(u)) + \int_0^t R(t-s)\left[F(s,\,\phi(s)) + H(u)(s) + (Bx)(s)\right]\mathrm{d}s.$$

Obviously, $(\Phi u)(T) = u_1$, which means that the control x steers system (5.1) from the given initial condition u_0 to u_1 in time T. Thus, if we show that Φ has a fixed point then the system (5.1) is controllable. The rest of the proof is similar to that of Theorem 3.5 and is omitted. □

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