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MAXIMAL INDEPENDENT, ANALYTIC SETS IN ABELIAN POLISH GROUPS

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ABSTRACT. The paper deals with the questions:
(a) whether a topological module admits maximal linearly independent subsets that are analytic;
(b) whether an Abelian topological group admits maximal independent subsets that are analytic;
(c) whether a topological field extension admits transcendence bases that are analytic.

1. Introduction

It dates back to Sierpiński [6] that a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ cannot be analytic (a fortiori, cannot be Borel). More recently, it has been noticed in [11] that the field-theoretic analogue of his result holds true as well: no transcendence basis of $\mathbb{R}$ over $\mathbb{Q}$ can be analytic. Moreover, Bartoszyński et al. [1, Theorem 3.9] have shown that in an infinite dimensional separable real Banach space no Hamel basis can be analytic. Starting from Sierpiński's original idea, and making repeated appeal to Pettis's Theorem 2.3, in this paper we aim to sharpen and extend these results to different topological-algebraic structures.

In the third section, we study the question whether a topological (unitary) module M over a topological integral domain R admits a maximal linearly independent, analytic subset. Specifically, we shall prove in Theorem 3.2 that, if M is a nondiscrete Polish space and R is analytic, then a maximal linearly

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independent subset S of M is analytic precisely when S is countable. Moreover, under the additional assumption that M is torsion-free, S is analytic iff it is finite. Theorem 3.2 can be further sharpened under the hypothesis that M is free: in this case we shall prove in Theorem 3.3 that S is analytic iff for some $n \in \mathbb{N}$ the topological R-modules M and R^n are isomorphic. In particular, if V is an infinite dimensional topological vector space over a topological field K, and both V and K are separable, metrizable and complete, then no Hamel basis of V can be analytic (see Corollary 3.5). On the one hand, for $K = \mathbb{R}$, this fact extends the afore-cited theorem in [1] for Banach spaces; on the other hand, for $K = \mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$, it is applied in the fourth section where the existence of maximal independent (instead of linearly independent), analytic subsets of Abelian topological groups is considered.

In Section 4 we show that no Abelian nondiscrete Polish group admits maximal independent, analytic subsets (Theorem 4.2). The proof of Theorem 4.2 is based to a large extent on the results of Section 3. First we apply Theorem 3.2, considering an Abelian topological group G as a topological $\mathbb{Z}$-module; second we use Corollary 3.5, considering each p-component of G as a topological vector space over the finite field $\mathbb{F}_p$.

Section 5 deals with topological field extensions $K|L$, where K is a nondiscrete Polish space and L is analytic. Theorem 5.2 states that, if $K|L$ is a transcendental extension, then no transcendence basis of $K|L$ can be analytic. If $K|L$ is a separable algebraic extension, then in Theorem 5.4(b) we prove that K must be a simple extension of L topologically isomorphic to $L^{(K:L)}$. Some information in the case that $K|L$ is an inseparable algebraic extension is given in Theorem 5.4(a).

2. Topological preliminaries

A *Polish space* is a topological space that is homeomorphic to a separable complete metric space. An *analytic space* is a Hausdorff topological space that is a continuous image of a Polish space. In particular, the empty set is analytic. A subset of a topological space is analytic if it is analytic in its relative topology. Obviously, any analytic space is separable.

Throughout, we shall use without any mention the following basic facts about analytic spaces the proof of which can be found in [2, Sections 6.6; 6.7], for instance.[1]

The product of finitely many analytic spaces is analytic.

[1]Analytic spaces are sometimes called "Souslin spaces", as in Bogachev's book.

If $A_1, A_2, \dots$ are analytic subsets of a Hausdorff space, then so are $\bigcup\limits_{n=1}^{\infty} A_n$ and $\bigcap\limits_{n=1}^{\infty} A_n$.

Every Borel subset of an analytic space is analytic.

A Hausdorff space that is a continuous image of an analytic set is analytic.

If X, Y are analytic spaces, A is an analytic subset of Y and $f\colon X \to Y$ is a continuous map, then $f^{-1}(A)$ is analytic.

Let G be a topological group. If G is a Polish space, then there exists a metric d on G inducing the topology of G such that (G, d) is a complete metric space. If G is Abelian, then, as proved by Christensen [3, Theorem 5.4], G is even a complete topological group, i.e. complete with respect to the uniformity generated by the sets $\bigl\{(x, y) \in G \times G :\ x - y \in U\bigr\}$ where U is a 0-neighbourhood. More precisely, Christensen proved:

Theorem 2.1. *Let G be an Abelian topological group. Then the following statements are equivalent:*

(1) *G is an analytic space of second category.*

(2) *G is a separable metrizable complete topological group.*

(3) *G is a Polish space.*

Proof.

(1) $\implies$ (2) follows from [3, Theorem 5.4] and the fact that the topology of an Abelian metrizable topological group is induced by an invariant metric (see [8, Theorem 6.4]).

(2) $\implies$ (3) is obvious and (3) $\implies$ (1) follows from the Baire category theorem. □

In our results we shall assume that a topological group (or module, or field) is a Polish space. The previous theorem gives conditions equivalent to this assumption.

Often we assume that a topological group G that is a Polish space is also nondiscrete. Equivalently, we may assume that G is uncountable. Indeed, if G is countable, then G has an isolated point by the Baire category theorem and therefore it is discrete. Vice versa, if G is discrete, then G is countable since it is separable.

The next facts are fundamental for all what follows. Recall that a subset A of a topological space X has the *Baire property* if there is an open subset O of X such that $A \triangle O$ is of first category in X.

Proposition 2.2. ([7, Theorem 2.2.9]) *An analytic subset of a Hausdorff space has the Baire property.*

In Corollary 2.4 we shall combine the previous proposition with the following theorem due to Pettis.

Theorem 2.3. ([5, Theorem 1], [3, Theorem 5.1]) *Let A be a subset of a Hausdorff topological group G. If A has the Baire property and is of second category in G, then $A - A$ is a 0-neighbourhood.*

Corollary 2.4 below shall be repeatedly applied, in this paper.

Corollary 2.4. *Let G be a topological group. If G is Polish and $A_1, A_2, \dots$ are analytic subsets of G with $G = \bigcup\limits_{n=1}^{\infty} A_n$, then for some $n \in \mathbb{N}$ the set $A_n - A_n$ is a 0-neighbourhood.*

Proof. By the Baire category theorem there exists $n \in \mathbb{N}$ such that A_n is of second category in G. Moreover, A_n has the Baire property by Proposition 2.2. Therefore, by Theorem 2.3, the set $A_n - A_n$ is a 0-neighbourhood. □

3. Linearly independent sets in topological modules

In this section R stands for an integral domain and M for a unitary[2] R-module.

If $S \subseteq M$, then $R\langle S\rangle$ denotes the submodule of M generated by S. Throughout, let $S^\times := S \setminus \{0\}$.

Recall that a subset S of M is *linearly independent (over R)* if for every finite number of distinct elements $s_1, s_2, \dots, s_n$ of S and every $\alpha_1, \alpha_2, \dots, \alpha_n \in R$, from $\sum\limits_{i=1}^{n} \alpha_i s_i = 0$ it follows $\alpha_i = 0$ for all $i = 1, 2, \dots, n$.

Linear independence can be characterized with the aid of the closure operator $M \supseteq S \mapsto A(S)$ defined by

$$A(S) := \big\{x \in M :\ (\exists \alpha \in R^\times)\big(\alpha x \in R\langle S\rangle\big)\big\};$$

$A(S)$ is a submodule of M containing S as well as the torsion submodule of M

$$\big\{x \in M :\ (\exists \alpha \in R^\times)(\alpha x = 0)\big\}.$$

One immediately sees that S is linearly independent iff $x \notin A\big(S \setminus \{x\}\big)$ for all $x \in S$.

[2] i.e. $1 \cdot x = x$ for all $x \in M$

The operator $S \mapsto A(S)$ satisfies the five axioms given on [10, p. 50]: A is monotone and idempotent; $S \subseteq A(S)$; $x \in A(S)$ implies $x \in A(F)$ for some finite subset F of S; $x \in A\big(S \cup \{y\}\big) \setminus A(S)$ implies $y \in A\big(S \cup \{x\}\big)$. It now follows from [10, Theorem I.20] that a subset S of M is a maximal linearly independent set iff it is linearly independent and $A(S) = M$ iff S is minimal with $A(S) = M$. Moreover, by [10, Theorem II.24], any two maximal linearly independent subsets of M have the same cardinality, which we denote by $\operatorname{rank}(M)$.

Lemma 3.1. *Let M be a Hausdorff topological module over a topological integral domain R and let $S \subseteq M$. Further suppose that R and S are analytic.*

(a) *Then $R\langle S\rangle$ is analytic.*

(b) *If also M is analytic, then the sets*

$$A_n := \bigcup_{\substack{F \subseteq S \\ |F| \le n}} A(F) \qquad (n \in \mathbb{N})$$

and $A(S)$ are analytic.

Proof.

(a) For $n \in \mathbb{N}$ the set

$$M_n := \Big\{ \sum_{i=1}^{n} \alpha_i s_i : \ \big(\forall i \in \{1, 2, \ldots, n\}\big)\big(\alpha_i \in R \ \ \& \ \ s_i \in S\big) \Big\}$$

is analytic as image of the analytic set $(R \times S)^n$ under the continuous map

$$(\alpha_1, s_1, \alpha_2, s_2, \ldots, \alpha_n, s_n) \mapsto \sum_{i=1}^{n} \alpha_i s_i.$$

Hence $R\langle S\rangle = \bigcup_{n=1}^{\infty} M_n$ is analytic.

(b) Let $p\colon R^{\times} \times M \to M$ and $\pi\colon R^{\times} \times M \to M$ be defined by $p(\alpha, x) := \alpha x$ and $\pi(\alpha, x) := x$, respectively. Since $R^{\times}$, M and M_n are analytic, so are

$$A_n = \pi(p^{-1}(M_n)) \qquad \text{and} \qquad A(S) = \bigcup_{n=1}^{\infty} A_n.$$

$\square$

The next theorem is the basic result of this section. An important tool in its proof is Corollary 2.4.

Theorem 3.2. *Let M be a topological module over a topological integral domain R and let S be a maximal linearly independent subset of M. Further assume that M is a Polish space, and S and R are analytic spaces.*

(a) *Then* $\operatorname{rank}(M)$ *is countable. Moreover, S contains a finite subset F such that $A(F)$ is open.*

(b) *If M is nondiscrete and torsion-free,*[3] *then R is nondiscrete; and if R is nondiscrete, then* $\operatorname{rank}(M)$ *is finite.*

Proof.

(a) For $n \in \mathbb{N}$ define A_n as in Lemma 3.1. By the maximality of S we have $M = A(S) = \bigcup_{n=1}^{\infty} A_n$. By Corollary 2.4 and Lemma 3.1(b) there is $n \in \mathbb{N}$ such that $A_n - A_n$ is a 0-neighbourhood. Since $A_n - A_n \subseteq A_{2n}$, also A_{2n} is a 0-neighbourhood. Therefore, by the separability of M, there exists a countable subset D of M such that $M = D + A_m$ where $m := 2n$. For any $d \in D$ there exists a finite subset F_d of S with $d \in A(F_d)$. Then $C := \bigcup_{d \in D} F_d$ is countable and $D \subseteq A(C)$.

Suppose now, by way of contradiction, that $\operatorname{rank}(M)$ is uncountable, i.e. S is uncountable. Then there exist $m+1$ distinct elements $z_1, z_2, \dots, z_{m+1}$ in $S \setminus C$. Put $x := \sum_{i=1}^{m+1} z_i$. Since $x \in D + A_m$, there are $s_1, s_2, \dots, s_m \in S$ such that $x \in A(C \cup \{s_1, s_2, \dots, s_m\})$. At least one of the elements $z_1, z_2, \dots, z_{m+1}$ cannot be contained in $\{s_1, s_2, \dots, s_m\}$. We may assume that $z_{m+1} \notin \{s_1, s_2, \dots, s_m\}$. Then $C \cup \{s_1, s_2, \dots, s_m\} \subseteq S \setminus \{z_{m+1}\}$, hence $x \in A(S \setminus \{z_{m+1}\})$ and thus

$$z_{m+1} = x - \sum_{i=1}^{m} z_i \in A\big(S \setminus \{z_{m+1}\}\big).$$

But this is in contrast with the linear independence of S.

As we have seen, S can be written as $S = \{s_n : n \in \mathbb{N}\}$. Let $F_n = \{s_i : i = 1, \dots, n\}$. Then $M = \bigcup_{n=1}^{\infty} A(F_n)$ and each $A(F_n)$ is analytic by Lemma 3.1. A further appeal to Corollary 2.4 shows that $A(F_m) - A(F_m)$ is a 0-neighbourhood for some $m \in \mathbb{N}$. Since $A(F_m)$ is a submodule of M, it follows that $A(F_m)$ is open.

(b) (i) Let M be nondiscrete and torsion-free. Suppose that R is discrete. Using that R is separable, S is countable and M is torsion-free, one obtains successively that R is countable, then that $R\langle S\rangle$ is countable, and finally that $M = A(S)$ is countable. So M must be discrete, in contrast with the assumption.

[3] i.e. $\alpha x \neq 0$ for $\alpha \in R^{\times}$ and $x \in M^{\times}$

(ii) To prove the second statement, assume that R is nondiscrete. As proved in (a), S contains a finite subset F such that $A(F)$ is open. We show that $x \in A(F)$ for any $x \in M^\times$; it then follows that $A(S) = A(F)$, hence $S = F$. Let now $x \in M^\times$. As R is nondiscrete and the map $R \ni \alpha \mapsto \alpha x$ is injective and continuous, 0 is an accumulation point of Rx and therefore $Rx \cap A(F) \neq \{0\}$. This implies that $x \in A(F)$. □

In Theorem 3.2, without further assumptions as for instance those in (b) one cannot deduce that $\operatorname{rank}(M)$ is finite: as an example take the $\mathbb{Z}$-module $C \times c_0(\mathbb{Z})$ with the product topology, where $C := \mathbb{F}_2^{\aleph_0}$ is the Cantor group and $c_0(\mathbb{Z})$ is the space of $\mathbb{Z}$-valued sequences that are eventually 0 endowed with the discrete topology.

The case of free modules (in particular, of vector spaces) deserves a separate treatment. Recall that M is *free* if it contains a linearly independent set S with $M = R\langle S\rangle$.

Theorem 3.3. *Let M be a nondiscrete topological free module over a topological integral domain R. Assume further that M is Polish and R is analytic. If M contains a maximal linearly independent subset that is analytic, then $n := \operatorname{rank}(M)$ is finite, R is a Polish space, and M and R^n are isomorphic as topological R-modules.*

Proof. By Theorem 3.2(b), $n = \operatorname{rank}(M)$ is finite.

Let now $f\colon R^n \to M$ be an arbitrary isomorphism and $\{e_1, e_2, \dots, e_n\}$ the canonical basis of R^n. Since $f(\alpha_1, \alpha_2, \dots, \alpha_n) = \sum_{i=1}^{n} \alpha_i f(e_i)$, it is obvious that f is continuous. Therefore, by the open mapping theorem due to Christensen [3, Theorem 5.2], f is open and thus a topological isomorphism.

This implies that also R^n and its projection R are Polish spaces. □

Corollary 3.4. *Let K be a field endowed with a nondiscrete ring topology and L a dense proper subfield of K. Further assume that K is Polish and L is analytic. Then no basis of K as vector space over L can be analytic.*

Proof. Suppose that K, as vector space over L, admits a basis that is analytic. It follows from Theorem 3.3 (and Theorem 2.1) that L is complete, hence closed; this implies $L = K$, since L is dense in K. □

This corollary includes as a very particular case, namely for $K = \mathbb{R}$ and $L = \mathbb{Q}$, the theorem of Sierpiński [6] mentioned at the beginning.

Another consequence of Theorem 3.3 is:

Corollary 3.5. *Let K be a field endowed with a ring topology and V a nondiscrete topological K-vector space. If K and V are Polish spaces and V admits a basis that is analytic, then V is finite dimensional.*

This was proved in [1, Theorem 3.9] for separable real Banach spaces. More generally, Corollary 3.5 implies that no separable metrizable complete real vector space of infinite dimension has analytic Hamel bases.

Let us briefly discuss the assumptions of this result. Separability is necessary since every analytic set is separable and so is the generated vector space.[4] Completeness cannot be canceled since a normed real vector space of countable dimension has of course an analytic basis and is separable. Also, metrizability is not superfluous: c_{00}, the space of all real sequences that are eventually 0, endowed with the box topology is a separable complete locally convex vector space with countable (hence analytic) basis.

4. Independent sets in topological groups

In this section G denotes an Abelian group.
For $n \in \mathbb{N}$ let

$$G[n] := \{g \in G : \; ng = 0\}$$

and let

$$t(G) := \bigcup_{n=1}^{\infty} G[n]$$

be the torsion subgroup of G. We shall sometimes consider G as a $\mathbb{Z}$-module and $G[p]$ as a vector space over the field $\mathbb{F}_p$ of p elements, p belonging to the set $\mathbb{P}$ of prime numbers. The order of $g \in t(G)$ is denoted by $\operatorname{ord}(g)$.

Recall that a subset S of G is *independent* if for every finite number of distinct elements $s_1, s_2, \ldots, s_n$ of S and every $k_1, k_2, \ldots, k_n \in \mathbb{Z}$, from $\sum_{i=1}^{n} k_i s_i = 0$ it follows $k_i s_i = 0$ for all $i = 1, 2, \ldots, n$.

In the proof of Theorem 4.2 we shall use the relationship between independence and linear independence stated in the following proposition; this allows us to make direct appeal to the results of the last section.

[4]However, an interesting result by Bartoszyński et al. says that there exist nonseparable Hilbert spaces with discrete (hence closed) Hamel bases (see [1, Theorem 3.7]).

Proposition 4.1.

(a) *A subset S of G is linearly independent*[5] *over $\mathbb{Z}$ iff $S \subseteq G \setminus t(G)$ and S is independent.*

(b) *If $p \in \mathbb{P}$ and $S \subseteq G[p]$, then S is linearly independent over $\mathbb{F}_p$ iff S is independent.*

(c) *Let S be a maximal independent subset of G. Then $S_0 := S \setminus t(G)$ is maximal linearly independent over $\mathbb{Z}$, and*

$$S_p := \left\{ \frac{\operatorname{ord}(s)}{p} s : \; s \in S \cap t(G) \; \textit{and } p \; \textit{divides} \; \operatorname{ord}(s) \right\}$$

is a basis of the $\mathbb{F}_p$-vector space $G[p]$.

Proof.

(a) and (b) are clear.

(c) The first statement of (c) easily follows from (a). To prove the second statement, we closely follow the arguments of [4, Section 16]. First observe: If $c(s) \in \mathbb{Z}$ with $c(s)s \neq 0$ for $s \in S$, then the elements $c(s)s$ are distinct and the set $\{c(s)s : \; s \in S\}$ is independent. In particular, S_p is independent. Moreover, $S_p \subseteq G[p]$ and thus S_p is linearly independent over $\mathbb{F}_p$ by (b). Therefore it remains to prove that $G[p] = \mathbb{F}_p\langle S_p \rangle$, i.e. that $G[p]$ is generated by S_p as an $\mathbb{F}_p$-vector space.

Let $x \in G[p]$. It follows from the maximality of S that there are distinct elements $s_1, s_2, \ldots, s_n \in S$ and integers $k, k_1, k_2, \ldots, k_n$ such that $\sum_{i=1}^{n} k_i s_i = kx \neq 0$ and $k_i s_i \neq 0$ for all $i = 1, 2, \ldots, n$. Since $\sum_{i=1}^{n} p k_i s_i = kpx = 0$ and S is independent, $pk_i s_i = 0$ and so $s_i \in t(G)$ for all $i = 1, 2, \ldots, n$. But $k_i s_i \neq 0$ and $pk_i s_i = 0$ imply that $\operatorname{ord}(s_i)$ divides pk_i and p divides $\operatorname{ord}(s_i)$. Consequently

$$kx = \sum_{i=1}^{n} a_i t_i,$$

where

$$a_i := pk_i / \operatorname{ord}(s_i) \in \mathbb{Z}$$

and

$$t_i := p^{-1} \operatorname{ord}(s_i) s_i \in S_p.$$

Since p does not divide k, it follows that $x \in \mathbb{F}_p\langle S_p \rangle$. □

[5]in the sense of Section 3

Theorem 4.2. *Let G be a nondiscrete topological Abelian group. If G is Polish, then no maximal independent subset of G can be analytic.*

Proof. Suppose that S is a maximal independent, analytic subset of G.

(i) Since $G[n]$ is closed for any $n \in \mathbb{N}$,

$$S_0 := S \setminus t(G) = S \cap \bigcap_{n=1}^{\infty} \big(G \setminus G[n]\big)$$

is analytic. We now apply Theorem 3.2(a) and Proposition 4.1(c) considering G as a topological module over the discrete ring $\mathbb{Z}$, and obtain that S_0 is countable, i.e. the free rank of G is countable.

(ii) Let $p \in \mathbb{P}$ and S_p be defined as in Proposition 4.1(c). For any $n \in \mathbb{N} \setminus \{1\}$

$$A(n) := \big\{g \in t(G) : \ \mathrm{ord}(g) = n\big\} = G[n] \cap \bigcap_{m=1}^{n-1} \big(G \setminus G[m]\big)$$

is analytic. Therefore

$$S_p = \bigcup_{n=1}^{\infty} n\big(A(pn) \cap S\big)$$

is analytic. Since, by Proposition 4.1(c), S_p is a basis of the $\mathbb{F}_p$-vector space $G[p]$ and $G[p]$ is a Polish space, S_p is countable by Theorem 3.2(a). It follows that the p-rank of G is countable, observing that the dimension of $G[p]$ over $\mathbb{F}_p$ coincides with the p-rank of G (see [4, p. 86]).

Finally, it follows from (i) and (ii) that the rank of G is countable. Hence G is countable and therefore cannot be a nondiscrete Polish space. $\square$

5. Algebraically independent sets in topological field extensions

Throughout this section, K stands for a field and L for a subfield of K.

For $S \subseteq K$, we denote by $L[S]$ and $L(S)$, respectively, the subring and the subfield of K generated by L and S. The dimension of K as vector space over L is denoted by $(K : L)$. If $x \in K$ is algebraic over L, then

$$\deg_L(x) := (L(x) : L)$$

is the degree of x over L.

The following lemma is the field-theoretic analogue of Lemma 3.1.

Lemma 5.1. *Let K be a Hausdorff topological field, and let L be a subfield and analytic subset of K.*

(a) *If S is an analytic subset of K, then $L[S]$ and $L(S)$ are analytic.*

(b) *If K is analytic, then the algebraic closure A of L in K is analytic; moreover,*
$$A_n := \{x \in A : \deg_L(x) \leq n\}$$
is analytic for every $n \in \mathbb{N}$.

Proof.

(a) If S is analytic, then so are the sets S_n defined by
$$S_1 := S \cup \{1\}, \qquad S_{n+1} := S_n \cdot S_1 \quad (n \in \mathbb{N})$$
and
$$S_\infty := \bigcup_{n=1}^{\infty} S_n.$$
Then $L[S] = L\langle S_\infty \rangle$ is analytic by Lemma 3.1(a). Moreover, $L(S)$ is the image of the analytic set $L[S] \times L[S]^\times$ under the continuous map $(x, y) \mapsto xy^{-1}$, hence analytic.

(b) For $n \in \mathbb{N}$, let $p_n : L^n \times K \to K$ and $\pi_n : L^n \times K \to K$ be defined by
$$p_n(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, x) := \sum_{i=0}^{n-1} \alpha_i x^i + x^n,$$
$$\pi_n(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, x) := x.$$
Then, for every $n \in \mathbb{N}$, $A_n = \bigcup_{m=1}^{n} \pi_m(p_m^{-1}(\{0\}))$ is analytic, and so must be $A = \bigcup_{n=1}^{\infty} A_n$. □

The first theorem of this section deals with transcendental field extensions.

Theorem 5.2. *Let L be a subfield of a topological field K. Assume that K is a nondiscrete Polish space and L is an analytic subspace of K. If $K|L$ is a transcendental extension, then no transcendence basis of $K|L$ can be analytic.*

Proof. Suppose, by way of contradiction, that there exists a transcendence basis T of K that is analytic. Let $t \in T$. Then $T \setminus \{t\}$ is analytic and, by Lemma 5.1(a), $L_0 := L(T \setminus \{t\})$ is analytic. Moreover, $L(T) = L_0(t)$. Therefore, replacing L by L_0, we may assume that $T = \{t\}$. This implies that L is not discrete: If L was discrete, then L (being separable) would be countable, therefore $L(t)$ and K (being an algebraic extension of $L(t)$) would be countable, hence discrete as well.

Let v be the normalized t-adic valuation on $L(t)$ (i.e. $v(\alpha) = 0$ for $\alpha \in L$ and $v(t) = 1$) and w a valuation on K extending v (such an extension exists, see [9, Theorem 2.22], for instance). The sets

$$A := \big\{ f \in L[t] : \ v(f) = 0 \big\} = tL[t] + L^{\times}$$

and

$$R := \{0\} \cup \bigcup_{n=0}^{\infty} t^n A A^{-1},$$

the valuation ring of v, are both analytic, and consequently, for $n \in \mathbb{N}$, the sets $t^{-n}R = \big\{ \alpha \in L(t) : \ v(\alpha) \geq -n \big\}$ are analytic. Therefore, arguing as in the proof of Lemma 5.1, one obtains that

$$A_n := \bigg\{ x \in K : \ \big(\exists m \in \mathbb{N}\big)\big(\exists \alpha_0, \alpha_1, \ldots, \alpha_{m-1} \in t^{-n}R\big)\Big(\sum_{i=0}^{m-1} \alpha_i x^i + x^m = 0 \Big) \bigg\}$$

is analytic. Let us prove that $x \in A_n$ implies $w(x) \geq -n$. We may of course assume $w(x) < 0$. Since $x^m = -\sum_{i=0}^{m-1} \alpha_i x^i$ where $\alpha_i \in t^{-n}R$, we have

$$\begin{aligned} mw(x) = w(x^m) &\geq \min\big\{ w(\alpha_i) + iw(x) : \ i = 0, 1, \ldots, m-1 \big\} \\ &\geq -n + (m-1)w(x). \end{aligned}$$

Observe now that for any $n \in \mathbb{N}$ the set

$$B_n := \big\{ x \in K : \ w(x) \geq -n \big\}$$

is a subgroup of K containing A_n, and that $K = \bigcup_{n=1}^{\infty} A_n$. So, by Corollary 2.4 we have that, for some $n \in \mathbb{N}$, B_n is a 0-neighbourhood. This implies, L being nondiscrete, that there exists $\epsilon \in L^{\times}$ such that $\epsilon t^{-(n+1)} \in B_n$. But this cannot be, for $w(\epsilon t^{-(n+1)}) = -(n+1) < -n$. $\square$

The following corollary was proved in [11, Theorem 1] in case of $K = \mathbb{R}$.

Corollary 5.3. *Let K be a nondiscrete topological field. If K is a Polish space, then no transcendence basis of K over its prime subfield can be analytic.*

The second theorem of this section, complementary to Theorem 5.2, deals with algebraic extensions.

Recall that a field L is called *perfect* if it is either of characteristic zero or is of characteristic $p \in \mathbb{P}$ and coincides with its subfield $L^p := \{x^p : \ x \in L\}$. By [10, Theorem II.6], any algebraic extension of a perfect field is separable.

Theorem 5.4. *Let K and L be as in* Theorem 5.2. *Moreover, let K be an algebraic extension of L.*

(a) *Then there exists $m \in \mathbb{N}$ such that $\deg_L x \leq m$ for all $x \in K$.*

(b) *If K is a separable extension of L (in particular, if L is perfect), then L is a Polish space and K is a simple extension of L isomorphic to $L^{(K:L)}$ as a topological vector space over L.*

Proof.

(a) For any $n \in \mathbb{N}$ let A_n be as in Lemma 5.1(b). Then $A_n - A_n \subseteq A_{n^2}$ by elementary algebra. Thus, by Corollary 2.4, we deduce that for some $m \in \mathbb{N}$ the set A_m is a 0-neighbourhood. As in the proof of Theorem 5.2 one sees that L is nondiscrete. This yields that for all $x \in K$ there exists $\epsilon \in L^\times$ such that $\epsilon x \in A_m$, which finally gives $\deg_L(x) = \deg_L(\epsilon x) \leq m$. This proves (a).

(b) If K is a separable extension of L, then, by the theorem of the primitive element [10, Theorem II.19], every finite field extension of L contained in K is simple and therefore its degree over L is not greater than m (m chosen as in (a)). It follows that K is a finite simple extension of L and $(K : L) \leq m$. The rest directly follows from Theorem 3.3. $\square$

Corollary 5.5. *Let K and L be as in* Theorem 5.2. *If L is a perfect field and a dense proper subset of K, then K is not algebraic over L.*

We end this section with a comment on Theorem 5.4. For any $n \in \mathbb{N}$ there are field extensions of degree n satisfying the assumptions of Theorem 5.4, for instance field extensions of the field $\mathbb{Q}_p$ of p-adic numbers. The following example shows that there are also field extensions $K|L$ of *infinite* degree satisfying the assumptions of Theorem 5.4; of course, such extensions must be inseparable by Theorem 5.4(b).

Let $p \in \mathbb{P}$ and $F := \mathbb{F}_p(x_1, x_2, \ldots)$ be a field with transcendence basis $x_1, x_2, \ldots$ over $\mathbb{F}_p$. Let $F(x)$ be a simple transcendental extension of F endowed with the x-adic topology. Its completion is the field $K := F((x))$ of all formal Laurent series $\sum_{n=i}^{\infty} a_n x^n$, where $i \in \mathbb{Z}$ and $a_n \in F$. Then K is a nondiscrete Polish space and its subfield $L := F^p((x))$ is closed, hence a Polish space, too. Moreover, $(L(z) : L) = p$ for every $z \in K \setminus L$. Thus the assumptions of Theorem 5.4 are satisfied. But $(K : L)$ is infinite.

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