VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-012-0047-4
Math. Slovaca **62** (2012), No. 5, 815–828

ON COMPLETE PSEUDO-BL-ALGEBRAS. A CANTOR-BERNSTEIN TYPE THEOREM

Andrzej Walendziak* — Magdalena Wojciechowska-Rysiawa**

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. Pseudo-BL-algebras are a noncommutative extention of BL-algebras. In this paper we consider polars in pseudo-BL-algebras and the class of complete pseudo-BL-algebras. In the final section, a version of the Cantor-Bernstein theorem will be proved.

1. Introduction

BL-algebras were introduced by Hájek [11] in 1998. Chang [2] introduced MV-algebras, which are contained in the class of BL-algebras. A noncommutative extention of MV-algebras, called pseudo-MV-algebras, were introduced by Georgescu and Iorgulescu [8] and independently by Rachůnek [20]. A concept of pseudo-BL-algebras was firstly introduced by Georgescu and Iorgulescu in 2000 as a noncommutative generalization of BL-algebras and pseudo-MV-algebras. In [5] and [6], there were given basic properties of pseudo-BL-algebras. The pseudo-BL-algebras correspond to a pseudo-basic fuzzy logic (see [12] and [13]).

Complete MV-algebras are investigated in Belluce [1] and Jakubík [14]; see also [3] and [17]. In this paper we consider complete pseudo-BL-algebras and characterize some type of filters in pseudo-BL-algebras, called polars. Finally, we prove a Cantor-Bernstein type theorem.

2010 Mathematics Subject Classification: Primary 03G25, 06F05.
Keywords: pseudo-BL-algebra, filter, polar, complete pseudo-BL-algebra, Cantor-Bernstein Theorem.

The Cantor-Bernstein theorem states that, if a set X can be embedded into a set Y, and vice versa, then there is a bijection of X onto Y. The theorem was extended to σ-complete Boolean algebras by Sikorski [21] and Tarski [22]. Some theorems of Cantor-Bernstein type for (pseudo-) MV-algebras and for pseudo-effect algebras were proved in [15], [4], [16] and [7]. In the final section of this paper we generalize the Cantor-Bernstein theorem to σ-dually Brouwerian pseudo-BL-algebras.

2. Preliminaries

A pseudo-BL-algebra was introduced in [9] as an algebra $(A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following axioms for any $a, b, c \in A$:

(C1) $(A; \wedge, \vee, 0, 1)$ is a bounded lattice;

(C2) $(A; \odot, 1)$ is a monoid;

(C3) $a \odot b \leq c \iff a \leq b \to c \iff b \leq a \rightsquigarrow c$;

(C4) $a \wedge b = (a \to b) \odot a = a \odot (a \rightsquigarrow b)$;

(C5) $(a \to b) \vee (b \to a) = (a \rightsquigarrow b) \vee (b \rightsquigarrow a) = 1$.

We shall agree that $\wedge$, $\vee$ and $\odot$ have higher priority then $\to$ and $\rightsquigarrow$. If the operation $\odot$ is commutative, then a pseudo-BL-algebra is called *commutative*. It is proved in [5] that in such case the implications $\to$ and $\rightsquigarrow$ coincide. Then commutative pseudo-BL-algebras are precisely BL-algebras.

Following [5], for any pseudo-BL-algebra A, the reduct $\mathrm{L}(A) := (A; \wedge, \vee)$ is a distributive lattice. Let $a, b \in A$ $(a \leq b)$. By $[a, b]$ we denote an interval, that is, the set of all $c \in A$ for which $a \leq c \leq b$. We say that b *covers* a if $a < b$ and $[a, b] = \{a, b\}$; in this case we write $a \prec b$. An element $a \in A$ is an *atom* (resp. a *coatom*) if $0 \prec a$ (resp. $a \prec 1$). By $\mathrm{C}(A)$ we denote the set of all coatoms of A. We say that A is *atomic* if the interval $[0, a]$ contains an atom for each $a > 0$. Dually, A is *coatomic* if for every element $a < 1$ the interval $[a, 1]$ contains a coatom. A is called *atomistic* if the lattice $\mathrm{L}(A)$ is atomistic, that is, every nonzero element of A is a join of atoms.

We set $\mathbb{N} := \{0, 1, 2, \dots\}$. For any element a of a pseudo-BL-algebra A we put $a^0 = 1$ and $a^{n+1} = a^n \odot a$ for $n \in \mathbb{N}$. We define $a^- := a \to 0$ and $a^\sim := a \rightsquigarrow 0$.

Throughout this paper A will denote a pseudo-BL-algebra.

Proposition 2.1. ([5]) *The following properties hold in A for all $a, b, c \in A$:*

(a) $a \odot b \le a$ *and* $a \odot b \le b$;

(b) $a \vee b = 1 \implies a \odot b = a \wedge b$;

(c) $a \le (a^-)^\sim$ *and* $a \le (a^\sim)^-$;

(d) $a^- \odot a = a \odot a^\sim = 0$;

(e) $a \le b \implies (b^- \le a^-$ *and* $b^\sim \le a^\sim)$.

Proposition 2.2. ([5]) *Let I be a nonempty set. In any pseudo-BL-algebra A we have:*

(a) $\left(\bigvee\limits_{i\in I} b_i\right) \odot a = \bigvee\limits_{i\in I} (b_i \odot a)$,

(b) $a \odot \left(\bigvee\limits_{i\in I} b_i\right) = \bigvee\limits_{i\in I} (a \odot b_i)$,

whenever the arbitrary unions exist.

Proposition 2.3. *Let I be a nonempty set. In any pseudo-BL-algebra A we have:*

(a) $\bigwedge\limits_{i\in I} a_i^- = \left(\bigvee\limits_{i\in I} a_i\right)^-$,

(b) $\bigwedge\limits_{i\in I} a_i^\sim = \left(\bigvee\limits_{i\in I} a_i\right)^\sim$,

whenever the arbitrary unions and meets exist.

Proof.

(a): Let $a = \bigwedge\limits_{i\in I} a_i^-$ and $b = \bigvee\limits_{i\in I} a_i$. For all $i \in I$ we have $a_i \le b$, whence $b^- \le a_i^-$ by Proposition 2.1(e). Hence $b^- \le a$. Similarly, for all $i \in I$ we obtain $a \le a_i^-$. Applying Proposition 2.1(c) we get

$$a_i \le (a_i^-)^\sim \le a^\sim$$

for $i \in I$. From this we see that $b \le a^\sim$. Therefore

$$a \le (a^\sim)^- \le b^-.$$

Consequently, $a = b^-$, which proves (a).

The proof of (b) is similar. $\square$

Following [5], a filter of A is a nonempty subset F of A such that for all $a, b \in A$:

(F1) $a \in F,\ \ a \leq b \implies b \in F$;

(F2) $a, b \in F \implies a \odot b \in F$.

Let $X \subseteq A$. The filter of A generated by X will be denoted by $[X)$. For any $a \in A$, put $[a) = [\{a\})$. In [5], it is proved that $[\emptyset) = \{1\}$ and if $X \neq \emptyset$, then

$$[X) = \bigl\{a \in A :\ x_0 \odot \cdots \odot x_n \leq a \text{ for some } n \in \mathbb{N} \text{ and } x_0, \ldots, x_n \in X\bigr\}.$$

Consequently,

$$[a) = \bigl\{b \in A :\ a^n \leq b \text{ for some } n \geq 1\bigr\} \tag{1}$$

for any $a \in A$. Let us denote by $\mathrm{B}(A)$ the Boolean algebra of all complemented elements in the lattice $\mathrm{L}(A)$.

Proposition 2.4. ([10]) *The following conditions are equivalent:*

(a) $e \in \mathrm{B}(A)$;

(b) $e \odot e = e$ *and* $e = (e^{\sim})^{-} = (e^{-})^{\sim}$;

(c) $e \vee e^{-} = 1$;

(d) $e \vee e^{\sim} = 1$.

Proposition 2.5. ([10]) *Let* $e \in \mathrm{B}(A)$. *Then:*

(a) $[e) = [e, 1]$;

(b) $e \odot a = e \wedge a$ *for any* $a \in A$;

(c) $e^{\sim} = e^{-}$ *is the complement of* e.

For each $a \in A$, let the functions $\rightarrow_a : A \times A \to A$, $\rightsquigarrow_a : A \times A \to A$ and $f_a : A \to A$ be defined by: $x \rightarrow_a y = a \vee (x \to y)$, $x \rightsquigarrow_a y = a \vee (x \rightsquigarrow y)$ and $f_a(x) = a \vee x$.

Proposition 2.6. *Let* $e \in \mathrm{B}(A)$ *and let* $F := [e, 1]$. *Then:*

(a) $(F; \wedge, \vee, \odot, \rightarrow_e, \rightsquigarrow_e, e, 1)$ *is a pseudo-BL-algebra;*

(b) f_e *is a homomorphism from* A *onto* F;

(c) $\mathrm{B}(F) = F \cap \mathrm{B}(A)$;

(d) $\mathrm{C}(F) = F \cap \mathrm{C}(A)$.

P r o o f. (a), (b) and (c) follow from [10, Proposition 1.16].

(d): Let a be a coatom in F. Suppose that there is $b \in A$ such that $a < b \leq 1$. Obviously, $b \in F$. Therefore $b = 1$, and hence $a \in \mathrm{C}(A)$. From this we have $\mathrm{C}(F) \subseteq F \cap \mathrm{C}(A)$. On the other hand, $F \cap \mathrm{C}(A) \subseteq \mathrm{C}(F)$. Thus (d) holds. $\square$

The following results are obvious.

Proposition 2.7. *Let A and B be pseudo-BL-algebras and let $f\colon A \to B$ be an isomorphism of A onto B. For any $a \in \mathrm{B}(A)$, the restriction of the map f to the interval $[a,1]$ of A is an isomorphism of the pseudo-BL-algebra $[a,1]$ onto the interval $[f(a),1]$ of B.*

Proposition 2.8. *Let A and B be pseudo-BL-algebras and let $f\colon A \to B$ be an isomorphism of A onto B. For $a_i \in A$ $(i \in I)$, if $\bigvee\limits_{i\in I} a_i$ exists in A, then $\bigvee\limits_{i\in I} f(a_i)$ exists in B and we have $f\left(\bigvee\limits_{i\in I} a_i\right) = \bigvee\limits_{i\in I} f(a_i)$.*

3. Polars

Let A be a pseudo-BL-algebra and let X be a nonempty subset of A. According to [18] (see also [5] and [19]), the set

$$X^{\perp} := \bigl\{a \in A :\ a \vee x = 1 \ \text{ for all } \ x \in X\bigr\}$$

is called the *polar* of X. We put $\emptyset^{\perp} := A$ and write $a^{\perp}$ instead of $\{a\}^{\perp}$. The polar of $\mathrm{C}(A)$ will be denoted by $\mathcal{A}$.

Proposition 3.1. ([5]) *Let $X, Y \subseteq A$. Then:*

(a) $X \subseteq Y \implies Y^{\perp} \subseteq X^{\perp}$;

(b) $X \subseteq X^{\perp\perp}$;

(c) $X^{\perp} = X^{\perp\perp\perp}$;

(d) $X^{\perp} = [X)^{\perp}$;

(e) $X^{\perp}$ *is a filter of* A;

(f) $[X) \cap X^{\perp} = \{1\}$.

Remark 3.2. $A^{\perp} = \{1\}$, $1^{\perp} = A$.

Proposition 3.3. *Let $X_i \subseteq A$ for $i \in I$. Then* $\bigcap\limits_{i\in I} X_i^{\perp} = \left(\bigcup\limits_{i\in I} X_i\right)^{\perp}$.

P r o o f. The proof is trivial. □

Let us define $\mathrm{Pol}(A) := \bigl\{X^{\perp} :\ X \subseteq A\bigr\}$.

Proposition 3.4. ([18])

(a) *$F \in \mathrm{Pol}(A)$ implies $F = F^{\perp\perp}$;*

(b) *$F, G \in \mathrm{Pol}(A)$ implies $F \cap G \in \mathrm{Pol}(A)$;*

(c) *$\{1\}, A \in \mathrm{Pol}(A)$;*

(d) *If we define an operation $F \vee' G = (F^{\perp} \cap G^{\perp})^{\perp}$ for $F, G \in \mathrm{Pol}(A)$, then $\big(\mathrm{Pol}(A); \cap, \vee', {}^{\perp}, \{1\}, A\big)$ is a Boolean algebra.*

Corollary 3.5. *$\mathrm{Pol}(A)$ is a complete Boolean algebra.*

Proof. Let F_i $(i \in I)$ be polars of A. Then for every $i \in I$ there is $X_i \subseteq A$ such that $X_i^{\perp} = F_i$. By Proposition 3.3, $\bigcap\limits_{i \in I} F_i = \bigcap\limits_{i \in I} X_i^{\perp} = \Big(\bigcup\limits_{i \in I} X_i\Big)^{\perp}$. Since $\Big(\bigcup\limits_{i \in I} X_i\Big)^{\perp} \in \mathrm{Pol}(A)$, we deduce that $\bigcap\limits_{i \in I} F_i \in \mathrm{Pol}(A)$, hence that the Boolean algebra $\mathrm{Pol}(A)$ is complete. □

Proposition 3.6. *Let $a, b \in A$. Then:*

(a) *$a \leq b \implies a^{\perp} \subseteq b^{\perp}$;*

(b) *$a^{\perp} = A \iff a = 1$.*

Proof.

(a): Let $a \leq b$ and $c \in a^{\perp}$. Then $c \vee b \geq c \vee a$ and $c \vee a = 1$. Hence $c \vee b = 1$ and consequently, $c \in b^{\perp}$.

(b): $a^{\perp} = A \iff 0 \vee a = 1 \iff a = 1$. □

Proposition 3.7. *If $a \in X^{\perp}$, then $a \odot x = a \wedge x$ for all $x \in X$.*

Proof. Obvious by the definition of a polar and Proposition 2.1(b). □

Proposition 3.8. *Let $a, b \in A$. Then:*

(a) *$a \in a^{\perp\perp}$;*

(b) *$a \leq b \implies a^{\perp\perp} \supseteq b^{\perp\perp}$;*

(c) *$b \in a^{\perp} \implies a^{\perp\perp} \subseteq b^{\perp}$.*

Proof.

(a): By Proposition 3.1(b).

(b): follows from Propositions 3.6(a) and 3.1(a).

(c): By Proposition 3.1(a). □

Proposition 3.9. *If* $e \in \mathrm{B}(A)$ *and* $F := [e, 1]$, *then* $F^{\perp} = [e^{-}, 1]$.

Proof. By Proposition 2.4, $e \vee e^{-} = 1$. From this it follows that $[e^{-}, 1] \subseteq F^{\perp}$. Assume that $x \in F^{\perp}$. Then $x \vee e = 1$ and hence

$$x = x \vee 0 = x \vee (e^{-} \odot e) = x \vee (e^{-} \wedge e) = \big(x \vee e^{-}\big) \wedge (x \vee e) = x \vee e^{-}.$$

Consequently, $e^{-} \leq x$. Therefore $x \in [e^{-}, 1]$. Thus $F^{\perp} \subseteq [e^{-}, 1] \subseteq F^{\perp}$ and so $F^{\perp} = [e^{-}, 1]$. □

4. Complete pseudo-BL-algebras

Let A be a pseudo-BL-algebra. We say that A is complete (σ-complete) if the lattice $\mathrm{L}(A)$ is complete (σ-complete).

Following [5], the identity

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i) \tag{2}$$

holds in A. Define a pseudo-BL-algebra A to be dually Brouwerian if A is complete and satisfies the dual to the identity (2), that is, for every $a \in A$ and every subset T of A,

$$a \vee \bigwedge T = \bigwedge \{a \vee t : \ t \in T\}. \tag{3}$$

Following [6], we denote

$$\mathrm{G}(A) := \big\{a \in A : \ a \odot a = a\big\},$$
$$\mathrm{M}(A) := \big\{a \in A : \ a = (a^{-})^{\sim} = (a^{\sim})^{-}\big\}.$$

In [6], it is proved that

$$\mathrm{B}(A) = \mathrm{G}(A) \cap \mathrm{M}(A). \tag{4}$$

Lemma 4.1. *Let* $a \in \mathrm{G}(A)$. *Then* $[a) = [a, 1]$.

Proof. By (1), $[a) = \{b \in A : \ a^n \leq b \ \text{ for some } \ n \geq 1\}$. Since $a^n = a$ for every $n \geq 1$, we have $[a) = [a, 1]$. □

Proposition 4.2. *Let* A *be a dually Brouwerian pseudo-BL-algebra and let* $e_i \in \mathrm{B}(A)$ *for* $i \in I$. *Then:*

(a) $\bigvee\limits_{i \in I} e_i \in \mathrm{B}(A)$;

(b) $\bigwedge\limits_{i \in I} e_i \in \mathrm{B}(A)$.

P r o o f.

(a): Set $a := \bigvee_{i\in I} e_i$ (the existence of a is ensured by the assumed of completeness of A). Since $e_i \in \mathrm{B}(A)$, we deduce from Proposition 2.4 that $e_i \vee e_i^- = 1$. Hence $a \vee e_i^- = 1$ for all $i \in I$. By Proposition 2.3(a), $a \vee a^- = a \vee \bigwedge_{i\in I} e_i^-$ and then dually Brouwerian implies

$$a \vee a^- = \bigwedge_{i\in I} \left(a \vee e_i^-\right) = 1.$$

Applying Proposition 2.4 we get $a \in \mathrm{B}(A)$.

(b): Let $b := \bigvee_{i\in I} e_i^-$. Since A is dually Brouwerian, we get

$$b \vee \bigwedge_{i\in I} e_i = \bigwedge_{i\in I} (b \vee e_i) = 1.$$

It is easy to see that $b \wedge \bigwedge_{i\in I} e_i = 0$. Thus $\bigwedge_{i\in I} e_i \in \mathrm{B}(A)$. □

Corollary 4.3. *If A is a dually Brouwerian pseudo-BL-algebra, then* $\mathrm{B}(A)$ *is a complete Boolean algebra.*

Proposition 4.4. *Let A be a dually Brouwerian pseudo-BL-algebra and let $\{e_i : i \in I\} \subseteq \mathrm{B}(A)$. If*

$$\bigwedge_{i\in I} e_i = 0 \qquad \textit{and} \qquad e_i \vee e_j = 1 \quad \textit{for all } i \neq j,$$

then A is isomorphic to the direct product $\prod_{i\in I} [e_i, 1]$.

P r o o f. Define $f\colon A \to \prod_{i\in I} [e_i, 1]$ by

$$f(a) = (a \vee e_i : i \in I) \qquad \text{for all} \quad a \in A.$$

By Proposition 2.6(b), f is a homomorphism. To see that f is one-to-one, suppose $f(a) = f(b)$. Then $a \vee e_i = b \vee e_i$ for all $i \in I$. Since A is dually Brouwerian, we have

$$a = a \vee 0 = a \vee \bigwedge_{i\in I} e_i = \bigwedge_{i\in I} (a \vee e_i) = \bigwedge_{i\in I} (b \vee e_i) = b \vee \bigwedge_{i\in I} e_i = b,$$

that is, $a = b$. Finally, to prove that f is onto $\prod_{i \in I} [e_i, 1]$ let $c_i \in [e_i, 1]$ for $i \in I$. Set $c := \bigwedge_{i \in I} c_i$. Obviously, $c_i \vee e_j \geq e_i \vee e_j = 1$ for $i \neq j$. Hence

$$f(c) = (c \vee e_j : j \in I) = \left(\bigwedge_{i \in I} c_i \vee e_j : j \in I \right)$$
$$= \left(\bigwedge_{i \in I} (c_i \vee e_j) : j \in I \right) = (c_i : i \in I) .$$

Thus f is an isomorphism. □

Proposition 4.5. *Let A be a complete atomistic pseudo-BL-algebra. Then A is commutative.*

Proof. Let a and b be atoms of A. Observe that $a \odot b = b \odot a$. It is obvious for $a = b$. If $a \neq b$, then $a \odot b \leq a \wedge b = 0$. Hence $a \odot b = 0 = b \odot a$.

Now let $x, y \in A$ be such that $x = \bigvee_{i \in I} a_i$ and $y = \bigvee_{j \in J} b_j$, where a_i, b_j are atoms. By Proposition 2.2,

$$x \odot a = \left(\bigvee_{i \in I} a_i \right) \odot a = \bigvee_{i \in I} (a_i \odot a) = \bigvee_{i \in I} (a \odot a_i) = a \odot \left(\bigvee_{i \in I} a_i \right) = a \odot x.$$

Then using Proposition 2.2 we obtain

$$x \odot y = x \odot \left(\bigvee_{j \in J} b_j \right) = \bigvee_{j \in J} (x \odot b_j) = \bigvee_{j \in J} (b_j \odot x) = \left(\bigvee_{j \in J} b_j \right) \odot x = y \odot x.$$

Thus A is commutative. □

Lemma 4.6. *Let A be a dually Brouwerian pseudo-BL-algebra and let $X \subseteq A$. Set $a := \bigwedge(X^\perp)$. Then:*

(a) $a \in \mathrm{G}(A)$;

(b) $[a, 1] = X^\perp$.

Proof.

(a): For all $x \in X$, we have

$$x \vee a = x \vee \bigwedge(X^\perp) = \bigwedge\{x \vee b : \ b \in X^\perp\} = 1.$$

Therefore $a \in X^\perp$. Since $X^\perp$ is a filter, we see that $a \odot a \in X^\perp$. But $a \odot a \leq a$ and $a = \bigwedge(X^\perp)$, so $a \odot a = a$. Consequently, $a \in \mathrm{G}(A)$.

(b): Since $a \in X^{\perp}$, we obtain $[a) \subseteq X^{\perp}$. Then, from Lemma 4.1 we have $[a, 1] \subseteq X^{\perp}$. On the other hand, let $y \in X^{\perp}$. By the definition of a, $y \geq a$ and hence $y \in [a, 1]$. It follows that $X^{\perp} \subseteq [a, 1]$. Thus (b) holds. $\square$

Theorem 4.7. *Let A be a dually Brouwerian pseudo-BL-algebra. If $\bigwedge \mathcal{A} \in \mathrm{M}(A) - \{0\}$, then $A \cong A_1 \times A_2$, where A_1 and A_2 are dually Brouwerian pseudo-BL-algebras such that $\mathrm{C}(A_1) = \emptyset$ and A_2 is coatomic.*

P r o o f. We set $a := \bigwedge \mathcal{A}$. From Lemma 4.6(a) we conclude that $a \in \mathrm{G}(A)$. By assumption, $a \in \mathrm{M}(A)$. Applying (4) yields $a \in \mathrm{B}(A)$.

Let $A_1 = [a, 1]$ and $A_2 = [a^-, 1]$. By Proposition 2.6(a), A_1 and A_2 are pseudo-BL-algebras. It is easy to see that they are dually Brouwerian. From Proposition 4.4 we conclude that $A \cong A_1 \times A_2$. Proposition 3.9 gives

$$A_2 = [a^-, 1] = [a, 1]^{\perp} = A_1^{\perp}. \tag{5}$$

Lemma 4.6(b) implies $A_1 = \mathrm{C}(A)^{\perp}$. From this and (5) we obtain $A_2 = \mathrm{C}(A)^{\perp\perp}$. Hence, by Proposition 3.1(b),

$$\mathrm{C}(A) \subseteq A_2. \tag{6}$$

In view of Proposition 2.6(d) we get

$$\mathrm{C}(A_1) \subseteq \mathrm{C}(A). \tag{7}$$

Applying (6) and (7) we see that $\mathrm{C}(A_1) \subseteq A_2$. Observe that $\mathrm{C}(A_1) = \emptyset$. Indeed, if $b \in \mathrm{C}(A_1)$, then $b \geq a$. On the other hand, $b \in A_2 = [a^-, 1]$, and hence $b \geq a^-$. Then $b \geq a \vee a^- = 1$, a contradiction.

Now we show that A_2 is coatomic. On the contrary suppose that

$$\big(\exists\, b \in A_2 - \{1\}\big)\big(\forall\, p \in \mathrm{C}(A_2)\big)\big(b \nleq p\big).$$

By assumption, $a \neq 0$, and therefore $\mathrm{C}(A) \neq \emptyset$. Let $q \in \mathrm{C}(A)$. We conclude from Proposition 2.6(d) that $\mathrm{C}(A_2) = A_2 \cap \mathrm{C}(A)$. Then, by (6), we get $\mathrm{C}(A) = \mathrm{C}(A_2)$. Thus $q \in \mathrm{C}(A_2)$ and, in consequence, $b \nleq q$. Hence $b \vee q = 1$. It follows that $b \in \mathrm{C}(A)^{\perp} = A_1$. Then $b \in A_1 \cap A_2 = A_1 \cap A_1^{\perp} = \{1\}$ (see (5) and Proposition 3.1). This contradicts the fact that $b \neq 1$. Thus A_2 is coatomic. $\square$

5. A Cantor-Bernstein type theorem

Let A be a σ-complete pseudo-BL-algebra. If (3) holds for every $a \in A$ and every countable subset T of A, then A is said to be σ-dually Brouwerian.

By the proof of Proposition 4.2 we have

Proposition 5.1. *If A is a σ-dually Brouwerian pseudo-BL-algebra, then* $\mathrm{B}(A)$ *is a σ-complete Boolean algebra.*

From the proof of Proposition 4.4 we obtain

Proposition 5.2. *Let A be a σ-dually Brouwerian pseudo-BL-algebra. Suppose that a sequence $e_0, e_1, \ldots \in \mathrm{B}(A)$ satisfies the following conditions:*

$$\bigwedge_{n\in\mathbb{N}} e_n = 0 \qquad \text{and} \qquad e_i \vee e_j = 1 \quad \text{for all } \ i \neq j.$$

Then A is isomorphic to the direct product $\prod\limits_{n\in\mathbb{N}} [e_n, 1]$.

If pseudo-BL-algebras A and B are isomorphic, then we denote it by $A \cong B$. In the present section we prove the following result.

Theorem 5.3. *Let A and B be σ-dually Brouwerian pseudo-BL-algebras. Let $a \in \mathrm{B}(A)$ and $b \in \mathrm{B}(B)$. If A is isomorphic to the interval algebra $[b, 1]$ of B and B is isomorphic to the interval algebra $[a, 1]$ of A, then $A \cong B$.*

Proof. Let $f\colon A \to [b, 1]$ and $g\colon B \to [a, 1]$ be isomorphisms. Skipping all trivialities, we may assume $0 < a, b < 1$. We construct sequences $a_0, a_1, \ldots \in A$ and $b_0, b_1, \ldots \in B$ by the following recursion:

$$\begin{aligned} a_0 &= 0, & b_0 &= 0, \\ a_{n+1} &= g(b_n), & b_{n+1} &= f(a_n). \end{aligned}$$

From the injectivity of f and g we have

$$a_0 < a_1 < \cdots < a_n < \ldots \qquad \text{and} \qquad b_0 < b_1 < \cdots < b_n < \ldots.$$

By Proposition 2.6(c), $a_n \in \mathrm{B}(A)$ and $b_n \in \mathrm{B}(B)$ for all $n \in \mathbb{N}$. Let $a_\infty = \bigvee\limits_{n\in\mathbb{N}} a_n$ and $b_\infty = \bigvee\limits_{n\in\mathbb{N}} b_n$. We conclude from Proposition 5.1 that $a_\infty \in \mathrm{B}(A)$ and $b_\infty \in \mathrm{B}(B)$. For each $n \in \mathbb{N}$, let us define $r_n = a_n \vee a_{n+1}^-$ and $s_n = b_n \vee b_{n+1}^-$. Let $n \neq m$, for example $n > m$. Then

$$r_n \vee r_m = a_n \vee a_{n+1}^- \vee a_m \vee a_{m+1}^- = a_n \vee a_{m+1}^- \geq a_n \vee a_n^- = 1.$$

Thus $r_n \vee r_m = 1$ for $n \neq m$, and similarly $s_n \vee s_m = 1$ for $n \neq m$. Obviously $a_\infty \vee r_n = b_\infty \vee s_n = 1$ for all $n \in \mathbb{N}$. Observe that

$$a_n \wedge r_0 \wedge r_1 \wedge \cdots \wedge r_{n-1} = 0 \qquad \text{for} \quad n \geq 1. \tag{8}$$

Indeed,

$$\begin{aligned} a_n \wedge (a_0 \vee a_1^-) \wedge \cdots \wedge (a_{n-1} \vee a_n^-) &= a_n \wedge a_1^- \wedge (a_1 \vee a_2^-) \wedge \cdots \wedge (a_{n-1} \vee a_n^-) \\ &= a_n \wedge a_2^- \wedge (a_2 \vee a_3^-) \wedge \cdots \wedge (a_{n-1} \vee a_n^-) \\ &= \cdots = a_n \wedge a_{n-1}^- \wedge (a_{n-1} \vee a_n^-) \\ &= a_n \wedge a_n^- = 0. \end{aligned}$$

Let $n \geq 1$. Since $a_{k+1} \leq a_n$ for $k = 0, 1, \ldots, n-1$, we have $a_n^- \leq a_{k+1}^- \leq r_k$ for $k = 0, 1, \ldots, n-1$. Then, applying (8) we obtain

$$\begin{aligned} r_0 \wedge \cdots \wedge r_n &= r_0 \wedge \cdots \wedge r_{n-1} \wedge \big(a_n \vee a_{n+1}^-\big) \\ &= (a_n \wedge r_0 \wedge \cdots \wedge r_{n-1}) \vee (a_{n+1}^- \wedge r_0 \wedge \cdots \wedge r_{n-1}) \\ &= 0 \vee a_{n+1}^- = a_{n+1}^-. \end{aligned}$$

Hence, by Proposition 2.3,

$$\bigwedge_{n\in\mathbb{N}} r_n = \bigwedge_{n\in\mathbb{N}} (r_0 \wedge \cdots \wedge r_n) = \bigwedge_{n\in\mathbb{N}} a_{n+1}^- = \left(\bigvee_{n\in\mathbb{N}} a_{n+1}\right)^- = a_\infty^-.$$

Therefore, $a_\infty \wedge \bigwedge_{n\in\mathbb{N}} r_n = a_\infty \wedge a_\infty^- = 0$. Applying Proposition 5.2 for the sequence $(a_\infty, r_0, \ldots, r_n, \ldots)$ we conclude that

$$A \cong [a_\infty, 1] \times \prod_{n\in\mathbb{N}} [r_n, 1]. \tag{9}$$

Similarly, the sequence $(b_\infty, s_0, \ldots, s_n, \ldots)$ satisfies the assumptions of Proposition 5.2, and hence

$$B \cong [b_\infty, 1] \times \prod_{n\in\mathbb{N}} [s_n, 1]. \tag{10}$$

Using Proposition 2.8 we get

$$f(a_\infty) = f\left(\bigvee_{n\in\mathbb{N}} a_n\right) = \bigvee_{n\in\mathbb{N}} f(a_n) = \bigvee_{n\in\mathbb{N}} b_{n+1} = b_\infty.$$

By Proposition 2.7, the restriction of f to $[a_\infty, 1]$ is an isomorphism of $[a_\infty, 1]$ onto $[f(a_\infty), 1]$, that is,

$$[a_\infty, 1] \cong [b_\infty, 1]. \tag{11}$$

Another application of Proposition 2.7 yields, for each $n \in \mathbb{N}$, isomorphisms

$$f\colon [r_{2n}, 1] \to [f(r_{2n}), 1]$$

and

$$g\colon [s_{2n}, 1] \to [g(s_{2n}), 1].$$

Since $f(r_{2n}) = s_{2n+1}$ and $g(s_{2n}) = r_{2n+1}$, we have

$$[r_{2n}, 1] \cong [s_{2n+1}, 1] \qquad \text{and} \qquad [r_{2n+1}, 1] \cong [s_{2n}, 1]. \tag{12}$$

From (9)–(12) we obtain $A \cong B$. □

Acknowledgement. The authors are indebted to the referee for his/her very careful reading and suggestions.

REFERENCES

[1] BELLUCE, L. P.: *Semi-simple and complete MV-algebras*, Algebra Universalis **29** (1992), 1–9.

[2] CHANG, C. C.: *Algebraic analysis of many valued logics*, Trans. Amer. Math. Soc. **88** (1958), 467–490.

[3] CIGNOLI, R.: *Complete and atomic algebras of the infinite valued Łukasiewicz logic*, Studia Logica **50** (1991), 375–384.

[4] DE SIMONE, A.—MUNDICI, D.—NAVARA, M.: *A Cantor-Bernstein theorem for σ-complete MV-algebras*, Czechoslovak Math. J. **53** (2003), 437–447.

[5] DI NOLA, A.—GEORGESCU, G.—IORGULESCU, A.: *Pseudo-BL algebras I*, Mult.-Valued Log. **8** (2002), 673–714.

[6] DI NOLA, A.—GEORGESCU, G.—IORGULESCU, A.: *Pseudo-BL algebras II*, Mult.-Valued Log. **8** (2002), 717–750.

[7] DVUREČENSKIJ, A.: *Central elements and Cantor-Bernstein's theorem for pseudo-effect algebras*, J. Aust. Math. Soc. **74** (2003), 121–143.

[8] GEORGESCU, G.—IORGULESCU, A.: *Pseudo MV-algebras: a noncommutative extension of MV-algebras*. In: The Proceedings of the Fourth International Symposium on Economic Informatics, Bucharest, Romania, 1999, pp. 961–968.

[9] GEORGESCU, G.—IORGULESCU, A.: *Pseudo BL-algebras: a noncommutative extension of BL-algebras*. In: Abstracts of the Fifth International Conference FSTA 2000, Slovakia, 2000, pp. 90–92.

[10] GEORGESCU, G.—LEUŞTEAN, L. L.: *Some classes of pseudo-BL algebras*, J. Aust. Math. Soc. **73** (2002), 127–153.

[11] HÁJEK, P.: *Metamathematics of Fuzzy Logic*, Kluwer, Amsterdam, 1998.

[12] HÁJEK, P.: *Fuzzy logics with noncommutative conjuctions*, J. Logic Comput. **13** (2003), 469–479.

[13] HÁJEK, P.: *Observations on non-commutative fuzzy logic*, Soft Comput. **8** (2003), 38–43.

[14] JAKUBÍK, J.: *On complete MV-algebras*, Czechoslovak Math. J. **45** (1995), 473–480.

[15] JAKUBÍK, J.: *Cantor-Bernstein theorem for MV-algebras*, Czechoslovak Math. J. **49** (1999), 517–526.

[16] JAKUBÍK, J.: *A theorem of Cantor-Bernstein type for orthogonally σ-complete pseudo-MV-algebras*, Tatra Mt. Math. Publ. **22** (2001), 91–103.

[17] JAKUBÍK, J.: *On the α-completeness of pseudo-MV-algebras*, Math. Slovaca **52** (2002), 511–516.

[18] KÜHR, J.: *Pseudo BL-algebras and DRℓ-monoids*, Math. Bohem. **128** (2003), 199–202.

[19] KÜHR, J.: *Prime ideals and polars in $DR\ell$-monoids and pseudo-BL-algebras*, Math. Slovaca **53** (2003), 233–246.

[20] RACHŮNEK, J.: *A non-commutative generalization of MV-algebras*, Czechoslovak Math. J. **52** (2002), 255–273.

[21] SIKORSKI, R.: *A generalization of a theorem of Banach and Cantor-Bernstein*, Colloq. Math. **1** (1948), 140–144 and 242.

[22] TARSKI, A.: *Cardinal Algebras*, Oxford University Press, New York, 1949.

Received 18. 5. 2010
Accepted 2. 10. 2010

* *Warsaw School of Information Technology*
Newelska 6
PL–01447 Warszawa
POLAND
E-mail: walent@interia.pl

** *Institute of Mathematics and Physics*
Siedlce University
3 Maja 54
PL–08110 Siedlce
POLAND
E-mail: magdawojciechowska6@wp.pl