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RESIDUAL FINITENESS, SUBGROUP SEPARABILITY AND CONJUGACY SEPARABILITY OF CERTAIN HNN EXTENSIONS

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ABSTRACT. In this note we shall give characterisations for HNN extensions of non-cyclic polycyclic-by-finite groups with normal infinite cyclic associated subgroups to be residually finite, subgroup separable and conjugacy separable.

1. Introduction

Let $G = \langle t, A; t^{-1}ht = k\rangle$ be an HNN extension where A is the base group and $\langle h\rangle$, $\langle k\rangle$ are cyclic associated subgroups. The residual properties of these HNN extensions G are difficult to determine in general. They seem to depend on the nature of the base group A and as well as the properties of h, k. Indeed one of the simplest type of these HNN extensions, the Baumslag-Solitar group, $G = \langle h, t; t^{-1}h^2t = h^3\rangle$, where the base group A is infinite cyclic, is not even residually finite. (See G. Baumslag and Solitar [2].)

However some important positive results had been obtained. Kim in [9] and Kim and Tang in [10], [11], [12] and [13] proved criteria for the residual finiteness, cyclic subgroup separability and conjugacy separability of these HNN extensions G. Applying their criteria to non-cyclic finitely generated abelian or non-cyclic finitely generated nilpotent or non-cyclic polycyclic-by-finite groups, they obtained the following results.

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If A is a non-cyclic finitely generated abelian group and h, k are elements of infinite order, then G is residually finite (conjugacy separable) if and only if $\langle h\rangle \cap \langle k\rangle = 1$ or $h^n = k^{\pm n}$ for some $n \geq 1$ ([11, Theorem 2.16] and [10, Theorem 3.6]). Thus these HNN extensions of finitely generated abelian groups are conjugacy separable if and only if they are residually finite. In addition they showed that if A is a non-cyclic finitely generated nilpotent group and h, k are elements of infinite order in the center of A, then G is cyclic subgroup separable if and only if $\langle h\rangle \cap \langle k\rangle = 1$ or $h^n = k^{\pm n}$ for some $n \geq 1$ ([11, Corollary 2.12]). Furthermore if A is a non-cyclic polycyclic-by-finite group and h, k are elements of infinite order in the center of A, then G is conjugacy separable if and only if $\langle h\rangle \cap \langle k\rangle = 1$ or $h^n = k^{\pm n}$ for some $n \geq 1$ ([12, Corollary 4.6] and [13, Corollary 3.12]).

So it is quite natural to ask if similar characterizations hold when A is a non-cyclic polycyclic-by-finite group, and $\langle h\rangle$ and $\langle k\rangle$ are normal infinite cyclic subgroups of A. We will give affirmative answers in Corollary 3.5, 3.7 and 5.4. These improve the results of Kim and Tang.

On the other hand, subgroup separability is a rather strong residually property and as such only few classes of groups have this property. However a characterisation for the subgroup separability of HNN extensions of finitely generated abelian groups was given by Metaftsis and Raptis in [15, Theorem 2]. This characterization was extended to HNN extensions of polycyclic-by-finite groups with central associated subgroups by Wong and Wong in [21, Corollary 3.2]. Applying Metaftsis and Raptis's result (or Wong and Wong's result), we will give a characterisation for subgroup separability when A is a non-cyclic polycyclic-by-finite group, and $\langle h\rangle$ and $\langle k\rangle$ are normal infinite cyclic subgroups of A (Corollary 4.4).

We shall begin by giving characterisations for these HNN extensions where the base group A is non-cyclic and subgroup separable. This will be achieved in Theorems 3.4, 3.6, 4.3 and 5.3.

The notation used here is standard. In addition, the following will be used for any group G:

$N \triangleleft_f G$ means N is a normal subgroup of finite index in G.

$G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ denotes an HNN extension where A is the base group, H, K are the associated subgroups and $\varphi\colon H \longrightarrow K$ is the associated isomorphism.

If $H = \langle h\rangle$ and $K = \langle k\rangle$, we write $G = \langle t, A; t^{-1}ht = k\rangle$.

$\|w\|$ denotes the t-length of w in the HNN extension G.

2. Preliminaries

We begin by recalling all the definitions.

Definition 2.1. A subset S of G is said to be *separable* if for each $x \in G \backslash S$, there exists $N \lhd_f G$ such that $x \notin SN$.

G is termed *subgroup separable* if every finitely generated subgroup of G is separable.

G is termed *cyclic subgroup separable* (or π_c for short) if every cyclic subgroup of G is separable.

Definition 2.2. A group G is said to be *conjugacy separable* if for each pair of elements $x, y \in G$ such that x and y are not conjugate in G, there exists $N \lhd_f G$, such that Nx and Ny are not conjugate in G/N.

Many families of groups, including the free groups, polycyclic groups and surface groups, are known to be subgroup separable and conjugacy separable. (See Blackburn [3], M. Hall [8], Mal'cev [14], Scott [19], [20].) Finite extensions of subgroup separable groups are again subgroup separable. Hence polycyclic-by-finite groups, free-by-finite groups and Fuchsian groups are subgroup separable and hence residually finite. In addition polycyclic-by-finite groups, free-by-finite groups and Fuchsian groups are conjugacy separable. (See Dyer [5], Fine and Rosenberger [6], Formanek [7] and Remeslennikov [18].)

3. Residual finiteness and cyclic subgroup separability

In this section we give characterizations for residual finiteness and cyclic subgroup separability. This will be achieved in Theorem 3.4, Theorem 3.6, Corollary 3.5 and Corollary 3.7. We begin with a lemma on subgroup separability which can be easily proved.

Lemma 3.1. *Let A be a subgroup separable group and H is a finitely generated normal subgroup of A. Then A/H is subgroup separable.*

Lemma 3.2. *Let A be a group and $\langle h \rangle$ be a normal infinite cyclic subgroup of A and $k \in A$. Suppose $\langle h \rangle \cap \langle k \rangle = \langle h^i \rangle = \langle k^j \rangle$ where $i, j \geq 1$. Then $hk = kh$.*

Proof. Since $k^{-1}\langle h \rangle k = \langle h \rangle$, we have $k^{-1}hk = h^{\pm 1}$. If $k^{-1}hk = h^{-1}$, then $k^{-1}h^i k = h^{-i}$ and hence $h^{2i} = 1$, for $h^i = k^{\pm j}$. This contradicts the fact that $\langle h \rangle$ is infinite cyclic. Hence $k^{-1}hk = h$ or equivalently $hk = kh$. $\square$

Lemma 3.3. *Let A be a subgroup separable group and $\langle h\rangle$ and $\langle k\rangle$ be normal infinite cyclic subgroups of A.*

(a) *If $\langle h\rangle \cap \langle k\rangle = 1$, then given any integer $\epsilon > 0$, there exists $M \lhd_f A$ such that $M \cap \langle h\rangle = \langle h^\epsilon\rangle$, $M \cap \langle k\rangle = \langle k^\epsilon\rangle$ and $M\langle h\rangle \cap M\langle k\rangle = M$.*

(b) *If $h^n = k^{\pm n}$ for some $n \geq 1$, then given any integer $\epsilon > 0$, there exists $M \lhd_f A$ such that $M \cap \langle h\rangle = \langle h^{n\epsilon}\rangle$, $M \cap \langle k\rangle = \langle k^{n\epsilon}\rangle$ and $M\langle h\rangle \cap M\langle k\rangle = M\langle h^n\rangle$.*

Proof.

(a) Suppose that $\langle h\rangle\cap\langle k\rangle = 1$. Let $\epsilon > 0$ be given. Note that $\langle h^\epsilon\rangle\langle k^\epsilon\rangle \lhd A$ and hence by Lemma 3.1, $\overline{A} = A/(\langle h^\epsilon\rangle\langle k^\epsilon\rangle)$ is subgroup separable. Since $\overline{\langle h\rangle\langle k\rangle} = \langle h\rangle\ \langle k\rangle/(\langle h^\epsilon\rangle\langle k^\epsilon\rangle)$ is finite, there exists $\overline{M} \lhd_f \overline{A}$ such that $\overline{M} \cap \overline{\langle h\rangle\langle k\rangle} = 1$. Let M be the preimage of $\overline{M}$. We claim that $M \cap \langle h\rangle = \langle h^\epsilon\rangle$, $M \cap \langle k\rangle = \langle k^\epsilon\rangle$ and $M\langle h\rangle \cap M\langle k\rangle = M$.

First we show that $M \cap \langle h\rangle = \langle h^\epsilon\rangle$. Clearly, $\langle h^\epsilon\rangle \subseteq M \cap \langle h\rangle$. Let $h^r \in M$ for some $r \in \mathbb{Z}$. Then $\overline{h}^r \in \overline{M} \cap \overline{\langle h\rangle\langle k\rangle} = 1$. This implies that $h^r \in \langle h^\epsilon\rangle\langle k^\epsilon\rangle$. Since $\langle k\rangle \cap \langle h\rangle = 1$ and $\langle h\rangle$ has infinite order, r must be a multiple of ϵ. So $M \cap \langle h\rangle \subseteq \langle h^\epsilon\rangle$. Hence $M \cap \langle h\rangle = \langle h^\epsilon\rangle$. Similarly one can show $M \cap \langle k\rangle = \langle k^\epsilon\rangle$.

Next we show that $M\langle h\rangle \cap M\langle k\rangle = M$. Clearly $M \subseteq M\langle h\rangle \cap M\langle k\rangle$. Let $y \in M\langle h\rangle \cap M\langle k\rangle$. Then $y = n_1h^{s_1} = n_2k^{s_2}$ for some $n_1, n_2 \in M$, $s_1, s_2 \in \mathbb{Z}$. This implies that $h^{s_1}k^{-s_2} = n_1^{-1}n_2 \in M$ and that $\overline{h}^{s_1}\overline{k}^{-s_2} \in \overline{M} \cap \overline{\langle h\rangle\langle k\rangle} = 1$. Therefore $h^{s_1}k^{-s_2} \in \langle h^\epsilon\rangle\langle k^\epsilon\rangle$. Since $\langle k\rangle \cap \langle h\rangle = 1$ and $\langle h\rangle$ has infinite order, s_1 must be multiple of ϵ. Thus $y \in M$, for $\langle h^\epsilon\rangle \subseteq M$. Hence $M\langle h\rangle \cap M\langle k\rangle = M$.

(b) Suppose that $h^n = k^{\pm n}$ for some $n \geq 1$. Let $\epsilon > 0$ be given. Note that $\langle h^{n\epsilon}\rangle \lhd A$ and hence by Lemma 3.1, $\overline{A} = A/(\langle h^{n\epsilon}\rangle)$ is subgroup separable. Since $\overline{\langle h\rangle} = \langle h\rangle/(\langle h^{n\epsilon}\rangle)$ and $\overline{\langle k\rangle} = \langle k\rangle/(\langle h^{n\epsilon}\rangle)$ are finite, there exists $\overline{M} \lhd_f \overline{A}$ such that $\overline{M} \cap \overline{\langle h\rangle}\ \overline{\langle k\rangle} = 1$. Let M be the preimage of $\overline{N}$. We can proceed as in part (a) above to show that $M \cap \langle h\rangle = \langle h^{n\epsilon}\rangle$, $M \cap \langle k\rangle = \langle k^{n\epsilon}\rangle$ and $M\langle h\rangle \cap M\langle k\rangle = M\langle h^n\rangle$ and the lemma follows. □

We now prove the main criterion of this section.

Theorem 3.4. *Let $G = \langle t, A; t^{-1}ht = k\rangle$ be an HNN extension where $\langle h\rangle$ and $\langle k\rangle$ are normal infinite cyclic subgroups of A. Suppose A is non-cyclic and subgroup separable. Then G is residually finite if and only if $\langle h\rangle \cap \langle k\rangle = 1$ or $h^n = k^{\pm n}$ for some $n \geq 1$.*

P r o o f.

($\Longrightarrow$) Suppose that G is residually finite. If $\langle h\rangle \cap \langle k\rangle = 1$, we are done. Suppose $\langle h\rangle \cap \langle k\rangle = \langle h^i\rangle = \langle k^j\rangle$ where $i, j \geq 1$.

Suppose $j = 1$. We shall show that $i = 1$. Assume $i > 1$. Then $k = h^{\pm i}$. Since A is not cyclic, $A \neq \langle h\rangle$. Let $c \in A - \langle h\rangle$. Then $c^{-1}hc = h^{\pm 1}$. Suppose that $c^{-1}hc = h^{-1}$. Let $w = c^{-1}tht^{-1}ctht^{-1}$. Then $w \neq 1$ since $\|w\| = 4$ in G. Since G is residually finite, there exists $N \lhd_f G$ such that $w \notin N$. Let $\overline{G} = G/N$.

We claim that $\langle \overline{h}\rangle = \langle \overline{k}\rangle$ in $\overline{G}$. Now $\langle \overline{k}\rangle \subseteq \langle \overline{h}\rangle$ since $k = h^{\pm i}$. Furthermore $\overline{h}$ and $\overline{k}$ have the same order in $\overline{G}$ (for $\overline{t^{-1}ht} = \overline{k}$). Thus $\langle \overline{h}\rangle = \langle \overline{k}\rangle$ and hence $\overline{tht^{-1}} = \overline{h}^r$ for some $r \in \mathbb{Z}$. Therefore $\overline{w} = \overline{c^{-1}tht^{-1}ctht^{-1}} = \overline{c}^{-1}\overline{h}^r\overline{c}\overline{h}^r = \overline{h}^{-r}\overline{h}^r = 1$, a contradiction.

Suppose $c^{-1}hc = h$. In this case we can use $w = c^{-1}th^{-1}t^{-1}ctht^{-1} \neq 1$ to obtain a contradiction. Hence $i = 1$. Similarly if $i = 1$ then $j = 1$.

Now let $i, j \geq 2$. Let B be the subgroup generated by $\langle h\rangle$ and $\langle k\rangle$. By Lemma 3.2, $hk = kh$. Therefore B is finitely generated abelian and $\langle h\rangle \neq B \neq \langle k\rangle$. Let $G_1 = \langle t, B; t^{-1}ht = k\rangle$. Then G_1 is a subgroup of G and therefore is residually finite. By [11, Theorem 2.16], (see also [1, Theorem]), $h^n = k^{\pm n}$ for some $n \geq 1$.

($\Longleftarrow$) Suppose $\langle h\rangle \cap \langle k\rangle = 1$ or $h^n = k^{\pm n}$ for some $n \geq 1$. By [11, Theorem 2.9] and Lemma 3.3 (see also [1, Theorem]), G is π_c. Thus G is residually finite. $\square$

Since polycyclic-by-finite groups are subgroup separable, we have the following corollary.

Corollary 3.5. *Let $G = \langle t, A; t^{-1}ht = k\rangle$ be an HNN extension where $\langle h\rangle$ and $\langle k\rangle$ are normal infinite cyclic subgroups of A. Suppose A is a non-cyclic polycyclic-by-finite group. Then G is residually finite if and only if $\langle h\rangle \cap \langle k\rangle = 1$ or $h^n = k^{\pm n}$ for some $n \geq 1$.*

By a careful examination of its proof Theorem 3.4 (and Corollary 3.5), it is not hard to see that it can be strengthened to the following theorem and corollary.

Theorem 3.6. *Let $G = \langle t, A; t^{-1}ht = k\rangle$ be an HNN extension where $\langle h\rangle$ and $\langle k\rangle$ are normal infinite cyclic subgroups of A. Suppose A is non-cyclic and subgroup separable. Then G is π_c if and only if $\langle h\rangle \cap \langle k\rangle = 1$ or $h^n = k^{\pm n}$ for some $n \geq 1$.*

Corollary 3.7. *Let $G = \langle t, A; t^{-1}ht = k\rangle$ be an HNN extension where $\langle h\rangle$ and $\langle k\rangle$ are normal infinite cyclic subgroups of A. Suppose A is a non-cyclic polycyclic-by-finite group. Then G is π_c if and only if $\langle h\rangle \cap \langle k\rangle = 1$ or $h^n = k^{\pm n}$ for some $n \geq 1$.*

4. Subgroup separability

In this section we give a characterization for subgroup separability. This will be achieved in Theorem 4.3 and Corollary 4.4.

Lemma 4.1. ([16, Lemma 2]) *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension. If A is subgroup separable and H and K are finite, then G is subgroup separable.*

The following lemma can be deduced from [15, Lemma 2].

Lemma 4.2. *Let $G = \langle t, A; t^{-1}ht = k\rangle$ be an HNN extension where $\langle h\rangle$ and $\langle k\rangle$ are normal infinite cyclic subgroups of A. Suppose A is non-cyclic and subgroup separable. If $h^n = k^{\pm n}$ for some, $n \geq 1$ then G is subgroup separable.*

We now prove the main criterion of this section.

Theorem 4.3. *Let $G = \langle t, A; t^{-1}ht = k\rangle$ be an HNN extension where $\langle h\rangle$ and $\langle k\rangle$ are normal infinite cyclic subgroups of A. Suppose A is non-cyclic. Then G is subgroup separable if and only if A is subgroup separable and $h^n = k^{\pm n}$ for some $n \geq 1$.*

Proof. Suppose that G is subgroup separable. Then G is π_c and A is subgroup separable. By Theorem 3.6, $\langle h\rangle \cap \langle k\rangle = 1$ or $h^n = k^{\pm n}$ for some $n \geq 1$.

Suppose that $\langle h\rangle \cap \langle k\rangle = 1$. Then the subgroup generated by $\langle h\rangle$ and $\langle k\rangle$ is the direct product $\langle h\rangle \times \langle k\rangle = Q$. Since $\langle h\rangle \neq Q \neq \langle k\rangle$, $G_1 = \langle t, Q; t^{-1}ht = k\rangle$ is a subgroup of G. Hence G_1 is subgroup separable. By [15, Theorem 2], $\langle h\rangle \cap \langle k\rangle$ must be a subgroup of finite index in $\langle h\rangle$. But $\langle h\rangle \cap \langle k\rangle = 1$. This implies that $\langle h\rangle$ is finite, a contradiction. Hence $h^n = k^{\pm n}$ for some $n \geq 1$.

The converse follows from Lemma 4.2. □

The following corollary follows from Theorem 4.3.

Corollary 4.4. *Let $G = \langle t, A; t^{-1}ht = k\rangle$ be an HNN extension where $\langle h\rangle$ and $\langle k\rangle$ are normal infinite cyclic subgroups of A. Suppose A is a non-cyclic polycyclic-by-finite group. Then G is subgroup separable if and only if $h^n = k^{\pm n}$ for some $n \geq 1$.*

5. Conjugacy separability

In this section we give a characterization for conjugacy separability. This will be achieved in Theorem 5.3 and Corollary 5.4.

Definition 5.1. A group A is said to be *double coset separable* at $\{h,k\}$ if, for each $u \in A$ and for each integer $\epsilon > 0$, $\langle h^\epsilon\rangle u\langle h^\epsilon\rangle$, $\langle h^\epsilon\rangle u\langle k^\epsilon\rangle$ and $\langle k^\epsilon\rangle u\langle k^\epsilon\rangle$ are separable in A.

We note here that double coset separability is similar to coset separability defined by Niblo in [17]. This property was first explored by him.

Definition 5.2. Let G be a group and $a \in G$. The conjugacy class of a will be denoted by $\{a\}^G$, that is, $\{a\}^G = \{gag^{-1} :\ g \in G\}$.

A group A is said to be *cyclic conjugacy separable* for $\langle h\rangle$ if, whenever $u \in A$ and $\{u\}^A \cap \langle h\rangle = \varnothing$, there exists $N \lhd_f A$ such that, in $\overline{A} = A/N$, $\{\overline{u}\}^{\overline{A}} \cap \langle \overline{h}\rangle = \varnothing$.

In the next theorem, we shall use the following notation:

Let G be a group and A be a subgroup of G. Let $u, v \in G$. The notation $u \sim_A v$ means $a^{-1}ua = v$ for some $a \in A$.

Theorem 5.3. *Let $G = \langle t, A; t^{-1}ht = k\rangle$ be an HNN extension where $\langle h\rangle$ and $\langle k\rangle$ are normal infinite cyclic subgroups of A. Suppose A is non-cyclic, subgroup separable and conjugacy separable. Then G is conjugacy separable if and only if $\langle h\rangle \cap \langle k\rangle = 1$ or $h^n = k^{\pm n}$ for some $n \geq 1$.*

Proof.

($\Longrightarrow$) Suppose that G is conjugacy separable. Then G is residually finite. By Theorem 3.4, we are done.

($\Longleftarrow$) By [13, Theorem 3.9], it is sufficient to show that A is double coset separable at $\{h,k\}$, A is cyclic conjugacy separable for $\langle h\rangle$ and $\langle k\rangle$, and for each $\epsilon > 0$, there is a $N_\epsilon \lhd A$ such that

(1) $N_\epsilon \cap \langle h\rangle = \langle h^\epsilon\rangle$ and $N_\epsilon \cap \langle k\rangle = \langle k^\epsilon\rangle$,

(2) $N_\epsilon\langle h\rangle \cap N_\epsilon\langle k\rangle = N_\epsilon(\langle h\rangle \cap \langle k\rangle)$,

(3) if $\overline{h}^i \sim_{\overline{A}} \overline{h}^j$ in $\overline{A} = A/N_\epsilon$, then $\overline{h}^j = \overline{h}^{\pm i}$,

(4) if $\overline{k}^i \sim_{\overline{A}} \overline{k}^j$ in $\overline{A} = A/N_\epsilon$, then $\overline{k}^j = \overline{k}^{\pm i}$,

(5) if $\langle h\rangle \cap \langle k\rangle = 1$ and $\overline{h}^i \sim_{\overline{A}} \overline{k}^j$ in $\overline{A} = A/N_\epsilon$, then $\overline{h}^i = \overline{k}^j = 1$, and if $\langle h\rangle \cap \langle k\rangle \neq 1$ and $\overline{h}^i \sim_{\overline{A}} \overline{k}^j$ in $\overline{A} = A/N_\epsilon$, then $\overline{h}^i = \overline{k}^{\pm j}$.

We note here that Conditions (1) up to (5) correspond to Conditions (1) up to (5) in [13, Definition 3.4] (where the λ_ϵ in [13] is chosen to be 1).

Since $\langle h\rangle$ and $\langle k\rangle$ are normal in A, and A is subgroup separable, it is not hard to see that A is double coset separable at $\{h,k\}$. Now the condition $\{u\}^A \cap \langle h\rangle = \varnothing$ is equivalent to $u \notin \langle h\rangle$, for $\langle h\rangle$ is normal in A. So A is subgroup separable implies that A is cyclic conjugacy separable for $\langle h\rangle$ and $\langle k\rangle$. It is now left to show that A satisfies Conditions (1) up to (5).

Suppose $\langle h\rangle \cap \langle k\rangle = 1$. Let $\epsilon > 0$ be given. By part (a) of Lemma 3.3, there exists $M \lhd_f A$ such that $M \cap \langle h\rangle = \langle h^\epsilon\rangle$, $M \cap \langle k\rangle = \langle k^\epsilon\rangle$ and $M\langle h\rangle \cap M\langle k\rangle = M$. So Conditions (1) and (2) are satisfied. Conditions (3) and (4) follow from the normality of $\langle h\rangle$ and $\langle k\rangle$ in A. For condition (5), if $\overline{h}^i \sim_{\overline{A}} \overline{k}^j$ in $\overline{A} = A/M$, then $\overline{h}^i = \overline{k}^j \in \langle \overline{h}\rangle \cap \langle \overline{k}\rangle = 1$.

Suppose $h^n = k^{\pm n}$ for some $n \geq 1$. Let $\epsilon > 0$ be given. By part (b) of Lemma 3.3, there exists $M \lhd_f A$ such that $M \cap \langle h\rangle = \langle h^{n\epsilon}\rangle$ and $M \cap \langle k\rangle = \langle k^{n\epsilon}\rangle$ and $M\langle h\rangle \cap M\langle k\rangle = M\langle h^n\rangle$. So conditions (1) and (2) are satisfied. Conditions (3) and (4) follow from the normality of $\langle h\rangle$ and $\langle k\rangle$ in A. For condition (5), if $\overline{h}^i \sim_{\overline{A}} \overline{k}^j$ in $\overline{A} = A/M$, then $\overline{h}^i = \overline{k}^j \in \langle \overline{h}\rangle \cap \langle \overline{k}\rangle = \langle \overline{h}^n\rangle = \langle \overline{k}^n\rangle$. Since the order of $\overline{k}$ is $n\epsilon$, we deduce that $j = j'n$ for some integer j'. This implies that $\overline{h}^i = \overline{k}^j = \overline{k}^{j'n} = \overline{h}^{\pm j'n} = \overline{h}^{\pm j}$. $\square$

Since a polycyclic-by-finite group is conjugacy separable and subgroup separable, we have the following corollary.

Corollary 5.4. *Let $G = \langle t, A; t^{-1}ht = k\rangle$ be an HNN extension where $\langle h\rangle$ and $\langle k\rangle$ are normal infinite cyclic subgroups of A. Suppose A is a non-cyclic polycyclic-by-finite group. Then G is conjugacy separable if and only if $\langle h\rangle \cap \langle k\rangle = 1$ or $h^n = k^{\pm n}$ for some $n \geq 1$.*

6. Residual finiteness, subgroup separability and conjugacy separability

By combining the results from Corollary 5.4 and Corollary 3.5 we have the following theorem.

Theorem 6.1. *Let $G = \langle t, A; t^{-1}ht = k\rangle$ be an HNN extension where $\langle h\rangle$ and $\langle k\rangle$ are normal infinite cyclic subgroups of A. Suppose that A is a non-cyclic polycyclic-by-finite group. Then G is conjugacy separable if and only if G is residually finite.*

If $\langle h\rangle \cap \langle k\rangle \neq 1$, we have the following stronger result which follows from Corollary 5.4, Corollary 4.4 and Corollary 3.5.

Theorem 6.2. *Let $G = \langle t, A; t^{-1}ht = k\rangle$ be an HNN extension where $\langle h\rangle$ and $\langle k\rangle$ are normal infinite cyclic subgroups of A, such that $\langle h\rangle \cap \langle k\rangle \neq 1$. Suppose A is a non-cyclic polycyclic-by-finite group. Then the following are equivalent:*

(a) *G is residually finite,*

(b) *G is subgroup separable,*

(c) *G is conjugacy separable.*

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