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A DIFFERENCE BETWEEN THE SETS OF ORDINARY AND STRONG DENSITY POINTS ON THE PLANE

Grażyna Horbaczewska* — Władysław Wilczyński**

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ABSTRACT. It is well known that the difference between the set of all ordinary density points and the set of all strong density points of an arbitrary measurable subset of the plane is a null set. It is of interest to check how large can a difference between sections of these sets be. Both measure and category cases are considered.

1. Introduction

In the paper we construct a measurable set A on the plane such that the difference between the set of all ordinary density points of A and of all strong density points of A includes a segment parallel to the $0x$ axis. The problem of full characterization of the difference between the two above mentioned sets seems to be much more difficult and remains open. In the category case we are able only to construct a set with the Baire property for which the analogous difference is residual (and of null measure) on some segment parallel to $0x$ axis.

2.

For the convenience of the reader we start with recalling the basic notions ([4]).

Definition 1. We say that $(x_0, y_0) \in \mathbb{R}^2$ is *an ordinary density point* of a measurable set $A \subset \mathbb{R}^2$ if

$$\lim_{h\to 0^+} \frac{m_2\left(A \cap ([x_0-h, x_0+h] \times [y_0-h, y_0+h])\right)}{4h^2} = 1.$$

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Definition 2. We say that $(x_0, y_0) \in \mathbb{R}^2$ is *a strong density point* of a measurable set $A \subset \mathbb{R}^2$ if

$$\lim_{\substack{h\to 0^+\\ k\to 0^+}} \frac{m_2\left(A \cap ([x_0 - h, x_0 + h] \times [y_0 - k, y_0 + k])\right)}{4hk} = 1.$$

Let $\mathcal{L}_2$ stand for the family of all Lebesgue measurable sets on the plane.

For $A \in \mathcal{L}_2$ we define

$$\Phi_o(A) := \left\{(x, y) \in \mathbb{R}^2 : \ (x, y) \ \text{ is an ordinary density point of } \ A\right\}$$

$$\Phi_s(A) := \left\{(x, y) \in \mathbb{R}^2 : \ (x, y) \ \text{ is a strong density point of } \ A\right\}$$

Obviously if (x_0, y_0) is a strong density point of A, then it is also an ordinary density point of A, but there exists a set $A \in \mathcal{L}_2$ for which $\Phi_o(A) \setminus \Phi_s(A)$ is nonempty.

The well known Lebesgue Density Theorem says that a symmetrical difference between a measurable set and the set of its density points is a null set. An analogous theorem holds for strong density points.

Theorem 1. ([4], [5]) *For any* $A \in \mathcal{L}_2$

$$m_2(A \triangle \Phi_o(A)) = 0$$

and

$$m_2(A \triangle \Phi_s(A)) = 0.$$

By virtue of the above theorem $m_2(\Phi_o(A) \triangle \Phi_s(A)) = 0$ for any $A \in \mathcal{L}_2$, so by [1, Fubini Theorem] we see that almost all sections of the symmetrical difference between the set of all ordinary density points of a measurable subset of the plane and the set of all its strong density points are of measure zero. Since the sets $\Phi_o(A)$ and $\Phi_s(A)$ are Borel sets, all sections of the considered difference are measurable.

We are going to use the following notation: for any $E \subset \mathbb{R}^2$, $y \in \mathbb{R}$ let

$$E^y := \left\{x \in \mathbb{R} : \ (x, y) \in E\right\}.$$

For abbreviation, let $S((x, y), r)$ stand for the square $(x-r, x+r)\times(y-r, y+r)$, where $x, y \in \mathbb{R}$, $r \in \mathbb{R}_+$.

Theorem 2. *There exists a measurable set* $A \subset \mathbb{R}^2$ *such that*

$$(\Phi_o(A) \setminus \Phi_s(A))^0 = [0, 1].$$

P r o o f. Let $a_1 = 1$, $a_2 = a_3 = 1/2$, $a_4 = a_5 = a_6 = 1/3$, etc.. We will use the following properties of $\{a_n\}_{n\in\mathbb{N}}$: it is a nonincreasing sequence of real numbers tending to 0, such that $a_1 \leq 1$ and there exists an increasing sequence of natural numbers $\{n_m\}_{m\in\mathbb{N}}$ such that $\sum\limits_{n=n_{m-1}+1}^{n_m} a_n = 1$ for every $m \in \mathbb{N}$, where

$n_0 = 0$. Moreover $\lim\limits_{m\to\infty}(n_m - n_{m-1}) = +\infty$ since $n_m - n_{m-1} = m$ and obviously $\sum\limits_{n=1}^{\infty} a_n = +\infty$.

Fix $m \in \mathbb{N}$. We divide the interval $[0,1]$ into $p_{(m)}$ — a sufficiently large number (described later) of intervals of the same length:

$$I_k^{(m)} := \left[\frac{k-1}{p_{(m)}}, \frac{k}{p_{(m)}}\right] \qquad k \in \{1, 2, \ldots, p_{(m)}\}$$

Now we divide each of these intervals into m intervals of the same length:

$$I_{k,j}^{(m)} := \left[\frac{k-1}{p_{(m)}} + \frac{j-1}{mp_{(m)}}, \frac{k-1}{p_{(m)}} + \frac{j}{mp_{(m)}}\right] \qquad j \in \{1, \ldots, m\}.$$

Define

$$E_m := \bigcup_{j=1}^{m} \bigcup_{k=1}^{p_{(m)}} \left(I_{k,j}^{(m)} \times \left[\frac{1}{2^{n_{m-1}+j}}, \frac{2}{2^{n_{m-1}+j}}\right]\right).$$

Now we repeat the same construction for each natural number m and let $E := \bigcup\limits_{m=1}^{\infty} E_m$. We shall show that the set $A = \mathbb{R}^2 \setminus E$ fulfils the desired condition.

Indeed, set $a \in [0,1]$. We first want to show that $(a,0) \in \Phi_o(A)$.

Let $\varepsilon > 0$. There exists $M \in \mathbb{N}$ such that for any $m \geq M$ we have $\frac{1}{m-1} < \varepsilon/6$ and $2^{-(m+1)} < \varepsilon/3$. Let $R = 2^{-n_{M-2}}$. Now fix $r < R$. There exists $m_0 > M$ such that $r \in [2^{-n_{m_0-1}}, 2^{-n_{m_0-2}})$. Obviously

$$\begin{aligned} &\frac{m_2(E \cap S((a,0),r))}{4r^2} \\ &= \frac{m_2(E_{m_0-1} \cap S((a,0),r))}{4r^2} + \frac{m_2(E_{m_0} \cap S((a,0),r))}{4r^2} \\ &\quad + \frac{m_2\Big(\bigcup\limits_{k>m_0} E_k \cap S((a,0),r)\Big)}{4r^2}. \end{aligned}$$

We start with an observation that

$$\frac{m_2(\bigcup\limits_{k>m_0} E_k \cap S((a,0),r))}{4r^2} < \frac{2r2^{-n_{m_0}}}{4r^2} < \frac{1}{2}\frac{2^{-n_{m_0}}}{2^{-n_{m_0-1}}} = \frac{1}{2}2^{-m_0} < \frac{\varepsilon}{3}$$

Consider now for a fixed j the number of intervals $I_{k,j}^{(m_0)}$ for which $I_{k,j}^{(m_0)} \cap [a-r, a+r]$ is nonempty. It is smaller than or equal to $\left[\frac{2r}{p_{(m_0)}^{-1}}\right] + 1$, so not greater than $2rp_{(m_0)} + 2$.

Hence for $h \in (0, 2^{1-(n_{m_0-1}+j)}]$ we get

$$\begin{aligned}
&\frac{m_2(E \cap ([a-r,a+r] \times [2^{-(n_{m_0-1}+j)}, 2^{-(n_{m_0-1}+j)}+h]))}{m_2([a-r,a+r] \times [2^{-(n_{m_0-1}+j)}, 2^{-(n_{m_0-1}+j)}+h])}\\
&\leq \frac{(2rp_{(m_0)}+2)(p_{(m_0)}m_0)^{-1}h}{2rh} = \frac{2rp_{(m_0)}}{2rp_{(m_0)}m_0} + \frac{2}{2rp_{(m_0)}m_0}\\
&\leq \frac{1}{m_0}\left(1 + \frac{2^{n_{m_0-1}}}{p_{(m_0)}}\right) < \frac{3}{2m_0}
\end{aligned}$$

if only $p_{(m_0)} > 2^{n_{m_0-2}}$ and $m_0 > 2$.

Therefore

$$\begin{aligned}
&\frac{m_2(E_{m_0} \cap S((a,0),r))}{4r^2}\\
&= \frac{\sum_{j=1}^{m_0} m_2\left(E_{m_0} \cap \left([a-r,a+r] \times [2^{-(n_{m_0-1}+j)}, 2^{1-(n_{m_0-1}+j)}]\right) \cap S((a,0),r)\right)}{4r^2}\\
&< \frac{3}{2m_0}\frac{\sum_{j=1}^{m_0} 2r2^{-(n_{m_0-1}+j)}}{4r^2} \leq \frac{3}{2m_0}\frac{1}{2}\sum_{j=1}^{m_0}\frac{2^{-(n_{m_0-1}+j)}}{2^{-(n_{m_0-1})}}\\
&= \frac{3}{4m_0}\sum_{j=1}^{m_0} 2^{-j} < \frac{3}{4m_0} < \frac{\varepsilon}{3}.
\end{aligned}$$

Clearly there exists $j \in \{0, \dots, m_0-1\}$ such that $r \in [2^{-(n_{m_0-1}-j)}, 2^{1-(n_{m_0-1}-j)})$, so

$$\begin{aligned}
&\frac{m_2(E_{m_0-1} \cap S((a,0),r))}{4r^2}\\
&= \frac{\sum_{l=0}^{j-1} m_2\left(E_{m_0-1} \cap ([a-r,a+r] \times [2^{-(n_{m_0-1}-l)}, 2^{1-(n_{m_0-1}-l)}])\right)}{4r^2}\\
&\quad + \frac{m_2\left(E_{m_0-1} \cap ([a-r,a+r] \times [2^{-(n_{m_0-1}-j)}, r])\right)}{4r^2}\\
&< \frac{3}{2(m_0-1)}\frac{1}{2}\left(\sum_{l=0}^{j-1}\frac{2^{-(n_{m_0-1}-l)}}{r} + \frac{r - 2^{-(n_{m_0-1}-j)}}{r}\right)\\
&< \frac{3}{4(m_0-1)}\left(\sum_{l=0}^{j-1}\frac{2^{-(n_{m_0-1}-l)}}{2^{-(n_{m_0-1}-j)}} + 1\right)\\
&= \frac{3}{4(m_0-1)}\left(\sum_{l=0}^{j-1} 2^{-(j-l)} + 1\right) < \frac{3}{2(m_0-1)} < \frac{\varepsilon}{3}.
\end{aligned}$$

Thus $\frac{m_2(E\cap S((a,0),r))}{4r^2} < \varepsilon$, so $(a,0) \in \Phi_o(A)$.

To show that $(a,0) \notin \Phi_s(A)$ we observe that for every $m \in \mathbb{N}$ there exists an interval $I_{k,j}^{(m)}$ containing the point a. Therefore the rectangle

$$P_m = \left[a - \frac{1}{mp(m)}, a + \frac{1}{mp(m)}\right] \times \left[-\frac{2}{2^{n_{m_0}+j}}, \frac{2}{2^{n_{m_0}+j}}\right]$$

contains the rectangle

$$I_{k,j}^{(m)} \times \left[\frac{1}{2^{n_{m_0}+j}}, \frac{2}{2^{n_{m_0}+j}}\right].$$

Hence $\frac{m_2(E\cap P_m)}{m_2(P_m)} \geq \frac{1}{8}$ for $m \in \mathbb{N}$ and the proof is finished. □

3.

It is natural to try to relate this result to the category case.

Write $(n,m)\cdot A := \big\{(nx,my):\ (x,y)\in A\big\}$

and $A-(x_0,y_0) := \big\{(x-x_0,y-y_0):\ (x,y)\in A\big\}$, where $A \subset \mathbb{R}^2$.

Definition 3. ([3]) We say that $(x_0,y_0)\in\mathbb{R}^2$ is *an ordinary $\mathcal{I}_2$-density point* (where $\mathcal{I}_2$ is the σ-ideal of meager sets on the plane) of a set A having the Baire property ($A \in \mathcal{B}_2$) if for every subsequence $\{n_m\}_{m\in\mathbb{N}}$ of the increasing sequence of all natural numbers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that $\chi_{[(n_{m_p},n_{m_p})\cdot(A-(x_0,y_0))]\cap[-1,1]^2} \underset{p\to\infty}{\longrightarrow} 1$ $\mathcal{I}_2$-a.e. on $[-1,1]^2$, i.e.

$$\limsup_{p\in\mathbb{N}}[(n_{m_p},n_{m_p})\cdot(A'-(x_0,y_0))]\cap[-1,1]^2 \in \mathcal{I}_2,$$

where $A' := \mathbb{R}^2 \setminus A$.

Definition 4. We say that $(x_0,y_0)\in\mathbb{R}^2$ is *a strong $\mathcal{I}_2$-density point* of a set $A\in\mathcal{B}_2$ if for every two subsequences $\{n'_m\}_{m\in\mathbb{N}}$ and $\{n''_m\}_{m\in\mathbb{N}}$ of the increasing sequence of all natural numbers there exists a subsequence $\{m_p\}_{p\in\mathbb{N}}$ such that $\chi_{[(n'_{m_p},n''_{m_p})\cdot(A-(x_0,y_0))]\cap[-1,1]^2} \underset{p\to\infty}{\longrightarrow} 1$ $\mathcal{I}_2$-a.e. on $[-1,1]^2$, i.e.

$$\limsup_{p\in\mathbb{N}}[(n'_{m_p},n''_{m_p})\cdot(A'-(x_0,y_0))]\cap[-1,1]^2 \in \mathcal{I}_2.$$

For $A\in\mathcal{B}_2$ we define

$$\Phi^o_{\mathcal{I}_2}(A) := \big\{(x,y)\in\mathbb{R}^2:\ (x,y) \text{ is an ordinary } \mathcal{I}_2\text{-density point of } A\big\}$$

and

$$\Phi^s_{\mathcal{I}_2}(A) := \big\{(x,y)\in\mathbb{R}^2:\ (x,y) \text{ is a strong } \mathcal{I}_2\text{-density point of } A\big\}$$

A Lebesgue density type theorem also holds here (see [2], [3]).

Theorem 3. *For any* $A \in \mathcal{B}_2$

$$A \triangle \Phi^o_{\mathcal{I}_2}(A) \in \mathcal{I}_2$$

and

$$A \triangle \Phi^s_{\mathcal{I}_2}(A) \in \mathcal{I}_2$$

Hence again $\Phi^o_{\mathcal{I}_2}(A) \triangle \Phi^s_{\mathcal{I}_2}(A) \in \mathcal{I}_2$ for any $A \in \mathcal{B}_2$, so by [1, Kuratowski-Ulam Theorem], we see that almost all (in the sense of category) sections of the symmetrical difference between the set of all ordinary $\mathcal{I}_2$-density points of a subset of the plane having the Baire property and the set of all its strong $\mathcal{I}_2$-density points are meager sets. Since the sets $\Phi^o_{\mathcal{I}_2}(A)$ and $\Phi^s_{\mathcal{I}_2}(A)$ are Borel sets, all sections of the considered difference have Baire property.

The main result we obtain for category is a little bit weaker than that in the measure case.

Theorem 4. *There exists a subset A of the plane having the Baire property such that the set* $(\Phi^o_{\mathcal{I}_2}(A) \setminus \Phi^s_{\mathcal{I}_2}(A))^0$ *is residual on* $[0,1]$ *and of measure zero.*

P r o o f. Let $\{q_n\}_{n\in\mathbb{N}}$ be a sequence of all rational numbers in $(0,1)$ and $\{a_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of real numbers tending to 0.

For $n \in \mathbb{N}$ let

$$E_n := ([q_n - a_n 2^{-n}, q_n + a_n 2^{-n}] \times [2^{-n}, 2^{-n+1}]) \cap [0,1]^2.$$

The set $A := \mathbb{R}^2 \setminus \bigcup_{n\in\mathbb{N}} E_n$ fulfils the desired condition. Indeed, it suffices to show that

$$\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} [q_m - a_m 2^{-m}, q_m + a_m 2^{-m}] \right) \cap [0,1] \subset (\Phi^o_{\mathcal{I}_2}(A) \setminus \Phi^s_{\mathcal{I}_2}(A))^0,$$

since for every $n \in \mathbb{N}$ the set $\bigcup_{m=n}^{\infty} [q_m - a_m 2^{-m}, q_m + a_m 2^{-m}]$ contains an open dense set in $[0,1]$, so $\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} [q_m - a_m 2^{-m}, q_m + a_m 2^{-m}]$ is residual in $[0,1]$.

Let $x \in \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} [q_m - a_m 2^{-m}, q_m + a_m 2^{-m}]$. Then there exists an increasing sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$ the point x belongs to $[q_{n_k} - a_{n_k} 2^{-n_k}, q_{n_k} + a_{n_k} 2^{-n_k}]$.

Consider the rectangle $P_k = [c_k, d_k] \times [-2^{-n_k+1}, 2^{-n_k+1}]$, where $[c_k, d_k]$ is the shortest interval containing $[q_{n_k} - a_{n_k} 2^{-n_k}, q_{n_k} + a_{n_k} 2^{-n_k}]$ with the center at the point x.

We can choose a subsequence $\{n_{k_l}\}_{l\in\mathbb{N}} \subset \{n_k\}_{k\in\mathbb{N}}$ such that $x \le q_{n_{k_l}}$ for every $l \in \mathbb{N}$ or $x > q_{n_{k_l}}$ for every $l \in \mathbb{N}$. For convenience we assume that $x \le q_{n_k}$ for every $k \in \mathbb{N}$.

Let $p_k = (d_k - x)^{-1}$ for $k \in \mathbb{N}$. Then

$$\big(p_k, 2^{n_k-1}\big)\cdot\big(A' - (x,0)\big) \supset \big(p_k, 2^{n_k-1}\big)\cdot\big([0, p_k^{-1}] \times [2^{-n_k}, 2^{-n_k+1}]\big) = [0,1]\times[1/2,1]$$

for $k \in \mathbb{N}$. Therefore

$$\limsup_{k\in\mathbb{N}} \big(p_k, 2^{n_k-1}\big)\cdot(A' - (x,0)) \supset [0,1]\times[1/2,1].$$

We conclude from [2, Lemma 1] that $(x,0)$ cannot be a point of strong $\mathcal{I}_2$-density of A.

The task is now to show that for any $x \in [0,1]$ the point $(x,0)$ is an $\mathcal{I}_2$-dispersion point of the set $\bigcup_{l\in\mathbb{N}} E_l$, i.e. that for each increasing sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ we can choose a subsequence $\{n_{k_p}\}_{p\in\mathbb{N}}$ such that $\limsup_{p\in\mathbb{N}}\Big[(n_{k_p}, n_{k_p})\cdot\Big(\bigcup_{l\in\mathbb{N}} E_l - (x,0)\Big)\Big] \cap [-1,1]^2$ is of the first category.

Fix an increasing sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ tending to infinity. For each square $S\big((x,0),\frac{1}{n_k}\big)$ we can choose the highest level which "touches" that square, it means for each $k \in \mathbb{N}$ there exists $i_k \in \mathbb{N}$ such that $\frac{1}{2^{i_k}} \le \frac{1}{n_k} < \frac{1}{2^{i_k-1}}$. Since $\frac{1}{2} < n_k 2^{-i_k} \le 1$ we can choose a subsequence $\{n_{k_t}\}_{t\in\mathbb{N}}$ of $\{n_k\}_{k\in\mathbb{N}}$ such that $n_{k_t}2^{-i_{k_t}}$ converges to $g \in \big[\frac{1}{2},1\big]$. For convenience we don't change the notation from $\{n_k\}_{k\in\mathbb{N}}$ to $\{n_{k_t}\}_{t\in\mathbb{N}}$.

Our next concern will be the behaviour of the set E_{i_k} on the square $S\big((x,0),\frac{1}{n_k}\big)$.

If inequalities $-\frac{1}{n_k} \le q_{i_k} + a_{i_k}\frac{1}{2^{i_k}} - x$ and $q_{i_k} - a_{i_k}\frac{1}{2^{i_k}} - x \le \frac{1}{n_k}$ hold only for a finite number of k, then we denote by $\{n_k^{(0)}\}_{k\in\mathbb{N}}$ the whole sequence $\{n_k\}_{k\in\mathbb{N}}$. If it is not the case and there is a subsequence of $\{n_k\}_{k\in\mathbb{N}}$ for which at least one of these inequalities holds we can choose a subsequence $\{n_k^{(0)}\}_{k\in\mathbb{N}}$ of $\{n_k\}_{k\in\mathbb{N}}$ such that $n_k^{(0)}\Big(q_{i_k^{(0)}} - a_{i_k^{(0)}}\frac{1}{2^{i_k^{(0)}}} - x\Big)$ and $n_k^{(0)}\Big(q_{i_k^{(0)}} + a_{i_k^{(0)}}\frac{1}{2^{i_k^{(0)}}} - x\Big)$ are convergent to the same point denoted by r_0. It is possible since the sequence $\{n_k 2^{-i_k}\}_{k\in\mathbb{N}}$ is bounded and $\{a_{i_k}\}_{k\in\mathbb{N}}$ tends to 0. We can now proceed analogously. If for almost all $k \in \mathbb{N}$ the set $\big[\big(n_k^{(0)}, n_k^{(0)}\big)\cdot(E_{i_k+1} - (x,0))\big] \cap [-1,1]^2$ is empty then we put $n_k^{(1)} = n_k^{(0)}$ for each $k \in \mathbb{N}$. In the opposite case we choose a subsequence $\{n_k^{(1)}\}_{k\in\mathbb{N}} \subset \{n_k^{(0)}\}_{k\in\mathbb{N}}$ such that $n_k^{(1)}\Big(q_{i_k^{(1)}} - a_{i_k^{(1)}}\frac{1}{2^{i_k^{(1)}}} - x\Big)$ and $n_k^{(1)}\Big(q_{i_k^{(1)}} + a_{i_k^{(1)}}\frac{1}{2^{i_k^{(1)}}} - x\Big)$ tend to r_1 and continue the same vein.

Then for each $z \in \mathbb{N}$ we obtain a subsequence $\big\{n_k^{(z)}\big\}_{k\in\mathbb{N}} \subset \big\{n_k^{(z-1)}\big\}_{k\in\mathbb{N}}$ such that $n_k^{(z)}\Big(q_{i_k^{(z)}+z} - a_{i_k^{(z)}+z}\frac{1}{2^{i_k^{(z)}+z}} - x\Big)$ and $n_k^{(z)}\Big(q_{i_k^{(z)}+z} + a_{i_k^{(z)}+z}\frac{1}{2^{i_k^{(z)}+z}} - x\Big)$ tend to r_z.

We claim that for each $z \in \mathbb{N}$

$$\limsup_{k\in\mathbb{N}}\big[\big(n_k^{(z)}, n_k^{(z)}\big)\cdot(E_{i_k^{(z)}+z}-(x,0))\big]\cap[-1,1]^2 \subset \{r_z\}\times\left[\frac{g}{2^z},\frac{g}{2^{z-1}}\right].$$

If there exists $k_1 \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k > k_1$ the intersection $\big[\big(n_k^{(z)}, n_k^{(z)}\big)\cdot(E_{i_k^{(z)}+z}-(x,0))\big]\cap[-1,1]^2$ is empty then the inclusion is obvious. Otherwise consider a point (a,b) which does not belong to $\{r_z\}\times\big[\frac{g}{2^z},\frac{g}{2^{z-1}}\big]$ for a fixed z. This gives that $a \neq r_z$ or $b \notin \big[\frac{g}{2^z},\frac{g}{2^{z-1}}\big]$. Assume that $a < r_z$. Then there exists $k_1 \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k > k_1$ we have

$$n_k^{(z)}\left(q_{i_k^{(z)}} - a_{i_k^{(z)}}\frac{1}{2^{i_k^{(z)}}} - x\right) > r_z - \frac{r_z - a}{2} > a,$$

so

$$a \notin n_k^{(z)}\left[q_{i_k^{(z)}} - a_{i_k^{(z)}}\frac{1}{2^{i_k^{(z)}}} - x, q_{i_k^{(z)}} + a_{i_k^{(z)}}\frac{1}{2^{i_k^{(z)}}} - x\right].$$

Hence $(a,b) \notin \limsup\limits_{k\in\mathbb{N}}\big(n_k^{(z)}, n_k^{(z)}\big)\cdot\big(E_{i_k^{(z)}+z}-(x,0)\big)$.

Similar arguments apply to the case when $a > r_z$ or $b \notin \big[\frac{g}{2^z},\frac{g}{2^{z-1}}\big]$.

Now we choose a subsequence $\{n_{k_p}\}_{p\in\mathbb{N}}$ using the diagonal method on the sequences $\{n_k^{(z)}\}_{k\in\mathbb{N}}$, $z \in \mathbb{N}$, thus $\{n_{k_p}\}_{p\geq z} \subset \{n_k^{(z)}\}_{k\in\mathbb{N}}$. It is evident that

$$\begin{aligned}
&\left[(n_{k_p}, n_{k_p})\cdot\left(\bigcup_{l\in\mathbb{N}} E_l - (x,0)\right)\right]\cap[-1,1]^2\\
&= \left[(n_{k_p}, n_{k_p})\cdot\left(\bigcup_{l=i_{k_p}}^{\infty} E_l - (x,0)\right)\right]\cap[-1,1]^2\\
&= \left[(n_{k_p}, n_{k_p})\cdot\left(\bigcup_{z=0}^{\infty} E_{i_{k_p}+z} - (x,0)\right)\right]\cap[-1,1]^2
\end{aligned}$$

for $p \in \mathbb{N}$.

It is sufficient to show that

$$\begin{aligned}
&\limsup_{p\in\mathbb{N}}\bigcup_{z=0}^{\infty}\Big[(n_{k_p}, n_{k_p})\cdot(E_{i_{k_p}+z}-(x,0))\Big]\cap[-1,1]^2\\
&\subset \bigcup_{z=0}^{\infty}\left[\left(\{r_z\}\times\left[\frac{g}{2^z},\frac{g}{2^{z-1}}\right]\right)\cup\left([0,1]\times\left\{\frac{g}{2^z}\right\}\right)\right].
\end{aligned}$$

Consider a point $(a,b) \in [0,1]^2\Big\backslash\Big(\bigcup\limits_{z=0}^{\infty}\big[\big(\{r_z\}\times\big[\frac{g}{2^z},\frac{g}{2^{z-1}}\big]\big)\cup\big([0,1]\times\big\{\frac{g}{2^z}\big\}\big)\big]\Big)$. There exists z_0 such that $b \in \big(\frac{g}{2^{z_0}},\frac{g}{2^{z_0-1}}\big)$ and there exists a number $p_1 \in \mathbb{N}$ such that for every $p \in N$, $p > p_1$ and for every $z \in \mathbb{N}$, $z \neq z_0$ the point (a,b) does not belong to the set $(n_{k_p}, n_{k_p})\cdot(E_{i_{k_p}+z_0}-(x,0))$. Moreover $a \neq r_{z_0}$, therefore $(a,b) \notin \limsup_{k\in\mathbb{N}} \big[\big(n_k^{(z_0)}, n_k^{(z_0)}\big) \cdot (E_{i_k^{(z_0)}+z_0} - (x,0))\big] \cap [-1,1]^2$, thus there exists $k_1 \in \mathbb{N}$, such that for each $k \in \mathbb{N}$, $k \geq k_1$ the point (a,b) does not belong to $(n_k^{(z_0)}, n_k^{(z_0)}) \cdot (E_{i_k^{(z_0)}+z_0} - (x,0))$, therefore there exists a number $p_2 \in \mathbb{N}$ such that for every $p \in N$, $p > p_2$ the point (a,b) does not belong to the set $(n_{k_p}, n_{k_p}) \cdot (E_{i_{k_p}+z_0} - (x,0))$.

Since for p greater than $p_0 = \max(p_1, p_2)$ the point (a,b) does not belong to the set $(n_{k_p}, n_{k_p}) \cdot (E_{i_{k_p}+z} - (x,0))$ for every $z \in \mathbb{N}$,

$$(a,b) \notin \limsup_{p\in\mathbb{N}} \bigcup_{z=0}^{\infty} \big[(n_{k_p}, n_{k_p}) \cdot (E_{i_{k_p}+z} - (x,0))\big] \cap [-1,1]^2,$$

which completes the proof. □

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University of Lodz
Department of Mathematics
and Computer Science
Banacha 22
90 238 Lodz
POLAND

E-mail: grhorb@math.uni.lodz.pl
wwil@uni.lodz.pl