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# ELIMINATION OF THE TYPE B UNCERTAINTY

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ABSTRACT. An uncertainty of an estimated parameter can be, in general, decomposed into two parts, i.e. an uncertainty caused by errors in the actual experiment (the type A uncertainty) and an uncertainty caused by errors in preceding experiment, where some wanted constants were estimated (the type B uncertainty). These constants are necessary for an estimation of the useful parameters.

The aim of the paper is to find a condition for elimination of the type B uncertainty.



## 1. Introduction

A direct/indirect measurement [3] of useful parameters needs often a knowledge of some constants. In many situations these constants are not known exactly, only their estimates are available before an experiment. Thus their uncertanties influence uncertainties of the parameter estimators (the type B uncertainties). The experiment raises the type A uncertainty.

This characterization is a great simplification. It is formulated in such a way in order to formulate one partial mathematical problem. Thus the A and B uncertainties have in the following text rather special meaning which will be clear from the context.

More about A and B uncertainties in metrology see in [2].

The aim of the paper is to find a condition for an elimination of the type B uncertainty.

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## 2. Symbols and preliminaries

The two stage model

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim_{n_1+n_2} \left[ \begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{D}, & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \gamma \\ \boldsymbol{\beta} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_1, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_2 \end{pmatrix} \right] \tag{1}$$

will be considered. Here $\mathbf{Y}_1$ is an $n_1$-dimensional random vector (observation vector of the first stage, where constants $\boldsymbol{\gamma}$ were estimated), with the mean value $E(\mathbf{Y}_1)$ equal to $\mathbf{X}_1\boldsymbol{\gamma}$, $\mathbf{X}_1$ is a known $n_1 \times k_1$ matrix with the rank $r(\mathbf{X}_1)$ equal to $k_1 < n_1$, and $\boldsymbol{\gamma}$ is a $k_1$-dimensional vector of constants, which must be estimated in the first stage. The $n_1 \times n_1$ matrix $\boldsymbol{\Sigma}_1$ is the covariance matrix of the vector $\mathbf{Y}_1$ and it is assumed that it is known and positive definite.

The best linear unbiased estimator (BLUE) of the vector $\boldsymbol{\gamma}$ in the first stage is well known (cf., e.g. [3]) and it is given by the relationship

$$\widehat{\gamma}(\mathbf{Y}_1) = \mathbf{C}_1^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1 \sim_{k_1} (\gamma, \mathbf{C}_1^{-1}),$$

where $\mathbf{C}_1 = \mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{X}_1$.

In the following text it is assumed that $\widehat{\gamma}(\mathbf{Y}_1)$ and $\mathbf{C}_1^{-1}$ are known before the second stage experiment

$$\mathbf{Y}_2 \sim_{n_2} \left[ (\mathbf{D}, \mathbf{X}_2) \begin{pmatrix} \gamma \\ \boldsymbol{\beta} \end{pmatrix}, \boldsymbol{\Sigma}_2 \right].$$

The symbol $\mathbf{Y}_2$ means the observation vector of the second stage and its mean value is $(\mathbf{D}, \mathbf{X}_2)\begin{pmatrix} \gamma \\ \boldsymbol{\beta} \end{pmatrix}$, where $\mathbf{D}$ is and $n_2 \times k_1$ known matrix, $\mathbf{X}_2$ is an $n_2 \times k_2$ known matrix and $r(\mathbf{X}_2) = k_2 < n_2$. The $k_2$-dimensional vector $\boldsymbol{\beta}$ is the unknown useful vector parameter. Its estimation is the aim of the experiment.

| | |
|---|---|
| $\mathbf{A}^-$ | means the $g$-inverse (generalized inverse) of the matrix $\mathbf{A}$, i.e. $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$ (in more detail cf. [4]), |
| $\mathbf{A}^+$ | means the Moore-Penrose $g$-inverse of the matrix $\mathbf{A}$, i.e. $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$, $\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$, $\mathbf{A}\mathbf{A}^+ = (\mathbf{A}\mathbf{A}^+)'$, $\mathbf{A}^+\mathbf{A} = (\mathbf{A}^+\mathbf{A})'$, |
| $\mathbf{A} \leq_L \mathbf{B}$ | means that $\mathbf{B} - \mathbf{A}$ is positive definite (the Loevner ordering), |
| $\mathbf{I}_n$ | means $n \times n$ identity matrix, $\mathbf{M}_X = \mathbf{I} - \mathbf{P}_X$, $\mathbf{P}_X = \mathbf{X}\mathbf{X}^+$, |
| $\mathbf{A}^-_{m(W)}$ | is the $\mathbf{W}$-seminorm minimum $g$-inverse of the matrix $\mathbf{A}$, i.e. $\mathbf{W}$ is at least positive semidefinite and $\mathbf{A}\mathbf{A}^-_{m(W)}\mathbf{A} = \mathbf{A} \quad \& \quad \mathbf{W}\mathbf{A}^-_{m(W)}\mathbf{A} = \left[\mathbf{W}\mathbf{A}^-_{m(W)}\mathbf{A}\right]'$. |

## 3. Estimation of the vector $\boldsymbol{\beta}$

There are at least two possibilities how to estimate the vector $\boldsymbol{\beta}$; to consider the estimator $\widehat{\gamma}(\mathbf{Y}_1)$ as $\boldsymbol{\gamma}$ and in the other case to use the estimator of $\boldsymbol{\beta}$ from Lemma 2.

**Lemma 1.** *The estimator in the first approach is*

$$\begin{aligned}\tilde{\boldsymbol{\beta}} &= (\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}[\mathbf{Y}_2 - \mathbf{D}\widehat{\gamma}(\mathbf{Y}_1)]\\ &\sim_{k_2} \big[\boldsymbol{\beta}, (\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2)^{-1}\big].\end{aligned}$$

*The type A uncertainty of this estimator is* $\mathbf{C}_2^{-1} = (\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2)^{-1}$ *and the type B uncertainty is*

$$\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}.$$

P r o o f. Proof is obvious. □

**Lemma 2.** *The BLUE of* $\boldsymbol{\beta}$ *in* (1) *can be expressed on the basis of* $\mathbf{Y}_2, \widehat{\gamma}(\mathbf{Y}_1), \mathbf{C}_1^{-1}$ *in the following way*

$$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) = \big[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\big]^{-1}\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\big[\mathbf{Y}_2 - \mathbf{D}\widehat{\gamma}(\mathbf{Y}_1)\big],$$

*where* $\widehat{\gamma}(\mathbf{Y}_1) = \mathbf{C}_1^{-1}\mathbf{X}'\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1$*;*

$$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) \sim_{k_2} \Big\{\boldsymbol{\beta}, \big[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\big]^{-1}\Big\}.$$

P r o o f. The BLUE of $\boldsymbol{\beta}$ in (1) is

$$\begin{aligned}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) = (\mathbf{0}, \mathbf{I})&\left[\begin{pmatrix} \mathbf{X}_1', & \mathbf{D}' \\ \mathbf{0}, & \mathbf{X}_2' \end{pmatrix}\begin{pmatrix} \boldsymbol{\Sigma}_1^{-1}, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_2^{-1} \end{pmatrix}\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{D}, & \mathbf{X}_2 \end{pmatrix}\right]^{-1}\\ &\times \begin{pmatrix} \mathbf{X}_1', & \mathbf{D}' \\ \mathbf{0}, & \mathbf{X}_2' \end{pmatrix}\begin{pmatrix} \boldsymbol{\Sigma}_1^{-1}, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_2^{-1} \end{pmatrix}\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}.\end{aligned}$$

Here

$$\begin{aligned}&\begin{pmatrix} \mathbf{X}_1', & \mathbf{D}' \\ \mathbf{0}, & \mathbf{X}_2' \end{pmatrix}\begin{pmatrix} \boldsymbol{\Sigma}_1^{-1}, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_2^{-1} \end{pmatrix}\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{D}, & \mathbf{X}_2 \end{pmatrix}\\ &\qquad = \begin{pmatrix} \mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}, & \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2 \\ \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}, & \mathbf{C}_2 \end{pmatrix}\end{aligned}$$

and

$$\begin{pmatrix} \mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}, & \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2 \\ \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}, & \mathbf{C}_2 \end{pmatrix}^{-1} = \begin{pmatrix} \boxed{11}, & \boxed{12} \\ \boxed{21}, & \boxed{22} \end{pmatrix}$$

$$
\begin{aligned}
\boxed{11} &= (\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1} + (\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\Big[\mathbf{C}_2 - \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D} \\
&\qquad \times (\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\Big]^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}, \\
\boxed{12} &= -(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2 \\
&\qquad \times \Big[\mathbf{C}_2 - \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\Big]^{-1} = \boxed{21}', \\
\boxed{22} &= \Big[\mathbf{C}_2 - \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\Big]^{-1} \\
&= \Big[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\Big]^{-1}.
\end{aligned}
$$

The last equality follows from the relationships

$$
\begin{aligned}
&\Big[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\Big]^{-1} \\
&\quad = \Big\{\mathbf{X}_2'\Big[\boldsymbol{\Sigma}_2^{-1} - \boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\Big]\mathbf{X}_2\Big\}^{-1} \\
&\quad = \Big[\mathbf{C}_2 - \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\Big]^{-1}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\widehat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2) &= \Big(\boxed{21}, \boxed{22}\Big)\begin{pmatrix} \mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\boldsymbol{Y}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}_2 \\ \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}_2 \end{pmatrix} \\
&= -\Big[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\Big]^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D} \\
&\qquad \times \Big[\mathbf{C}_1^{-1} - \mathbf{C}_1^{-1}\mathbf{D}'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\mathbf{C}_1^{-1}\Big](\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\boldsymbol{Y}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}_2) \\
&\qquad + \Big[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\Big]^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}_2 \\
&= \Big[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\Big]^{-1}\Big\{\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}_2 - \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D} \\
&\qquad \times \Big[\mathbf{C}_1^{-1} - \mathbf{C}_1^{-1}\mathbf{D}'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\mathbf{C}_1^{-1}\Big]\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}_2 \\
&\qquad - \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\Big[\mathbf{C}_1^{-1} - \mathbf{C}_1^{-1}\mathbf{D}'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\mathbf{C}_1^{-1}\Big]\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\boldsymbol{Y}_1.\Big\}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}_2 - \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}_2 \\
&\quad + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}_2
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}_2 - \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}_2 \\
&\quad + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\Big[(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}') - \boldsymbol{\Sigma}_2\Big](\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}_2 \\
&= \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}_2 - \mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}_2 \\
&= \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}_2 - \mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\Big[(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}') - \boldsymbol{\Sigma}_2\Big]\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}_2 \\
&= \mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\boldsymbol{Y}_2
\end{aligned}
$$

and

$$
\begin{aligned}
&\quad - \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\Big[\mathbf{C}_1^{-1} - \mathbf{C}_1^{-1}\mathbf{D}'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\mathbf{C}_1^{-1}\Big]\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\boldsymbol{Y}_1 \\
&= -\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\mathbf{C}_1^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\boldsymbol{Y}_1 \\
&\quad + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\mathbf{C}_1^{-1}\mathbf{X}'\boldsymbol{\Sigma}_1^{-1}\boldsymbol{Y}_1 \\
&= -\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\mathbf{C}_1^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\boldsymbol{Y}_1 \\
&\quad + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\Big[(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}') - \boldsymbol{\Sigma}_2\Big](\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\mathbf{C}_1^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\boldsymbol{Y}_1 \\
&= \mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\widehat{\gamma}(\boldsymbol{Y}_1),
\end{aligned}
$$

we have

$$
\widehat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2) = \Big[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_2^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\Big]^{-1}\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_2^{-1}\mathbf{D}')^{-1}\Big[\boldsymbol{Y}_2 - \mathbf{D}\widehat{\gamma}(\boldsymbol{Y}_1)\Big].
$$

The relation

$$
\mathrm{Var}\Big[\widehat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)\Big] = \Big[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\Big]^{-1}
$$

is obvious. □

**Lemma 3.** *It is valid that*

$$
\begin{aligned}
&\mathrm{Var}\Big[\widehat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)\Big] \\
&\quad = \mathbf{C}_2^{-1} + \mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\Big[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{D}\Big]^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}.
\end{aligned}
$$

*The type A uncertainty is* $\mathbf{C}_2^{-1}$ *and the type B uncertainty is*

$$
\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\Big[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{D}\Big]^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}.
$$

Proof.

$$
\begin{aligned}
&\Big[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\Big]^{-1} \\
&\quad = \left\{\mathbf{X}_2'\Big[\boldsymbol{\Sigma}_2^{-1} - \boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\Big]^{-1}\mathbf{X}_2\right\}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big[\mathbf{C}_2 - \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\Big]^{-1} \\
&= \mathbf{C}_2^{-1} + \mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D} - \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1} \\
&\quad \times \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1} \\
&= \mathbf{C}_2^{-1} + \mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\Big[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{D}\Big]^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}.
\end{aligned}
$$

□

Evidently the type B uncertainty in this case is smaller than the type B uncertainty from Lemma 1, i.e.

$$
\begin{aligned}
\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\Big[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{D}\Big]^{-1}&\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1} \\
&\leq_L \mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}.
\end{aligned}
$$

Since $\widehat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$ from Lemma 2 is the BLUE in the model (1), another linear unbiased estimator based on $\boldsymbol{Y}_1, \boldsymbol{Y}_2$ cannot be better. However it can be interesting whether some linear function $\boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in R^{k_2}$, can be estimated on the basis $\boldsymbol{Y}_2$ with the same dispersion as $\mathrm{Var}\left[\boldsymbol{h}'\widehat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)\right]$.

**Lemma 4.** *The class of linear unbiasedly estimable functions of the parameter $\boldsymbol{\beta}$ on the basis of $\boldsymbol{Y}_2$ is characterized by the subspace*

$$\mathcal{M}(\mathbf{X}_2'\mathbf{M}_D) = \{\mathbf{X}_2'\mathbf{M}_D\boldsymbol{u} :\ \boldsymbol{u} \in R^{n_2}\},$$

*i.e. $\boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in R^{k_2}$ is unbiasedly estimable on the basis $\boldsymbol{Y}_2$ iff $\boldsymbol{h} \in \mathcal{M}(\mathbf{X}_2'\mathbf{M}_D)$.*

Proof. If $\widehat{\boldsymbol{h}\boldsymbol{\beta}} = \boldsymbol{l}'\boldsymbol{Y}_2$, then

$$
\begin{aligned}
&\forall\{\boldsymbol{\beta} \in R^{k_2}, \boldsymbol{\gamma} \in R^{k_1}\} \\
&\boldsymbol{h}'\boldsymbol{\beta} = \boldsymbol{l}'(\mathbf{X}_2\boldsymbol{\beta} + \mathbf{D}\boldsymbol{\gamma}) = \boldsymbol{l}'\mathbf{X}_2\boldsymbol{\beta} + \boldsymbol{l}'\mathbf{D}\boldsymbol{\gamma} \iff \mathbf{X}_2'\boldsymbol{l} = \boldsymbol{h}\ \&\ \mathbf{D}'\boldsymbol{l} = \mathbf{0} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \iff \mathbf{M}_D\boldsymbol{u} = \boldsymbol{l},
\end{aligned}
$$

thus $\boldsymbol{h} \in \mathcal{M}(\mathbf{X}_2'\mathbf{M}_D)$.

If $\boldsymbol{h} \in \mathcal{M}(\mathbf{X}_2'\mathbf{M}_D)$, the equation $\boldsymbol{u}'\mathbf{M}_D\mathbf{X}_2 = \boldsymbol{h}'$ has a solution $\boldsymbol{u} = (\mathbf{X}_2'\mathbf{M}_D)^{-}\boldsymbol{h}$ and

$$E_{\beta,\gamma}\Big\{\boldsymbol{h}'\Big[(\mathbf{X}_2'\mathbf{M}_D)^{-}\Big]'\mathbf{M}_D\boldsymbol{Y}_2\Big\} = \boldsymbol{h}'\Big[(\mathbf{X}_2'\mathbf{M}_D)^{-}\Big]'\mathbf{M}_D\mathbf{X}_2\boldsymbol{\beta} = \boldsymbol{h}'\boldsymbol{\beta}.$$

□

**Lemma 5.** *In the model*

$$\mathbf{M}_D \boldsymbol{Y}_2 \sim_{n_2} (\mathbf{M}_D \mathbf{X}_2 \boldsymbol{\beta}, \mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D) \tag{2}$$

*the BLUE* $\widehat{\mathbf{M}_D \mathbf{X}_2 \boldsymbol{\beta}}$ *of the vector* $\mathbf{M}_D \mathbf{X}_2 \boldsymbol{\beta}$ *is*

$$\widehat{\mathbf{M}_D \mathbf{X}_2 \boldsymbol{\beta}} = \mathbf{M}_D \mathbf{X}_2 \Big[\mathbf{X}_2'(\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \mathbf{X}_2\Big]^- \mathbf{X}_2'(\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \boldsymbol{Y}_2$$

*and*

$$\widehat{\mathbf{M}_D \mathbf{X}_2 \boldsymbol{\beta}} \sim_{n_2} \Big\{\mathbf{M}_D \mathbf{X}_2 \boldsymbol{\beta}, \mathbf{M}_D \mathbf{X}_2 \Big[\mathbf{X}_2'(\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \mathbf{X}_2\Big]^- \mathbf{X}_2' \mathbf{M}_D\Big\}.$$

P r o o f. In a singular linear model [1] $\boldsymbol{\eta} \sim_n (\mathbf{A}\boldsymbol{\beta}, \mathbf{W})$ the blue of $\mathbf{A}\boldsymbol{\beta}$ is $\mathbf{A}\big[(\mathbf{A}')^-_{m(W)}\big]'\boldsymbol{\eta}$. Thus in the model (2)

$$\begin{aligned} &\widehat{\mathbf{M}_D \mathbf{X}_2 \boldsymbol{\beta}} \\ &\quad= \mathbf{M}_D \mathbf{X}_2 \Big[(\mathbf{X}_2' \mathbf{M}_D)^-_{m(M_D \Sigma_2 M_D)}\Big]' \mathbf{M}_D \boldsymbol{Y}_2 \\ &\quad= \mathbf{M}_D \mathbf{X}_2 \big[\mathbf{X}_2' \mathbf{M}_D (\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \mathbf{M}_2 \mathbf{X}_2\big]^- \mathbf{X}_2' \mathbf{M}_D (\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \mathbf{M}_D \boldsymbol{Y}_2, \end{aligned}$$

since

$$\Big[(\mathbf{X}_2' \mathbf{M}_D)^-_{m(M_D \Sigma_2 M_D)}\Big]' = \big[\mathbf{X}_2' \mathbf{M}_D (\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \mathbf{M}_2 \mathbf{X}_2\big]^- \mathbf{X}_2' \mathbf{M}_D (\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+.$$

The last equality is implied by the inclusion $\mathcal{M}(\mathbf{M}_D \mathbf{X}_2) \subset \mathcal{M}(\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)$ and by the equalities

$$\begin{aligned} \mathbf{M}_D (\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \mathbf{M}_D &= (\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \mathbf{M}_D \\ &= \mathbf{M}_D (\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ = (\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+. \end{aligned}$$

Thus

$$\widehat{\mathbf{M}_D \mathbf{X}_2 \boldsymbol{\beta}} = \mathbf{M}_D \mathbf{X}_2 \Big[\mathbf{X}_2'(\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \mathbf{X}_2\Big]^- \mathbf{X}_2'(\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \boldsymbol{Y}_2$$

and

$$\begin{aligned} &\mathrm{Var}(\widehat{\mathbf{M}_D \mathbf{X}_2 \boldsymbol{\beta}}) \\ &= \mathbf{M}_D \mathbf{X}_2 \big[\mathbf{X}_2'(\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \mathbf{X}_2\big]^- \mathbf{X}_2'(\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \boldsymbol{\Sigma}_2 (\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \mathbf{X}_2' \mathbf{M}_D \\ &= \mathbf{M}_D \mathbf{X}_2 \big[\mathbf{X}_2'(\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \mathbf{X}_2\big]^- \mathbf{X}_2' \mathbf{M}_D. \end{aligned}$$

In the last equality the relationships

$$\begin{aligned} (\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \boldsymbol{\Sigma}_2 (\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ &= (\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D (\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \\ &= (\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \end{aligned}$$

and the fact that the expression

$$\mathbf{M}_D \mathbf{X}_2 \big[\mathbf{X}_2'(\mathbf{M}_D \boldsymbol{\Sigma}_2 \mathbf{M}_D)^+ \mathbf{X}_2\big]^- \mathbf{X}_2' \mathbf{M}_D$$

is invariant on the choice of the $g$-inverse were utilized. $\square$

**Lemma 6.** *It is valid that*

$$\begin{aligned}
&\mathbf{M}_D\mathbf{X}_2\big[\mathbf{X}_2'(\mathbf{M}_D\boldsymbol{\Sigma}_2\mathbf{M}_D)^+\mathbf{X}_2\big]^-\mathbf{X}_2'\mathbf{M}_D\\
&=\mathbf{M}_D\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\mathbf{M}_D+\mathbf{M}_D\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\big[\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}\big]^+\\
&\quad\times\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\mathbf{M}_D.
\end{aligned}$$

Proof.

$$\begin{aligned}
&\mathbf{M}_D\mathbf{X}_2\big[\mathbf{X}_2'(\mathbf{M}_D\boldsymbol{\Sigma}_2\mathbf{M}_D)^+\mathbf{X}_2\big]^-\mathbf{X}_2'\mathbf{M}_D\\
&=\mathbf{M}_D\mathbf{X}_2\Big\{\mathbf{X}_2'\Big[\boldsymbol{\Sigma}_2^{-1}-\boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^-\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\Big]\mathbf{X}_2\Big\}^-\mathbf{X}_2'\mathbf{M}_D\\
&=\mathbf{M}_D\mathbf{X}_2\Big[\mathbf{C}_2-\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^-\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\Big]^-\mathbf{X}_2'\mathbf{M}_D\\
&=\mathbf{M}_D\mathbf{X}_2\Big\{\mathbf{C}_2^{-1}+\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\Big[\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}-\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\Big]^-\\
&\quad\times\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}\Big\}\mathbf{X}_2'\mathbf{M}_D\\
&=\mathbf{M}_D\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\mathbf{M}_D+\mathbf{M}_D\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\big[\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}\big]^+\\
&\quad\times\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\mathbf{M}_D.
\end{aligned}$$

Here the relationships

$$(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+=\boldsymbol{\Sigma}_2^{-1}-\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1},$$
$$\mathcal{M}(\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})\subset\mathcal{M}(\mathbf{C}_2)\quad\text{and}\quad\mathcal{M}(\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2)\subset\mathcal{M}(\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})$$

were utilized. □

**Theorem 1.** *We have the following necessary and sufficient condition*

$$\mathbf{M}_D\mathbf{X}_2\widehat{\boldsymbol{\beta}}(\boldsymbol{Y}_2)=\mathbf{M}_D\mathbf{X}_2\widehat{\boldsymbol{\beta}}(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\iff\mathbf{M}_D\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}=\mathbf{0}.$$

Proof. Let $\mathbf{M}_D\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}=\mathbf{0}$. Then

$$\begin{aligned}
&\mathbf{M}_D\mathbf{X}_2\widehat{\boldsymbol{\beta}}(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\\
&=\mathbf{M}_D\mathbf{X}_2\big[\mathbf{X}_2'(\boldsymbol{\Sigma}_2+\mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\big]^{-1}\mathbf{X}_2'(\boldsymbol{\Sigma}_2+\mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\big[\boldsymbol{Y}_2-\mathbf{D}\widehat{\boldsymbol{\gamma}}(\boldsymbol{Y}_1)\big]\\
&=\mathbf{M}_D\mathbf{X}_2\Big\{\mathbf{X}_2'\big[\boldsymbol{\Sigma}_2^{-1}-\boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{C}_1+\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\big]\mathbf{X}_2\Big\}^{-1}\\
&\quad\times\mathbf{X}_2'(\boldsymbol{\Sigma}_2+\mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\big[\boldsymbol{Y}_2-\mathbf{D}\widehat{\boldsymbol{\gamma}}(\boldsymbol{Y}_1)\big]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{M}_D\mathbf{X}_2\big[\mathbf{C}_2 - \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\big]^{-1} \\
&\quad \times \mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\big[\boldsymbol{Y}_2 - \mathbf{D}\widehat{\gamma}(\boldsymbol{Y}_1)\big] \\
&= \mathbf{M}_D\mathbf{X}_2\Big\{\mathbf{C}_2^{-1} + \mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D} - \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1} \\
&\quad \times \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}\Big\}\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\big[\boldsymbol{Y}_2 - \mathbf{D}\widehat{\gamma}(\boldsymbol{Y}_1)\big] \\
&= \mathbf{M}_D\mathbf{X}_2\Big\{\mathbf{C}_2^{-1} - \mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\big[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{D}\big]^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}\Big\} \\
&\quad \times \mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\big[\boldsymbol{Y}_2 - \mathbf{D}\widehat{\gamma}(\boldsymbol{Y}_1)\big] \\
&= \mathbf{M}_D\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\big[\boldsymbol{Y}_2 - \mathbf{D}\widehat{\gamma}(\boldsymbol{Y}_1)\big] \\
&= \mathbf{M}_D\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\big[\boldsymbol{\Sigma}_2^{-1} - \boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\big]\big[\boldsymbol{Y}_2 - \mathbf{D}\widehat{\gamma}(\boldsymbol{Y}_1)\big] \\
&= \mathbf{M}_D\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}_2.
\end{aligned}
$$

On the other hand let $\mathbf{M}_D\mathbf{X}_2\widehat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2) = \widehat{\mathbf{M}_D\mathbf{X}_2\boldsymbol{\beta}}(\boldsymbol{Y}_2)$. Then it must be valid that

$$
\begin{aligned}
\operatorname{Var}\big[\mathbf{M}_D\mathbf{X}_2\widehat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)\big] &= \mathbf{M}_D\mathbf{X}_2\big[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\big]^{-1}\mathbf{X}_2'\mathbf{M}_D \\
&= \mathbf{M}_D\mathbf{X}_2\big[\mathbf{X}_2'(\mathbf{M}_D\boldsymbol{\Sigma}_2\mathbf{M}_D)^{+}\mathbf{X}_2\big]^{-}\mathbf{X}_2'\mathbf{M}_D \\
&= \operatorname{Var}\big[\widehat{\mathbf{M}_D\mathbf{X}_2\boldsymbol{\beta}}(\boldsymbol{Y}_2)\big].
\end{aligned}
$$

Since (cf. Lemma 3)

$$
\begin{aligned}
&\mathbf{M}_D\mathbf{X}_2\big[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\big]^{-1}\mathbf{X}_2'\mathbf{M}_D \\
&= \mathbf{M}_D\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\mathbf{M}_D + \mathbf{M}_D\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\big[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{D}\big]^{-1} \\
&\quad \times \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\mathbf{M}_D
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{M}_D\mathbf{X}_2\big[\mathbf{X}_2'(\mathbf{M}_D\boldsymbol{\Sigma}_2\mathbf{M}_D)^{+}\mathbf{X}_2\big]^{-}\mathbf{X}_2'\mathbf{M}_D \\
&= \mathbf{M}_D\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\mathbf{M}_D + \mathbf{M}_D\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}\big[\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^{+}\mathbf{D}\big]^{-} \\
&\quad \times \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\mathbf{M}_D,
\end{aligned}
$$

it must be valid that

$$\mathbf{M}_D\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D} = \mathbf{0}.$$

□

**Remark 1.** If $\mathbf{M}_D\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D} = \mathbf{0}$, then also the estimator $\tilde{\boldsymbol{\beta}}(\boldsymbol{Y}_2)$ from Lemma 1 can be used for the BLUE of $\mathbf{M}_D\mathbf{X}_2\boldsymbol{\beta}$.

## 4. Numerical example

Let the two stage model be

$$\begin{pmatrix} Y_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim_3 \left[ \begin{pmatrix} 1, & 0, & 0 \\ 1, & 1, & 0 \\ 1, & 1, & 1 \end{pmatrix} \begin{pmatrix} \gamma \\ \beta_1 \\ \beta_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2, & 0, & 0 \\ 0, & \sigma_2^2, & 0 \\ 0, & 0, & \sigma_2^2 \end{pmatrix} \right],$$

i.e.

$$X_1 = 1, \quad \mathbf{D} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} 1, & 0 \\ 1, & 1 \end{pmatrix}, \quad C_1^{-1} = \sigma_1^2, \quad \mathbf{C}_2^{-1} = \sigma_2^2 \begin{pmatrix} 1, & -1 \\ -1, & 2 \end{pmatrix},$$

$$\mathbf{M}_D = \frac{1}{2} \begin{pmatrix} 1, & -1 \\ -1, & 1 \end{pmatrix}, \qquad \mathbf{M}_D \mathbf{X}_2 = \frac{1}{2} \begin{pmatrix} 0, & -1 \\ 0, & 1 \end{pmatrix},$$

$$\mathbf{M}_D \mathbf{X}_2 \big[\mathbf{X}_2'(\mathbf{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2\big]^{-1} \mathbf{X}_2' \mathbf{M}_D = \frac{\sigma_2^2}{2} \begin{pmatrix} 1, & -1 \\ -1, & 1 \end{pmatrix},$$

$$\mathbf{M}_D \mathbf{X}_2 \big[\mathbf{X}_2'(\mathbf{M}_D \mathbf{\Sigma}_2 \mathbf{M}_D)^{+}\mathbf{X}_2\big]^{-} \mathbf{X}_2' \mathbf{M}_D = \frac{\sigma_2^2}{2} \begin{pmatrix} 1, & -1 \\ -1, & 1 \end{pmatrix}.$$

Thus, in this case, all functions $\boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in R^2$, $\boldsymbol{h} \in \mathcal{M}\begin{pmatrix} 0, & 0 \\ -1, & 1 \end{pmatrix}$, i.e. $\boldsymbol{h} = \begin{pmatrix} 0 \\ a \end{pmatrix}$, $a \in R^1$, can be estimated on the basis of $\boldsymbol{Y}_2$.

$$\mathbf{M}_D \mathbf{X}_2 \mathbf{C}_2^{-1} \mathbf{X}_2' \mathbf{\Sigma}_2^{-1} \mathbf{D} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

### REFERENCES


[1] FIŠEROVÁ, E.—KUBÁČEK, L.—KUNDEROVÁ, P.: *Linear Statistical Models; Regularity and Singularities*, Academia, Praha, 2007.

[2] *Guide to the Expression of Uncertainty in Measurement*, International Organization for Standartization, Switzerland, 1993.

[3] KUBÁČEK, L.—KUBÁČKOVÁ, L.—VOLAUFOVÁ, J.: *Statistical Models with Linear Structures*, Veda, Bratislava, 1995.

[4] RAO, C. R.—MITRA, S. K.: *Generalized Inverse of Matrices and Its Applications*, John Wiley & Sons, Inc. New York-London-Sydney-Toronto, 1971.



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