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BASIC MATRIX THEOREMS FOR $\mathcal{I}$-CONVERGENCE IN (ℓ)-GROUPS

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ABSTRACT. Some aspects of the theory of order and (D)-convergence in (ℓ)-groups with respect to ideals are investigated. Moreover some new Basic Matrix Theorems are proved.

1. Introduction

The classical Basic Matrix Theorem of Antosik-Mikusiński-Swartz (see [2]) was extended by P. Antosik and C. Swartz to the context of Riesz spaces (see [3]), where the so-called "$(*)$-convergence" is used, and was further generalized by A. Aizpuru et al. (see [1]) in the case of statistical convergence, introduced in 1951 independently by H. Steinhaus and H. Fast (see [13]). Further recent studies about measures and integrals in the context of (ℓ)-groups and Riesz spaces can be found, for example, in [7, 8, 9].

In general, the nature of $(*)$-convergence is topological. However, there are Riesz spaces, that can be viewed as metrizable groups (with respect to a suitable topology), but such that order convergence is not generated by *any* topology: for example, $L^0(X, \mathcal{B}, \mu)$, where μ is a σ-additive and σ-finite non-atomic positive $\widetilde{\mathbb{R}}$-valued measure. Indeed, these spaces can be metrized in order to obtain convergence in measure, though order convergence (which coincides with (D)-convergence) means almost everywhere convergence and is not topological.

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Here, we extend the notion of order and (D)-convergence involving ideals introduced in [5] to the setting of (ℓ)-groups, and in particular we consider a class of ideals endowed with suitable properties. We study properties of these convergences, dealing in particular with double sequences, and prove a version of the Basic Matrix Theorem for (ℓ)-group-valued double sequences and with respect to suitable ideals. Furthermore, we show that our hypotheses, even in the real case, cannot be in general weakened, giving some counterexamples.

2. Preliminaries

Definitions 2.1. An Abelian group $(R,+)$ is called (ℓ)-*group* if it is a lattice and the following implication holds:

$$[a \le b] \Longrightarrow [a+c \le b+c]$$

for all $a,b,c \in R$. We denote by the symbols $\vee$ and $\wedge$ the supremum and the infimum in R respectively.

An (ℓ)-group R is said to be *Dedekind complete* if every nonempty subset of R, bounded from above, has supremum in R. A Dedekind complete (ℓ)-group is said to be *super Dedekind complete* if every subset $R_1 \subset R$, $R_1 \neq \emptyset$ bounded from above contains a countable subset having the same supremum as R_1.

Let R be an (ℓ)-group. Given an element $x \in R$, we call *absolute value* of x the element $|x|$ defined by setting $|x| := x \vee (-x)$. We say that a sequence $(p_n)_n$ of positive elements of R is an (O)-*sequence* if it is decreasing and $\bigwedge_n p_n = 0$. A sequence $(x_n)_n$ in R is said to be *order-convergent* (or (O)-*convergent*) to $x \in R$ if there exists an (O)-sequence $(p_n)_n$ in R with $|x_n - x| \le p_n$ for all $n \in \mathbb{N}$, and in this case we will write $(O)\lim_n x_n = x$. If Λ is any nonempty set, $\{(x_{n,\lambda})_n : \lambda \in \Lambda\}$ is a family of sequences in R and $x_\lambda \in R$ for all $\lambda \in \Lambda$, we say that $(O)\lim_n x_{n,\lambda} = x_\lambda$ *uniformly with respect to* $\lambda \in \Lambda$ if there exists an (O)-sequence $(q_n)_n$ in R with $|x_{n,\lambda} - x_\lambda| \le q_n$ for all $n \in \mathbb{N}$ and $\lambda \in \Lambda$. We say that the sequence $(x_n)_n$ is (O)-*Cauchy* if $(O)\lim_n (x_n - x_{n+p}) = 0$ uniformly with respect to $p \in \mathbb{N}$.

A bounded double sequence $(a_{t,l})_{t,l}$ in R is called (D)-*sequence* or *regulator* if for all $t, l \in \mathbb{N}$ we have $a_{t,l} \ge a_{t,l+1}$ and $\bigwedge_l a_{t,l} = 0$ for all $t \in \mathbb{N}$. A sequence $(x_n)_n$ in R is said to be (D)-*convergent* to $x \in R$ (and we write $(D)\lim_n x_n = x$) if there exists a (D)-sequence $(a_{t,l})_{t,l}$ in R, such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds $n_0 \in \mathbb{N}$ such that $|x_n - x| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ for all $n \in \mathbb{N}$, $n \ge n_0$. If $x_{n,\lambda}$ and x_λ,

$n \in \mathbb{N}$, $\lambda \in \Lambda$, are as above, we say that $(D)\lim_n x_{n,\lambda} = x_\lambda$ *uniformly with respect to* $\lambda \in \Lambda$ if there exists a (D)-sequence $(a_{t,l})_{t,l}$ in R, such that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $n_0 \in \mathbb{N}$ such that $|x_{n,\lambda} - x_\lambda| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ for all $n \in \mathbb{N}$, $n \geq n_0$ and $\lambda \in \Lambda$. The sequence $(x_n)_n$ is said to be (D)-*Cauchy* if $(D)\lim_n (x_n - x_{n+p}) = 0$ uniformly with respect to $p \in \mathbb{N}$.

We say that an (ℓ)-group is (O)-*complete* if every (O)-Cauchy sequence is (O)-convergent, and (D)-*complete* if every (D)-Cauchy sequence is (D)-convergent. We recall that every Dedekind complete (ℓ)-group is (O)-complete and (D)-complete (see also [10, Chapter 2]).

An (ℓ)-group R is said to be *weakly* σ-*distributive* if for every (D)-sequence $(a_{t,l})_{t,l}$ we have:

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \Big) = 0.$$

In general, the limit of a sequence (with respect to (D)-convergence) is not unique. However, (O)-convergence of sequences implies always (D)-convergence; moreover, if R is weakly σ-distributive, then a sequence is (D)-convergent if and only if it is (O)-convergent, and in this case the limit is unique.

We now denote by $l^1(R)$ the set of all sequences of the type $(x_j)_j$, with $x_j \in R$ for all $j \in \mathbb{N}$ and such that $\bigvee_q \Big(\sum_{j=1}^{q} |x_j| \Big) \in R$. As R is complete, if $(x_j)_j$ belongs to $l^1(R)$, then $S := (O)\lim_n \sum_{j=1}^{n} x_j$ exists in R. For every element $(x_j)_j$ in $l^1(R)$, we shall also write $S = (O)\lim_n \sum_{j=1}^{n} x_j = \sum_{j=1}^{\infty} x_j$, and say that S is the *sum* of the sequence $(x_j)_j$. Similarly as in the classical case, it is easy to check that, if the sum of a series $\sum_{j=1}^{\infty} x_j$ exists in R, then $(D)\lim_j x_j = 0$.

The following well-known result will be useful in the sequel (see [10]).

Lemma 2.2. *Let R be a Dedekind complete (ℓ)-group (not necessarily weakly σ-distributive), $(a_{t,l}^{(n)})_{t,l}$, $n \in \mathbb{N}$, be a sequence of regulators in R. Then for every $u \in R$, $u \geq 0$ there exists a (D)-sequence $(a_{t,l})_{t,l}$ in R such that:*

$$u \wedge \Big[\sum_{n=1}^{\infty} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}^{(n)} \Big) \Big] \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

for all $\varphi \in \mathbb{N}^{\mathbb{N}}$.

Definition 2.3. Let X be any nonempty set. A family of sets $\mathcal{I} \subset \mathcal{P}(X)$ is called an *ideal* of X if $A \cup B \in \mathcal{I}$ whenever $A, B \in \mathcal{I}$ and for each $A \in \mathcal{I}$ and $B \subset A$ we get $B \in \mathcal{I}$. An ideal is said to be *non-trivial* if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is said to be *admissible* if it contains all singletons.

An admissible ideal $\mathcal{I}$ is said to be a *P-ideal* if for any sequence $(A_j)_j$ in $\mathcal{I}$ there are sets $B_j \subset X$, $j \in \mathbb{N}$, such that the symmetric difference $A_j \Delta B_j$ is finite for all $j \in \mathbb{N}$ and $\bigcup\limits_{j=1}^{\infty} B_j \in \mathcal{I}$ (see also [13]).

Let $X = \mathbb{N}$, and for every $A \subset \mathbb{N}$ and $j \in \mathbb{N}$ set

$$d_j(A) = \frac{\sharp(A \cap \{1, \dots, j\})}{j},$$

where $\sharp$ means the cardinality of the set in brackets. The limit $d(A) := \lim\limits_j d_j(A)$ is called the (*asymptotic*) *density* of A. It is known that the ideal

$$\mathcal{I}_d := \big\{A \subset \mathbb{N} : \ d(A) = 0\big\}$$

is a P-ideal, as well as the ideal $\mathcal{I}_{\text{fin}}$ of all finite subsets of $\mathbb{N}$, while there are also other examples of P-ideals, known in the literature (see for example [13]).

Remark 2.4. It is also known (see [11]) that, if $X = \mathbb{N}^2$, then every P-ideal $\mathcal{I}$ is such that for every sequence $(A_j)_j$ in $\mathcal{I}$ there is a sequence $(B_j)_j$ such that for all $j \in \mathbb{N}$ the set $A_j \Delta B_j$ is included in a finite union of rows and columns in $\mathbb{N}^2$ and $B = \bigcup\limits_{j=1}^{\infty} B_j \in \mathcal{I}$.

Now, given a *fixed* admissible ideal $\mathcal{I}$, together with its dual filter

$$\mathcal{F} = \mathcal{F}(\mathcal{I}) := \{X \setminus I : \ I \in \mathcal{I}\},$$

we introduce the order and the (D)-convergence related with it.

When we deal with an ideal $\mathcal{I}$, we always suppose that $\mathcal{I}$ is admissible, without saying it explicitly.

An ideal $\mathcal{I}$ is said to be *maximal* if its dual filter $\mathcal{F}$ is an ultrafilter.

If $\mathcal{I}$ is an ideal of $\mathbb{N}$, we say that a sequence $(x_n)_n$ in R *$(O\mathcal{I})$-converges to* $x \in R$ if there exists an (O)-sequence $(\sigma_p)_p$ with the property that

$$\big\{n \in \mathbb{N} : \ |x_n - x| \leq \sigma_p\big\} \in \mathcal{F} \tag{1}$$

for all $p \in \mathbb{N}$; $(x_n)_n$ is said to be *$(O\mathcal{I})$-Cauchy* if there is an (O)-sequence $(\sigma_p)_p$ such that for each $p \in \mathbb{N}$ there is $m \in \mathbb{N}$ with

$$\big\{n \in \mathbb{N} : \ |x_n - x_m| \leq \sigma_p\big\} \in \mathcal{F}.$$

Similarly, if $\mathcal{I}$ is an ideal of $\mathbb{N}^2$, a double sequence $(x_{i,j})_{i,j}$ in R is $(O\mathcal{I})$-*convergent to* $\xi \in R$ if there is an (O)-sequence $(\sigma_p)_p$ with the property that

$$\big\{(i,j) \in \mathbb{N}^2 : \ |x_{i,j} - \xi| \le \sigma_p\big\} \in \mathcal{F}$$

for all $p \in \mathbb{N}$; $(x_{i,j})_{i,j}$ is said to be $(O\mathcal{I})$-*Cauchy* if there is an (O)-sequence $(\sigma_p)_p$ such that to every $p \in \mathbb{N}$ there corresponds $(m,n) \in \mathbb{N}$ with

$$\big\{(i,j) \in \mathbb{N}^2 : \ |x_{i,j} - x_{m,n}| \le \sigma_p\big\} \in \mathcal{F}.$$

Note that condition (1) is equivalent to say

$$\bigwedge_{U \in \mathcal{F}} \Big(\bigvee_{n \in U} x_n\Big) = x = \bigvee_{U \in \mathcal{F}} \Big(\bigwedge_{n \in U} x_n\Big).$$

Analogously we can formulate the concept of (D)-convergence with respect to an ideal.

A sequence $(x_n)_n$ in R $(D\mathcal{I})$-*converges to* $x \in R$ if there exists a (D)-sequence $(a_{t,l})_{t,l}$ with the property that

$$\Big\{n \in \mathbb{N} : \ |x_n - x| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\Big\} \in \mathcal{F}$$

for all $\varphi \in \mathbb{N}^{\mathbb{N}}$; $(x_n)_n$ is called $(D\mathcal{I})$-*Cauchy* if there exists a regulator $(a_{t,l})_{t,l}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $m \in \mathbb{N}$ with

$$\Big\{n \in \mathbb{N} : \ |x_n - x_m| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\Big\} \in \mathcal{F}.$$

Similarly, a double sequence $(x_{i,j})_{i,j}$ in R is $(D\mathcal{I})$-*convergent to* $\xi \in R$ if there is a regulator $(a_{t,l})_{t,l}$ such that

$$\Big\{(i,j) \in \mathbb{N}^2 : \ |x_{i,j} - \xi| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\Big\} \in \mathcal{F}$$

for any $\varphi \in \mathbb{N}^{\mathbb{N}}$; $(x_{i,j})_{i,j}$ is said to be $(D\mathcal{I})$-*Cauchy* if there exists a regulator $(a_{t,l})_{t,l}$ such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds a pair $(m,n) \in \mathbb{N}^2$ with

$$\Big\{(i,j) \in \mathbb{N}^2 : \ |x_{i,j} - x_{m,n}| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\Big\} \in \mathcal{F}.$$

If Λ is any arbitrary nonempty set, we can formulate the concepts of $(D\mathcal{I})$-convergence and $(D\mathcal{I})$-Cauchy uniformly with respect to $\lambda \in \Lambda$ as follows (this will be useful in the sequel).

We say that $\big\{(x_{n,\lambda})_n : \ \lambda \in \Lambda\big\}$ in R $(D\mathcal{I})$-*converges to* $x_\lambda \in R$ *uniformly with respect to* $\lambda \in \Lambda$ if there is a (D)-sequence $(a_{t,l})_{t,l}$ such that

$$\Big\{n \in \mathbb{N} : \ \bigvee_{\lambda \in \Lambda} |x_{n,\lambda} - x_\lambda| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\Big\} \in \mathcal{F}$$

for any $\varphi \in \mathbb{N}^{\mathbb{N}}$; $\{(x_{n,\lambda})_n : \lambda \in \Lambda\}$ is said to be $(D\mathcal{I})$-*Cauchy uniformly with respect to* $\lambda \in \Lambda$ if there is a regulator $(a_{t,l})_{t,l}$ such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a positive integer m with

$$\Big\{n \in \mathbb{N} : \bigvee_{\lambda \in \Lambda} |x_{n,\lambda} - x_{m,\lambda}| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\Big\} \in \mathcal{F}.$$

We now state the following result.

Proposition 2.5. *Every* $(O\mathcal{I})$*-convergent* ($(O\mathcal{I})$*-Cauchy*) *(double) sequence is* $(D\mathcal{I})$*-convergent to the same limit* ($(D\mathcal{I})$*-Cauchy*). *Moreover, if* R *is a super Dedekind complete and weakly* σ*-distributive* (ℓ)*-group, then the converse implication holds, too.*

Proof. Without loss of generality, we prove the proposition only for the case of single sequences $(x_n)_n$ and order convergence, since the cases involving double sequences and/or the Cauchy properties are analogous.

We begin with the first part. Let $(\sigma_p)_p$ be an (O)-sequence, satisfying the definition of $(O\mathcal{I})$-convergence of $(x_n)_n$ to the element $x \in R$, and for all $t, l \in \mathbb{N}$ set $a_{t,l} := \sigma_l$. Fix arbitrarily $\varphi \in \mathbb{N}^{\mathbb{N}}$, and let $n_0 := \min\{\varphi(n) : n \in \mathbb{N}\}$. We get:

$$\sigma_{n_0} \leq \bigvee_{n=1}^{\infty} \sigma_{\varphi(n)} = \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}.$$

Hence, taking into account the $(O\mathcal{I})$-convergence, we obtain

$$\mathcal{F}(\mathcal{I}) \ni \{n \in \mathbb{N} : |x_n - x| \leq \sigma_{n_0}\} \subset \Big\{n \in \mathbb{N} : |x_n - x| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\Big\},$$

and therefore

$$A_{\varphi} := \Big\{n \in \mathbb{N} : |x_n - x| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\Big\} \in \mathcal{F}(\mathcal{I}). \tag{2}$$

Thus the first implication is proved.

We now turn to the second part. We know the existence of a (D)-sequence $(a_{t,l})_{t,l}$, satisfying (2). Thanks to super Dedekind completeness and weak σ-distributivity of R, by [4, Theorem 3.1] there exists an (O)-sequence $(\sigma_p)_p$ in R such that to every $p \in \mathbb{N}$ there corresponds $\varphi_p \in \mathbb{N}^{\mathbb{N}}$ with $\bigvee_{t=1}^{\infty} \alpha_{t,\varphi_p(t)} \leq \sigma_p$. For each $p \in \mathbb{N}$, set $F_p := \{n \in \mathbb{N} : |x_n - x| \leq \sigma_p\}$. For every $p \in \mathbb{N}$, we get $F_p \supset A_{\varphi_p}$, and hence $F_p \in \mathcal{F}(\mathcal{I})$. This concludes the proof. □

From now on, we always suppose that R is a super Dedekind complete weakly σ-distributive (ℓ)-group. Examples of such spaces are $\mathbb{R}^{\mathbb{N}}$ and $L^0(X, \mathcal{B}, \mu)$ with μ positive, σ-additive and σ-finite (see also [10]). So, in our setting $(O\mathcal{I})$- and

$(D\mathcal{I})$-convergence coincide. If $R = \mathbb{R}$, instead of $(O\mathcal{I})$ and $(D\mathcal{I})$ we will write simply $(\mathcal{I})$.

Moreover, let us define

$$(\mathcal{I})\sum_{j=1}^{\infty} x_j := (O\mathcal{I})\lim_n \sum_{j=1}^{n} x_j = (D\mathcal{I})\lim_n \sum_{j=1}^{n} x_j.$$

Proposition 2.6. *Let $\mathcal{I}$ be any fixed admissible ideal of $\mathbb{N}$. If $(D)\lim_n x_n = x$, then $(D\mathcal{I})\lim_n x_n = x$. Similar results hold for double sequences $(x_{i,j})_{i,j}$.*

Moreover, if $(x_n)_n$ is an increasing sequence in R and $x \in R$, then $(D\mathcal{I})\lim_n x_n = x$ if and only if $(D)\lim_n x_n = x$.

Proof. The first part is straightforward.

We now turn to the final part. It is enough to prove the "only if" implication. By hypothesis there is a (D)-sequence $(a_{t,l})_{t,l}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ an integer $n^* \in \mathbb{N}$ can be found, with

$$0 \le x - x_{n^*} \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}.$$

By monotonicity we get:

$$0 \le x - x_n \le x - x_{n^*} \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

for any $n \ge n^*$. So the sequence $(x_n)_n$ (D)-converges monotonically to x, according to the usual concept of (D)-convergence. This concludes the proof. $\square$

Observe that an easy consequence of Proposition 2.6 is that, if a series $\sum_{j=1}^{\infty} x_j$ is of positive terms in R and S is its sum, then $(\mathcal{I})\sum_{j=1}^{\infty} x_j = S$ (and vice-versa).

We now introduce another kind of convergence in the context of ideals.

Definition 2.7. We say that a sequence $(x_n)_n$ in R $(O\mathcal{I}^*)$-$[(D\mathcal{I}^*)]$-*converges to* $x \in R$ if there exists $A \in \mathcal{F}(\mathcal{I})$ with

$$(\mathrm{O})\lim_{\substack{n\to+\infty\\ n\in A}} x_n = x \quad \Big[(\mathrm{D})\lim_{\substack{n\to+\infty\\ n\in A}} x_n = x\Big].$$

The following result, which will be useful in the sequel, extends the corresponding one given in [13].

Proposition 2.8. *Let $(x_n)_n$ be a sequence in R, $(O\mathcal{I})$-convergent to $x \in R$. If $\mathcal{I}$ is a P-ideal, then $(x_n)_n$ $(O\mathcal{I}^*)$-converges to x.*

P r o o f. Let $\mathcal{I}$ be a P-ideal, and $(\sigma_p)_p$ be an (O)-sequence existing by virtue of $(O\mathcal{I})$-convergence, as in (1). For each $p \in \mathbb{N}$, set $A_p := \big\{n \in \mathbb{N} : \ |x_n - x| \not\leq \sigma_p\big\}$: then $A_p \in \mathcal{I}$ for all p. Since $\mathcal{I}$ is a P-ideal, there exists a sequence of sets $(B_p)_p$ such that the symmetric difference $A_p \Delta B_p$ is a finite set for any $p \in \mathbb{N}$ and $B := \bigcup\limits_{p=1}^{\infty} B_p \in \mathcal{I}$.

So, in order to prove the proposition, it is enough to check that

$$(\mathrm{O}) \lim_{\substack{n \to +\infty \\ n \in \mathbb{N} \setminus B}} x_n = x. \tag{3}$$

Let $p \in \mathbb{N}$. Since $A_p \Delta B_p$ is a finite set, there is $n_p \in \mathbb{N}$, without loss of generality with $n_p \in \mathbb{N} \setminus B$, $n_p > p$, such that

$$(\mathbb{N} \setminus B_p) \cap \{n \in \mathbb{N} : \ n \geq n_p\} = (\mathbb{N} \setminus A_p) \cap \{n \in \mathbb{N} : \ n \geq n_p\}. \tag{4}$$

If $n \in \mathbb{N} \setminus B$ and $n \geq n_p$, then $n \notin B_p$, and, by (4), $n \notin A_p$. Thus $|x_n - x| \leq \sigma_p$.

Thus we have proved that for all $p \in \mathbb{N}$ there is $n_p \in \mathbb{N} \setminus B$, $n_p > p$, such that $|x_n - x| \leq \sigma_p$ for each $n \geq n_p$: without loss of generality, we can suppose that $n_{p+1} > n_p$ for every $p \in \mathbb{N}$. Let $n_0 = 0$, and for each $n \in \mathbb{N} \setminus B$ set $b_n := \sigma_p$, where $p = p(n)$ is the unique natural number such that $n_{p-1} + 1 \leq n \leq n_p$. We get that $(b_n)_n$ is an (O)-sequence and $|x_n - x| \leq b_n$ for all $n \in \mathbb{N} \setminus B$, and so (3) is proved. This concludes the proof. □

A consequence of Proposition 2.8 is the following:

Proposition 2.9. *Under the same hypotheses as in* Proposition 2.8, *let $\mathcal{I}$ be a P-ideal and $(x_n)_n$ be a sequence in R, such that $(D\mathcal{I}) \lim\limits_n x_n = x \in R$.*

Then there exists a subsequence $(x_{n_q})_q$ of $(x_n)_n$, such that $(D) \lim\limits_q x_{n_q} = x$.

Remark 2.10. Proposition 2.8 holds even if we consider a double sequence $(x_{i,j})_{i,j}$ instead of a sequence $(x_n)_n$. Indeed the proof, considering ideals of $\mathbb{N}^2$ rather than of $\mathbb{N}$, is substantially analogous to the one of Proposition 2.8, with the only difference that, instead of formula (4), using Remark 2.4, one considers the fact that, since $A_p \Delta B_p$, $p \in \mathbb{N}$, is included in a finite union of rows and columns in $\mathbb{N}^2$, there is $n_p \in \mathbb{N}$, without loss of generality with $(n_p, n_p) \in \mathbb{N}^2 \setminus B$, $n_p > p$, such that

$$(\mathbb{N}^2 \setminus B_p) \cap \big\{(m,n) \in \mathbb{N}^2 : \ m,n \geq n_p\big\} = (\mathbb{N}^2 \setminus A_p) \cap \big\{(m,n) \in \mathbb{N}^2 : \ m,n \geq n_p\big\}$$

(see also [11, Theorem 3]).

The following proposition holds for any admissible ideal and extends [13, Proposition 3.2]. For the sake of simplicity, we prove it in the case of single sequences and ideals of $\mathbb{N}$.

Proposition 2.11. *Suppose that* $(D\mathcal{I}^*)\lim_n x_n = x$. *Then* $(D\mathcal{I})\lim_n x_n = x$.

Proof. By hypothesis, there is $A \in \mathcal{I}$ such that for $M := \mathbb{N} \setminus A$, $M =: \{m_1 < \dots < m_h < \dots\}$ we get

$$(\mathrm{D})\lim_{h\to+\infty} x_{m_h} = x \tag{5}$$

with respect to a suitable regulator $(a_{t,l})_{t,l}$. Fix arbitrarily $\varphi \in \mathbb{N}^{\mathbb{N}}$. Then by (5) there exists $h_0 \in \mathbb{N}$ with

$$|x_{m_h} - x| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

whenever $h \ge h_0$. Thus the set

$$A_\varphi := \left\{n \in \mathbb{N} : |x_n - x| \not\le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\right\} \subset A \cup \left\{m_1, \dots, m_{h_0-1}\right\} \in \mathcal{I},$$

since $\mathcal{I}$ is admissible. Thus $A_\varphi \in \mathcal{I}$. This concludes the proof. □

We now give the following:

Proposition 2.12. *Let* $(x_{i,j})_{i,j}$ *be a bounded double sequence in* R, $\mathcal{I}$ *be any* P*-ideal,* $\mathcal{F} = \mathcal{F}(\mathcal{I})$ *be its dual filter, and let us suppose that* $(D\mathcal{I})\lim_i x_{i,j} = x_j$ *for every* $j \in \mathbb{N}$.

Then there exists $B_0 \in \mathcal{F}$ *such that*

$$(\mathrm{D})\lim_{\substack{h\to+\infty \\ h\in B_0}} x_{h,j} = x_j$$

for all $j \in \mathbb{N}$ *and with respect to a same* (D)*-sequence* $(\alpha_{t,l})_{t,l}$.

Proof. By hypotheses and Propositions 2.5, 2.8 we get that $(D\mathcal{I}^*)\lim_i x_{i,j} = x_j$ for all $j \in \mathbb{N}$. This means that there exists a sequence $(A_j)_j$ in the dual filter $\mathcal{F}$, such that $(\mathrm{D})\lim_{i\in A_j} x_{i,j} = x_j$ for all $j \in \mathbb{N}$. As $\mathcal{I}$ is a P-ideal, there is a sequence of sets $(B_j)_j$ in $\mathcal{F}$, such that $A_j \Delta B_j$ is finite for all $j \in \mathbb{N}$ and $B_0 := \bigcap_{j=1}^{\infty} B_j \in \mathcal{F}$. Thus, since $(\mathrm{D})\lim_{i\in A_j} x_{i,j} = x_j$ for all $j \in \mathbb{N}$, we get also $(\mathrm{D})\lim_{i\in B_j} x_{i,j} = x_j$ for every j.

So for each $j \in \mathbb{N}$ a regulator $(\alpha^{(j)}_{t,l})_{t,l}$ can be found, with the property that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $\overline{i} \in B_j$ such that

$$|x_{i,j} - x_j| \le \bigvee_{t=1}^{\infty} \alpha^{(j)}_{t,\varphi(t+j)} \tag{6}$$

for all $i \in B_j$, $i \ge \overline{i}$. Let $u := \bigvee_{i,j} |x_{i,j}|$. By virtue of the Fremlin Lemma 2.2, there exists a regulator $(\alpha_{t,l})_{t,l}$ with the property that

$$u \wedge \left[\sum_{j=1}^{\infty} \left(\bigvee_{t=1}^{\infty} \alpha^{(j)}_{t,\varphi(t+j)} \right) \right] \le \bigvee_{t=1}^{\infty} \alpha_{t,\varphi(t)} \tag{7}$$

for all $\varphi \in \mathbb{N}^{\mathbb{N}}$. By (6) and (7), the regulator $(\alpha_{t,l})_{t,l}$ is such that for every $j \in \mathbb{N}$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\overline{i} \in B_j$ with

$$|x_{i,j} - x_j| \le \bigvee_{t=1}^{\infty} \alpha_{t,\varphi(t)} \tag{8}$$

for every $i \in B_j$, $i \ge \overline{i}$. Let $B_0 = \{p_1 < \dots < p_h < \dots\}$ and choose arbitrarily $j \in \mathbb{N}$: then, since $B_0 \subset B_j$, from (8) it follows that in correspondence with every $\varphi \in \mathbb{N}^{\mathbb{N}}$ an integer $\overline{h} = \overline{h}(j, \varphi)$ can be found, with

$$|x_{p_h,j} - x_j| \le \bigvee_{t=1}^{\infty} \alpha_{t,\varphi(t)}$$

whenever $h \ge \overline{h}$. This concludes the proof. □

We now prove a Cauchy-type condition, which extends [14, Theorem 5.1], given in the case $R = \mathbb{R}$, and will be useful in the sequel. We formulate it in the context of double sequences and ideals of $\mathbb{N}^2$: an analogous result holds, if we deal with ordinary sequences and ideals of $\mathbb{N}$. We begin with the following:

Proposition 2.13. *Let $\mathcal{I}$ be any admissible ideal of $\mathbb{N}^2$. A double sequence $(x_{i,j})_{i,j}$ is $(D\mathcal{I})$-convergent if and only if it is $(D\mathcal{I})$-Cauchy.*

Proof. We begin with the "if" part. Let $(x_{i,j})_{i,j}$ be a $(D\mathcal{I})$-Cauchy double sequence. Then, by Proposition 2.5, $(x_{i,j})_{i,j}$ is $(O\mathcal{I})$-Cauchy. Let $(\varepsilon_p)_p$ be an (O)-sequence, related with the $(O\mathcal{I})$-Cauchy condition. So there exist two sequences $(m_p)_p$, $(n_p)_p$ in $\mathbb{N}$ with

$$\{(i,j) \in \mathbb{N}^2 : \ |x_{i,j} - x_{m_p,n_p}| \le \varepsilon_p\} \in \mathcal{F}(\mathcal{I}) \tag{9}$$

for any $p \in \mathbb{N}$. Let now $p, q \in \mathbb{N}$, $p \ne q$. Since $\mathcal{F}(\mathcal{I})$ is a filter in $\mathbb{N}^2$, we get

$$\{(i,j) \in \mathbb{N}^2 : \ |x_{i,j} - x_{m_p,n_p}| \le \varepsilon_p\} \cap \{(i,j) \in \mathbb{N}^2 : \ |x_{i,j} - x_{m_q,n_q}| \le \varepsilon_q\} \in \mathcal{F}(\mathcal{I}).$$

Thus for every pair $(p,q) \in \mathbb{N}^2$ with $p \neq q$ there is $(i_{p,q}, j_{p,q}) \in \mathbb{N}^2$ with $|x_{i_{p,q},j_{p,q}} - x_{m_p,n_p}| \leq \varepsilon_p$ and $|x_{i_{p,q},j_{p,q}} - x_{m_q,n_q}| \leq \varepsilon_q$, so that $|x_{m_p,n_p} - x_{m_q,n_q}| \leq \varepsilon_p + \varepsilon_q$. As $(\varepsilon_p)_p$ is an (O)-sequence, then $(x_{m_p,n_p})_p$ is an (O)-Cauchy sequence (in the classical sense). Since every Dedekind complete (ℓ)-group is (O)-complete (see [10]), there exists an element $\xi \in R$ with $\xi = (O)\lim_p x_{m_p,n_p}$. Thanks to (9) and the main properties of filters, for every $p \in \mathbb{N}$ we get:

$$\begin{aligned}
&\big\{(i,j) \in \mathbb{N}^2 : \ |x_{i,j} - \xi| \leq 2\varepsilon_p\big\} \\
\supset &\big\{(i,j) \in \mathbb{N}^2 : \ |x_{i,j} - x_{m_p,n_p}| + |x_{m_p,n_p} - \xi| \leq 2\varepsilon_p\big\} \\
\supset &\big\{(i,j) \in \mathbb{N}^2 : \ |x_{m_p,n_p} - \xi| \leq \varepsilon_p\big\} \\
&\cap \big\{(i,j) \in \mathbb{N}^2 : \ |x_{i,j} - x_{m_p,n_p}| \leq \varepsilon_p\big\} \in \mathcal{F}(\mathcal{I}).
\end{aligned}$$

This concludes the proof of the "if" part.

We now turn to the "only if" part. Since, by hypothesis, $(x_{i,j})_{i,j}$ is $(D\mathcal{I})$-convergent to an element $\xi \in R$, there is a regulator $(a_{t,l})_{t,l}$ with the property that

$$A_\varphi := \Big\{(i,j) \in \mathbb{N}^2 : \ |x_{i,j} - \xi| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\Big\} \in \mathcal{F}(\mathcal{I})$$

for every $\varphi \in \mathbb{N}^{\mathbb{N}}$. Fixed arbitrarily such a function φ, there exist positive integers m, n such that

$$|x_{m,n} - \xi| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}.$$

Let now $(i,j) \in A_\varphi$: then

$$|x_{i,j} - x_{m,n}| \leq |x_{i,j} - \xi| + |x_{m,n} - \xi| \leq 2 \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}.$$

Hence

$$B_\varphi := \Big\{(i,j) \in \mathbb{N}^2 : \ |x_{i,j} - x_{m,n}| \leq 2 \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\Big\} \supset A_\varphi,$$

and thus $B_\varphi \in \mathcal{F}(\mathcal{I})$. The assertion follows by arbitrariness of $\varphi \in \mathbb{N}^{\mathbb{N}}$. This concludes the proof. □

Analogously as Proposition 2.13 it is possible to prove the following:

Proposition 2.14. *If $\mathcal{I}$ is an admissible ideal of $\mathbb{N}$, then a sequence $(x_n)_n$ in R is $(D\mathcal{I})$-convergent if and only if it is $(D\mathcal{I})$-Cauchy. Moreover, if Λ is any abstract nonempty set, then a family $\big\{(x_{n,\lambda})_n : \ \lambda \in \Lambda\big\}$ is $(D\mathcal{I})$-convergent uniformly with respect to $\lambda \in \Lambda$ if and only if it is $(D\mathcal{I})$-Cauchy uniformly with respect to $\lambda \in \Lambda$.*

The following lemma deals with interchange of limits with respect to $(D\mathcal{I})$-convergence and holds without assuming necessarily that the involved ideal is a P-ideal (for the classical version in the real setting, see [12, Lemma I.7.6]).

Lemma 2.15. *Let $(x_{i,j})_{i,j}$ be a bounded double sequence of R, $\mathcal{I}$ be any admissible ideal in $\mathbb{N}$, $\mathcal{F}$ be its dual filter and K be any fixed element of $\mathcal{F}$. Set $\mathcal{I} \times \mathcal{I} := \{D_1 \times D_2 : \ D_1, D_2 \in \mathcal{I}\}$. Suppose that*

(i) $(D\mathcal{I}) \lim\limits_{i} x_{i,j} = y_j$ *exists in R for all $j \in \mathbb{N}$.*

(ii) $(D\mathcal{I}) \lim\limits_{j} \big[\sup\limits_{i \in K} |x_{i,j} - x_i|\big] = 0$.

Then the following results hold with respect to a same (D)-sequence $(b_{t,l})_{t,l}$.

(iii) *In R, there exists the limit $a := (D\mathcal{I}) \lim\limits_{j} y_j$.*

(iv) *In R, there exists $b := (D\mathcal{I}) \lim\limits_{i} x_i$.*

(v) *There exist an ideal $\mathcal{J} \subset \mathcal{P}(\mathbb{N} \times \mathbb{N})$ and $c \in R$ such that $\mathcal{I} \times \mathcal{I} \subset \mathcal{J}$ and $(D\mathcal{J}) \lim\limits_{i,j} x_{i,j} = c$.*

(vi) *There exists in R $d := (D\mathcal{I}) \lim\limits_{i} x_{i,i}$.*

(vii) *We get: $a = b = c = d$.*

Proof. First of all note that by (i), arguing analogously as in the proof of Proposition 2.12, thanks to boundedness of the given sequence and Lemma 2.2 there exists a regulator $(a_{t,l})_{t,l}$ such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N}$ there corresponds $D_j \in \mathcal{I}$ with the property that $|x_{i,j} - y_j| \leq \bigvee\limits_{t=1}^{\infty} a_{t,\varphi(t)}$ whenever $i \notin D_j$.

By (ii), without loss of generality, the regulator $(a_{t,l})_{t,l}$ can be chosen in such a way that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $D \in \mathcal{I}$ such that

$$|x_{i,j} - x_i| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \tag{10}$$

for all $j \notin D$ and $i \in K$.

We now prove (v). Let $j_0 := \min(\mathbb{N} \setminus D)$. Then by (10) we have:

$$|x_{i,j_0} - x_i| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \tag{11}$$

for all $i \in K$. By (10) and (11) we get that

$$|x_{i,j} - x_{i,j_0}| \leq 2 \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \qquad j \notin D, \quad i \in K. \tag{12}$$

By (i) we have the existence of the limit $(D\mathcal{I})\lim_{i} x_{i,j_0} = y_{j_0}$ and so there is $D_{j_0} \in \mathcal{I}$ such that

$$|x_{i,j_0} - y_{j_0}| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \qquad i \notin D_{j_0}, \quad i \in K. \tag{13}$$

Let $i_0 := \min(\mathbb{N} \setminus D_{j_0})$. Then by (13) we get:

$$|x_{i_0,j_0} - y_{j_0}| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}. \tag{14}$$

By (13) and (14) we obtain:

$$|x_{i,j_0} - x_{i_0,j_0}| \leq 2 \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}, \qquad i \notin D_{j_0}, \quad i \in K. \tag{15}$$

By (12) and (15) we get that

$$|x_{i,j} - x_{i_0,j_0}| \leq 4 \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \qquad i \notin D_{j_0}, \quad i \in K, \quad j \notin D. \tag{16}$$

Let now $i' \notin D_{j_0}$, $i' \in K$, $j' \notin D$. Then by (16) we have:

$$|x_{i_0,j_0} - x_{i',j'}| \leq 4 \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}. \tag{17}$$

Let $S := (D_{j_0} \cup (\mathbb{N} \setminus K)) \times D \in \mathcal{I} \times \mathcal{I}$ and

$$\mathcal{J} := \Big\{ \bigcup_{s=1}^{k} (A_s \times B_s) : \; A_s, B_s \in \mathcal{I}, \;\; s = 1, \dots, k; \;\; k \in \mathbb{N} \Big\}.$$

Then $\mathcal{J}$ is an admissible ideal in $\mathbb{N} \times \mathbb{N}$ and $S \in \mathcal{J}$. By (17) we obtain that

$$|x_{i,j} - x_{i',j'}| \leq 8 \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \tag{18}$$

for all $(i,j), (i',j') \notin S$, and by (18) the double sequence $(x_{i,j})_{i,j}$ is $(D\mathcal{J})$-Cauchy. By virtue of Proposition 2.13 it follows that the limit

$$c := (D\mathcal{J}) \lim_{i,j} x_{i,j}$$

exists in R. Thus (v) is proved.

(vi) With the same notations as in the proof of (v), if $i, i' \notin D_{j_0} \cup D \cup (\mathbb{N} \setminus K) \in \mathcal{I}$, then from (16) and (17) it follows that

$$|x_{i,i} - x_{i',i'}| \leq 8 \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}.$$

Thus the sequence $(x_{i,i})_i$ is $(D\mathcal{I})$-Cauchy, and hence the limit $(D\mathcal{I})\lim\limits_i x_{i,i}$ exists in R and is equal to c.

We now prove (iii). By (v) there exists a (D)-sequence $(\alpha_{t,l})_{t,l}$ such that to any $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds $S \in \mathcal{J}$ with the property that

$$|x_{i,j} - c| \le \bigvee_{t=1}^{\infty} \alpha_{t,\varphi(t)} \tag{19}$$

for all $(i,j) \notin S$. But $S = \bigcup\limits_{s=1}^{k_0} (A_s \times B_s)$, where $k_0 \in \mathbb{N}$ and $A_s,\ B_s \in \mathcal{I}$ for all $s = 1, \dots, k_0$. Moreover, by (i), for every $j \in \mathbb{N}$ we have the existence of $D_j \in \mathcal{I}$ with

$$|x_{i,j} - y_j| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \tag{20}$$

for all $i \notin D_j$. So, for each $i \notin \Big(\bigcup\limits_{s=1}^{k_0} A_s\Big) \cup D_j \in \mathcal{I}$ and $j \notin \Big(\bigcup\limits_{s=1}^{k_0} B_s\Big) \in \mathcal{I}$, by (19) and (20) we get:

$$|y_j - c| \le |x_{i,j} - y_j| + |x_{i,j} - c| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} \alpha_{t,\varphi(t)}. \tag{21}$$

By (21), thanks to weak σ-distributivity of R, we get that the element a as in (iii) exists in R and $a = c$.

(iv) Similarly as in (iii).

(vii) It is an obvious consequence of (iii), (iv), (v) and (vi). $\square$

3. The basic matrix theorem

We now turn to the main result.

Theorem 3.1. *Let $(x_{i,j})_{i,j}$ be a bounded double sequence in R, and $\mathcal{I}$ be a P-ideal of $\mathbb{N}$. Suppose that:*

(i) $(D\mathcal{I})\lim\limits_i x_{i,j} =: x_j$ *exists in R for all $j \in \mathbb{N}$;*

(ii) $(D\mathcal{I})\lim\limits_j x_{i,j} = 0$ *for all $i \in \mathbb{N}$;*

(iii) *there exists a regulator $(d_{t,l})_{t,l}$ such that for every infinite subset $B \subset \mathbb{N}$ there is an infinite subset $C \subset B$ such that the sequence*

$$\Big((\mathcal{I}) \sum_{j \in C} x_{i,j}\Big)_i$$

(D)-converges (with respect to the same regulator $(d_{t,l})_{t,l}$).

Then the following hold:

(I) *There exists* $K \in \mathcal{F} = \mathcal{F}(\mathcal{I})$ *such that* $(D\mathcal{I}) \lim_i \big[\bigvee_{j \in K} |x_{i,j} - x_j| \big] = 0$.

(II) $(D\mathcal{I}) \lim_j x_j = 0$.

(III) *If* $\mathcal{J} \subset \mathcal{P}(\mathbb{N}^2)$ *is the ideal of* $\mathbb{N}^2$ *generated by the finite unions of the Cartesian products of the elements of* $\mathcal{I}$, *then* $(D\mathcal{J}) \lim_{i,j} x_{i,j} = 0$.

(IV) $(D\mathcal{I}) \lim_i x_{i,i} = 0$.

(V) *There is* $A \in \mathcal{F} = \mathcal{F}(\mathcal{I})$ *with* $(D\mathcal{I}) \lim_j \big[\bigvee_{i \in A} |x_{i,j}| \big] = 0$.

Proof.

(I) First of all note that, by virtue of (ii) and Proposition 2.12, a set $K \in \mathcal{F}$ can be found, with

$$(\mathrm{D}) \lim_{\substack{j \to +\infty \\ j \in K}} x_{i,j} = 0 \tag{22}$$

for all $i \in \mathbb{N}$, with respect to a same regulator $(\beta_{t,l})_{t,l}$.

Let $b_{t,l} = 2\beta_{t,l}$, $t, l \in \mathbb{N}$. From (22) it follows that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $i, k \in \mathbb{N}$ there is $s = s(i,k) \in K$ with the property that

$$|x_{i,j} - x_{k,j}| \le \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \tag{23}$$

for all $j \ge s$, $j \in K$. Let $u := \bigvee_{i,j} |x_{i,j}|$. By virtue of the Fremlin Lemma 2.2, there exists a regulator $(b^*_{t,l})_{t,l}$ with the property that

$$(2u) \wedge \Big[\sum_{q=1}^{\infty} \Big(\bigvee_{t=1}^{\infty} b_{t,\varphi(t+q)} \Big) \Big] \le \bigvee_{t=1}^{\infty} b^*_{t,\varphi(t)} \tag{24}$$

for all $\varphi \in \mathbb{N}^{\mathbb{N}}$. Moreover, by (i) and Proposition 2.12 again, we get the existence of a set $A \in \mathcal{F}$ such that

$$(\mathrm{D}) \lim_{\substack{i \to +\infty \\ i \in A}} x_{i,j} = x_j$$

for all $j \in \mathbb{N}$ and with respect to a same (D)-sequence $(\alpha_{t,l})_{t,l}$.

Let $A = \{q_1 < \dots < q_i < \dots\}$: for the sake of simplicity, put $q_i = i$ for all i.

Again by Lemma 2.2, proceeding analogously as above, taking into account boundedness of the double sequence $(x_{i,j})_{i,j}$, a regulator $(a^*_{t,l})_{t,l}$ can be found, such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $s \in \mathbb{N}$ there is $p \in \mathbb{N}$ with

$$\sum_{\substack{j \in K \\ j=1}}^{s} |x_{i,j} - x_j| \leq \bigvee_{t=1}^{\infty} a^*_{t,\varphi(t)} \tag{25}$$

for all $i \geq p$.

Let now $a_{t,l} = 2a^*_{t,l}$, $t, l \in \mathbb{N}$. For all $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $s \in \mathbb{N}$ there is $p = p(s) \in \mathbb{N}$ with the property that

$$\sum_{\substack{j \in K \\ j=1}}^{s} |x_{i,j} - x_{h,j}| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \tag{26}$$

for all $i, h \geq p$. Let $(d_{t,l})_{t,l}$ be as in (iii) and set

$$d^*_{t,l} := 2d_{t,l}, \quad c_{t,l} := 2(a_{t,l} + b^*_{t,l} + d^*_{t,l}), \qquad t, l \in \mathbb{N}.$$

Let K be as in (22): we will prove that

$$(\mathrm{D}) \lim_{i \in A} \Big[\bigvee_{j \in K} |x_{i,j} - x_j| \Big] = 0. \tag{27}$$

This, thanks to Proposition 2.11, is enough to prove (I).

Before proving (27), we claim that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $i \in A$ such that the set

$$\Big\{ k \in A : \bigvee_{j \in K} |x_{i,j} - x_{k,j}| \not\leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \Big\} \tag{28}$$

is finite. Otherwise, there is $\varphi \in \mathbb{N}^{\mathbb{N}}$ with the property that for every $i \in A$ there exist $k = k(i) \in A$, $k > i$ and $j \in K$ with

$$|x_{i,j} - x_{k,j}| \not\leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}. \tag{29}$$

Choose arbitrarily $i_1 \in A$: in correspondence with i_1 there exist $k_1 = k(i_1) \in A$, $k_1 > i_1$ and $j_1 \in K$ with

$$|x_{i_1,j_1} - x_{k_1,j_1}| \not\leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}. \tag{30}$$

Let $s_1 := s(i_1, k_1) \in K$ be as in (23): without loss of generality, we can choose $s_1 > j_1$. We get

$$|x_{i_1,j} - x_{k_1,j}| \leq \bigvee_{t=1}^{\infty} b_{t,\varphi(t+1)}$$

whenever $j \geq s_1$, $j \in K$. Let $p_1 := p(s_1)$ be as in (26). We obtain

$$\sum_{j \in K, j=1}^{s_1} |x_{p,j} - x_{q,j}| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \tag{31}$$

for all $p, q \geq p_1$.

Let now $i_2 \in A$, with $i_2 > p_1$. Without loss of generality, we can choose $i_2 \in \{k(i) : \ i \in \mathbb{N}\}$. In correspondence with i_2 there are $k_2 = k(i_2) \in A$, $k_2 > i_2$, and $j_2 \in K$ such that

$$|x_{i_2,j_2} - x_{k_2,j_2}| \not\leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}. \tag{32}$$

Note that, by construction, $j_2 > s_1$. Let $s_2 := s(i_2, k_2) \in K$ be as in (23): without loss of generality, we can choose $s_2 > j_2$. We get

$$|x_{i_1,j} - x_{k_1,j}| \vee |x_{i_2,j} - x_{k_2,j}| \leq \bigvee_{t=1}^{\infty} b_{t,\varphi(t+2)}$$

whenever $j \geq s_2$, $j \in K$.

Proceeding by induction, we get the existence of four strictly increasing sequences: $(i_r)_r$ and $(k_r)_r$ in A; $(j_r)_r$ and $(s_r)_r$ in K, with the properties that $i_r < k_r < i_{r+1}$, $j_r < s_r < j_{r+1}$ for all $r \in \mathbb{N}$; $i_r \in \{k(i) : \ i \in \mathbb{N}\}$ for any $r \geq 2$, and:

j) $\sum_{\substack{j \in K \\ j=1}}^{s_{r-1}} |x_{i_r,j} - x_{k_r,j}| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$;

jj) $|x_{i_r,j_r} - x_{k_r,j_r}| \not\leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}$;

jjj) $|x_{i_r,j_{r+h}} - x_{k_r,j_{r+h}}| \leq \bigvee_{t=1}^{\infty} b_{t,\varphi(t+h)}$ for all $r \geq 2$ and $h \in \mathbb{N}$.

By virtue of (iii), in correspondence with $B := \{j_r : \ r \geq 2\}$ there exist an infinite set $C \subset B$ and a natural number n_0 such that

$$\left| (\mathcal{I}) \sum_{j \in C} (x_{i_r,j} - x_{k_r,j}) \right| \leq \bigvee_{t=1}^{\infty} d^*_{t,\varphi(t)} \tag{33}$$

for all $r \geq n_0$. From j), jj), jjj), if $s > r \geq n_0$ and $j_r \in C$, then we get:

$$\begin{aligned} |x_{i_r,j_r} - x_{k_r,j_r}| \leq & \Big| \sum_{\substack{j \in C \\ j \in \{j_1,\dots,j_s\}}} (x_{i_r,j} - x_{k_r,j}) \Big| \\ & + \sum_{\substack{j \in C \\ j \in \{j_1,\dots j_{r-1}\}}} |x_{i_r,j} - x_{k_r,j}| + \sum_{\substack{j \in C \\ j \in \{j_{r+1},\dots,j_s\}}} |x_{i_r,j} - x_{k_r,j}| \qquad (34) \\ \leq & \Big| \sum_{\substack{j \in C \\ j \in \{j_1,\dots,j_s\}}} (x_{i_r,j} - x_{k_r,j}) \Big| + \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} + \sum_{h=1}^{s} \Big(\bigvee_{t=1}^{\infty} b_{t,\varphi(t+h)} \Big). \end{aligned}$$

By passing to the $(O\mathcal{I})$-limit as s tends to $+\infty$ in (34), and taking into account Proposition 2.6, we obtain

$$\begin{aligned} |x_{i_r,j_r} - x_{k_r,j_r}| & \leq \Big| (\mathcal{I}) \sum_{j \in C} (x_{i_r,j} - x_{k_r,j}) \Big| \\ & \leq + \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} + \bigvee_{s=1}^{\infty} \Big(\sum_{h=1}^{s} \Big(\bigvee_{t=1}^{\infty} b_{t,\varphi(t+h)} \Big) \Big) \\ & \leq \bigvee_{t=1}^{\infty} d^*_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} + \bigvee_{s=1}^{\infty} \Big(\sum_{h=1}^{s} \Big(\bigvee_{t=1}^{\infty} b_{t,\varphi(t+h)} \Big) \Big), \end{aligned}$$

that is

$$|x_{i_r,j_r} - x_{k_r,j_r}| - \bigvee_{t=1}^{\infty} d^*_{t,\varphi(t)} - \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \leq \bigvee_{s=1}^{\infty} \Big(\sum_{h=1}^{s} \Big(\bigvee_{t=1}^{\infty} b_{t,\varphi(t+h)} \Big) \Big). \qquad (35)$$

We have also

$$\begin{aligned} & |x_{i_r,j_r} - x_{k_r,j_r}| - \bigvee_{t=1}^{\infty} d^*_{t,\varphi(t)} - \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \\ \leq \; & |x_{i_r,j_r} - x_{k_r,j_r}| \leq 2u. \end{aligned} \qquad (36)$$

From (24), (35) and (36) it follows that

$$|x_{i_r,j_r} - x_{k_r,j_r}| - \bigvee_{t=1}^{\infty} d^*_{t,\varphi(t)} - \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \leq \bigvee_{t=1}^{\infty} b^*_{t,\varphi(t)}, \qquad (37)$$

and finally, if $r \geq n_0$ and $j_r \in C$, then we have:

$$|x_{i_r,j_r} - x_{k_r,j_r}| \leq \bigvee_{t=1}^{\infty} d^*_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} b^*_{t,\varphi(t)} \leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}. \qquad (38)$$

So (38) holds for infinitely many indexes r. This contradicts (29) and proves the claim (28).

We now prove (27). From (28) it follows that the family $\{(x_{i,j})_{i\in A} : j \in K\}$ is $(D\mathcal{I}_{\rm fin})$-Cauchy uniformly with respect to $j \in K$. Thus (27) follows from the last part of Proposition 2.14, since $(D\mathcal{I}_{\rm fin})$ convergence coincides with usual (D)-convergence. This ends the proof of (I).

(II) We have just proved that $(D\mathcal{I}) \lim_i \big[\bigvee_{j\in K} |x_{i,j} - x_j| \big] = 0$, and by (ii) we know that $(D\mathcal{I}) \lim_j x_{i,j} = 0$ for every $i \in \mathbb{N}$. Thus by (iii), (iv) and (vii) of Lemma 2.15, interchanging the role of the variables i and j, we get that $(D\mathcal{I}) \lim_j x_j = 0$, that is (II).

(III) It is an immediate consequence of (I), (II) and Lemma 2.15.

(IV) It follows from (I) and (vi) of Lemma 2.15.

(V) In the proof of (I) we proved the existence of $A, K \in \mathcal{F}$ and a regulator $(c_{t,l})_{t,l}$ such that

$$\text{(D)} \lim_{i\in A} \Big[\bigvee_{j\in K} |x_{i,j} - x_j| \Big] = 0 \tag{39}$$

with respect to the (D)-sequence $(c_{t,l})_{t,l}$. Moreover, by (II), $0 = (D\mathcal{I}) \lim_j x_j = (O\mathcal{I}) \lim_j x_j$. Since $\mathcal{I}$ is a P-ideal, by Proposition 2.8 we get: $0 = (O\mathcal{I}^*) \lim_j x_j = (D\mathcal{I}^*) \lim_j x_j$. Thus a set $K_0 \in \mathcal{F}(\mathcal{I})$ can be found, with $\text{(D)} \lim_{j\in K_0} x_j = 0$. Let $K' := K \cap K_0$: then $K' \in \mathcal{F}(\mathcal{I})$. In order to prove the assertion, thanks to Proposition 2.11 it is enough to show that

$$\text{(D)} \lim_{j\in K'} \Big[\bigvee_{i\in A} |x_{i,j}| \Big] = 0. \tag{40}$$

To this aim observe that by (39), in correspondence with $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $\overline{i} \in A$ with

$$|x_{i,j} - x_j| \leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}$$

whenever $i \in A$, $i \geq \overline{i}$ and $j \in K$ (and a fortiori $j \in K'$). Since $\text{(D)} \lim_{j\in K_0} x_j = 0$, there is a regulator $(\xi_{t,l})_{t,l}$ with the property that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds $\overline{j} \in K_0$ such that

$$|x_j| \leq \bigvee_{t=1}^{\infty} \xi_{t,\varphi(t)}$$

for all $j \geq \overline{j}$, $j \in K_0$ (and a fortiori $j \in K'$). Note that, without loss of generality, the integer $\overline{j}$ can be taken in K'.

Since (ii) holds, proceeding analogously as in the proof of (I) we get

$$(\mathrm{D})\lim_{j\in K'} x_{i,j}=0$$

for all $i\in\mathbb{N}$, with respect to a same regulator $(\beta_{t,l})_{t,l}$. From this it follows that for every $\varphi\in\mathbb{N}^{\mathbb{N}}$ and $i=1,\dots,\overline{i}-1$, $i\in A$, there exists $j_i\in K'$ with

$$|x_{i,j}|\le\bigvee_{t=1}^{\infty}\beta_{t,\varphi(t)}$$

whenever $j\ge j_i$, $j\in K'$.

Let now $j^*:=\max\{\overline{j},\max_{\substack{i=1,\dots,\overline{i}-1,\\ i\in A}} j_i\}$, and choose arbitrarily $i\in A$, $j\in K'$, $j\ge j^*$. If $i\ge\overline{i}$, then

$$\begin{aligned}|x_{i,j}| &\le |x_{i,j}-x_j|+|x_j|\\ &\le \bigvee_{t=1}^{\infty}c_{t,\varphi(t)}+\bigvee_{t=1}^{\infty}\xi_{t,\varphi(t)}\\ &\le \bigvee_{t=1}^{\infty}c_{t,\varphi(t)}+\bigvee_{t=1}^{\infty}\xi_{t,\varphi(t)}+\bigvee_{t=1}^{\infty}\beta_{t,\varphi(t)}.\end{aligned}$$

If $i\le\overline{i}-1$, then

$$|x_{i,j}|\le\bigvee_{t=1}^{\infty}\beta_{t,\varphi(t)}\le\bigvee_{t=1}^{\infty}c_{t,\varphi(t)}+\bigvee_{t=1}^{\infty}\xi_{t,\varphi(t)}+\bigvee_{t=1}^{\infty}\beta_{t,\varphi(t)}.$$

This proves the claim (40) and hence (V), and completes the proof of the theorem. □

Remark 3.2. Theorem 3.1 is an extension to the context of (ℓ)-groups and P-ideals of [1, Theorem 4], which was formulated for normed spaces and $\mathcal{I}=\mathcal{I}_d$.

Furthermore observe that, if in the hypotheses of Theorem 3.1 we keep (i) and (iii), *fix* $K\in\mathcal{F}$ and replace (ii) with the condition

$$(\mathrm{D})\lim_{\substack{j\to+\infty\\ j\in K}} x_{i,j}=0 \tag{41}$$

for all $i\in\mathbb{N}$ (without loss of generality with respect to a same regulator, thanks to Proposition 2.12), then the thesis of the theorem continues to hold, and the set K for which (I) is satisfied is just the element K of $\mathcal{F}$ fixed *a priori* in (41): indeed, it will be enough to repeat the same arguments of the proof of Theorem 3.1.

In particular, if we take $K = \mathbb{N}$, (ii) becomes

(ii') $(D)\lim_{j} x_{i,j} = 0$ for all $i \in \mathbb{N}$

(without loss of generality, with respect to a same regulator). Note that, by arguing analogously as in the proof of Theorem 3.1 it is possible to prove that (i), (ii') and (iii) imply that

(I') $(D\mathcal{I})\lim_{i}\big[\bigvee_{j\in\mathbb{N}} |x_{i,j} - x_j|\big] = 0.$

Similarly, if in 3.1 we keep (ii) and (iii), fix $A \in \mathcal{F}$ and replace (i) with

$$(\mathrm{D})\lim_{\substack{i\to+\infty\\ i\in A}} x_{i,j} = x_j$$

for all $j \in \mathbb{N}$ (again without loss of generality with respect to a same regulator), then the set A for which (V) holds is just the mentioned element A of $\mathcal{F}$. In particular, if we choose $A = \mathbb{N}$, (i) becomes

(i') $(D)\lim_{i} x_{i,j} = x_j$ exists in R for all $j \in \mathbb{N}$

(without loss of generality, with respect to a same regulator). Note that, by proceeding analogously as in the proof of Theorem 3.1, we can prove that (i'), (ii) and (iii) imply:

(V') $(D\mathcal{I})\lim_{j}\big[\bigvee_{i\in\mathbb{N}} |x_{i,j}|\big] = 0.$

Remark 3.3. We now claim that Theorem 3.1 holds (with $K = \mathbb{N}$) even if we assume (i), (ii') and replace condition (iii) with the following hypothesis:

(iii') *there exists a regulator $(d_{t,l})_{t,l}$ such that for every strictly increasing sequence $(n_h)_h$ in $\mathbb{N}$ the sequence*

$$\Big((\mathcal{I})\sum_{j=1}^{\infty} x_{n_h,j}\Big)_h$$

$(D\mathcal{I})$-converges (with respect to the same regulator $(d_{t,l})_{t,l}$).

We now sketch only the proof of (I), since the proof of the other parts is similar as above.

Let us define the sequence $(k_i)_i$ by setting $k_i = k(i)$, $i \in \mathbb{N}$, where $k(i)$ is as in (29). By (iii') and Proposition 2.9 there is a subsequence $(k_{i_s})_s$ of $(k_i)_i$ such that the sequence

$$\Big((\mathcal{I})\sum_{j=1}^{\infty} x_{k_{i_s},j}\Big)_s$$

(D)-converges (with respect to the regulator $(d_{t,l})_{t,l}$). Let $d^*_{t,l} := 2d_{t,l}$, $t, l \in \mathbb{N}$. In the argument leading to a contradiction, we take the natural numbers i_r, k_r, in such a way that $i_r \in \{k_{i_s} : s \in \mathbb{N}\}$ for each $r \geq 2$ and $k_r = k(i_r)$ for any $r \in \mathbb{N}$. From (iii') and the particular choice of the i_r's, k_r's it follows that a positive integer n_0 can be found, such that

$$\Big|(\mathcal{I})\sum_{j=1}^{\infty}(x_{i_r,j} - x_{k_r,j})\Big| \leq \bigvee_{t=1}^{\infty} d^*_{t,\varphi(t)}$$

for all $r \geq n_0$. So, if $s > r \geq n_0$ and $j \in C_r$, then we have

$$\begin{aligned} |x_{i_r,j_r} - x_{k_r,j_r}| &\leq \Big|\sum_{j=1}^{s}(x_{i_r,j} - x_{k_r,j})\Big| + \sum_{j\in\{j_1,\dots,j_{r-1}\}} |x_{i_r,j} - x_{k_r,j}| \\ &\quad + \sum_{j\in\{j_{r+1},\dots,s\}} |x_{i_r,j} - x_{k_r,j}|. \end{aligned}$$

Arguing analogously as in (34)–(38), we obtain

$$|x_{i_r,j_r} - x_{k_r,j_r}| \leq \bigvee_{t=1}^{\infty} d^*_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} b^*_{t,\varphi(t)} \leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}, \tag{42}$$

getting a contradiction with (29) and proving the claim.

It is not difficult to find an example of bounded real-valued double sequence $(x_{i,j})_{i,j}$, such that for every $B \subset \mathbb{N}$ and for each strictly increasing sequence $(n_h)_h$ the sequence $\Big(\sum_{j\in B} x_{n_h,j}\Big)_h$ is bounded. If $\mathcal{I}$ is maximal, then such sequences admit always $\mathcal{I}$-limit (see also [5]).

Example 3.4. In general, even when $R = \mathbb{R}$, if we drop (iii) or (iii'), conditions (i') and (ii') do not imply (V').

Indeed, let $\mathcal{I}$ be *any* admissible ideal, different from $\mathcal{I}_{\text{fin}}$, and let

$$H := \{h_1 < \dots < h_j < h_{j+1} < \dots\}$$

be an infinite set belonging to $\mathcal{I}$: since $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$, H does exist. For every $j \in \mathbb{N}$, let us define $x_{h_j,j} := 1$; for the other choices of i and j, put $x_{i,j} := 0$. As $x_{i,j} = 0$ whenever $j \in \mathbb{N}$ and $i \in \mathbb{N} \setminus H$, and since $H \in \mathcal{I}$, then $\lim_i x_{i,j} = 0$ for all $j \in \mathbb{N}$ and $\lim_j x_{i,j} = 0$ for all $i \in \mathbb{N}$. So, the hypotheses (i') and (ii') related to Theorem 3.1 (see Remark 3.2) are satisfied.

However, $(x_{i,j})_{i,j}$ does not fulfil (V'), since $\sup_{i\in\mathbb{N}} |x_{i,j}| = 1$ for each $j \in \mathbb{N}$.

Furthermore, neither (iii) nor (iii') hold, since for *each* strictly increasing sequence $(j_s)_s$ in $\mathbb{N}$ we get: $\sum_{s=1}^{\infty} x_{h_{j_s},j_s} = +\infty$.

Definitions 3.5. Let R, $\{(x_{n,\lambda})_n : \lambda \in \Lambda\}$, $\{x_\lambda : \lambda \in \Lambda\}$, $(q_n)_n$, φ and $(a_{t,l})_{t,l}$ be as in Definitions 2.1. We now state the following:

a) $x_{n,\lambda} \xrightarrow{\#\text{-}u} x_\lambda$ if there exists an $n_0 \in \mathbb{N}$ (depending on the sequence $(q_n)_n$) such that $\#\big(\{n \in \mathbb{N} :\ x_{n,\lambda} - x_\lambda \notin [-q_n, q_n]\}\big) \leq n_0$, for every $\lambda \in \Lambda$, where the symbol $\#$ denotes the cardinality of the set in brackets.

b) $x_{n,\lambda} \xrightarrow{\#\text{-}D\text{-}u} x_\lambda$ if to every φ corresponds an $n_0 \in \mathbb{N}$ such that

$$\#\Big(\Big\{n \in \mathbb{N} :\ x_{n,\lambda} - x_\lambda \notin \Big[-\bigvee_{t=1}^{\infty} a_{t,\varphi(t)}, \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\Big]\Big\}\Big) \leq n_0,$$

for every $\lambda \in \Lambda$.

Remarks 3.6. Obviously if $(O)\lim_n x_{n,\lambda} = x_\lambda$ uniformly with respect to $\lambda \in \Lambda$ then also $x_{n,\lambda} \xrightarrow{\#\text{-}u} x_\lambda$. Thus the $\#$-u-convergence is weaker than the (O)-uniform convergence.

Similarly $\#$-D-u-convergence is weaker than (D)-uniform convergence.

Moreover it is easy to see that if R is weakly σ-distributive then $\#$-u-convergence and $\#$-D-u-convergence coincide.

Open Problems.

(a) Is $\#$-u-convergence strictly weaker than the (O)-uniform convergence?

(b) Is $\#$-D-u-convergence strictly weaker than (D)-uniform convergence?

(c) Formulate and prove new basic matrix theorems of Antosík-Swartz-type using these new types of convergence.

(d) Formulate definitions similar to Definitions 3.5 in the ideal case and investigate problems analogous to (a), (b) and (c) in this context.

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