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ANTI-PERIODIC SOLUTIONS OF NONLINEAR FIRST ORDER IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The existence of anti-periodic solutions of the following nonlinear impulsive functional differential equations

$$\begin{aligned} x'(t) + a(t)x(t) &= f(t, x(t), x(\alpha_1(t)), \dots, x(\alpha_n(t))), \qquad t \in \mathbb{R}, \\ \Delta x(t_k) &= I_k(x(t_k)), \qquad k \in \mathbb{Z} \end{aligned}$$

is studied. Sufficient conditions for the existence of at least one anti-periodic solution of the mentioned equation are established. Several new existence results are obtained.

1. Introduction

Anti-periodic boundary value problems for ordinary differential equations with or without impulses effects were studied extensively in the last ten years since these problems appear in a variety of applications. For example, for first order ordinary differential equations without impulses effects, a Massera's criterion is presented in [6], quasilinearization methods are applied in [22] and in [7], [10]–[12], [17, 19], [21]–[23] the validity of lower and upper solution methods coupled with the monotone iterative technique is shown.

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Recently, in paper [17], the existence of solutions of of a class of anti-periodic boundary value problems for impulsive ordinary differential equations was studied by Luo, Shen and Neito under the existence of pair of coupled lower and upper solutions of the corresponding system. The methods used in [17] are based upon the lower and upper solutions methods and monotone iterative technique. In [9], the existence of solutions for a class of first order functional differential equation with anti-periodic boundary value conditions was studied by introducing the concept of lower and upper solutions using monotone iterative technique coupled with lower and upper solutions too.

The anti-periodic boundary problems for partial differential equations and abstract differential equations were considered in [4, 5, 8]. The solvability of the anti-periodic problems for higher order differential equations were studied in papers [1, 2, 3, 6, 20] and the references cited there.

We note that, in above mentioned papers, the anti-periodic boundary value problems were discussed, and the methods used are lower and upper solution methods and monotone iterative techniques. There seems to be no paper discussed the existence of anti-periodic solutions of the corresponding impulsive functional differential equations on infinite line. Furthermore, the assumptions in the known theorems imposed on the nonlinear functions or the impulses functions are at most linear in their variables. So the solvability problem have not been well solved when the nonlinear functions or the impulses functions are supper-linear.

In this paper, we are concerned with the existence of anti-periodic solutions of the nonlinear impulsive functional differential equations

$$\begin{cases} x'(t)+a(t)x(t)=f(t,x(t),x(\alpha_1(t)),\dots,x(\alpha_n(t))), & t\in\mathbb{R},\\ \Delta x(t_k)=I_k(x(t_k)), \qquad k\in\mathbb{Z}, \end{cases} \tag{1.1}$$

where $\mathbb{Z}$, $\mathbb{R}$ denote the integer set and real number set respectively, $T>0$ a constant, $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$, $I_k\colon\mathbb{R}\to\mathbb{R}$ is continuous, $a\colon\mathbb{R}\to\mathbb{R}$, $f\colon\mathbb{R}^{n+2}\to\mathbb{R}$ and $\alpha_i\colon\mathbb{R}\to\mathbb{R}$ $(i=1,\dots,n)$ are functions. Sufficient conditions for the existence of at least one anti-periodic solutions of (1) are established. The proofs of the main theorems are based upon the Schauder's fixed point theorem [13].

To the author's knowledge, the existence of periodic solutions of the impulsive functional differential equations was studied extensively, see the text book [18] and papers [15, 16]. It is easy to see that, an anti-periodic solution with anti-period T of the impulsive functional differential equation is a periodic solution of the same equation with period $2T$. So the studies on the existence of anti-periodic solutions has more importance and significance. On the other hand, when one studies the existence of periodic solutions of the impulsive functional differential equations, the Mawhin's continuation theorem is used to establish the existence criteria, see [16]; the existence of multiple positive periodic solutions of the impulsive functional differential equations is obtained by using the fixed point theorems on cones on the suitable Banach spaces [15].

The remainder of this paper is divided into two sections, the main results are established in Section 2 and two examples are given in Section 3 to illustrate the main results.

2. Main results

In this paper, the following assumption is supposed:

(A1) $\dots < t_{-m} < \dots < t_0 < t_1 < \dots < t_m < \dots$ are constants and there exists a positive integer l such that $t_k + T = t_{k+l}$ and $I_k(x) = -I_{k+l}(-x)$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}$; denote

$$l_0 = \min\big\{l > 0 : \ t_k + T = t_{k+l} \ \text{ and } \ I_k(x) = -I_{k+l}(-x) \ \text{ for all } \ k \in \mathbb{Z}, \ x \in \mathbb{R}\big\}.$$

Let X be defined by

$$X = \Big\{u\colon \mathbb{R} \to \mathbb{R} : \ u|_{(t_k,t_{k+1})} \in C^0(t_k,t_{k+1}), \quad k \in \mathbb{Z} \ \text{there exist the limits } \lim_{t\to t_k^-} x(t) = x(t_k), \ \lim_{t\to t_k^+} x(t), \ k \in \mathbb{Z} \ \text{ and } \ x(t) = -x(t+T) \ \text{ for all } \ t \in \mathbb{R}\Big\}.$$

Define the norm $\|u\| = \sup_{t\in\mathbb{R}} |u(t)|$ for all $u \in X$. It is easy to show that X is a real Banach space.

Definition 2.1. A function $f\colon \mathbb{R}\times\mathbb{R}^{n+1} \to \mathbb{R}$ is called an impulsive continuous function if

- $f(\cdot, u_0, u_1, \dots, u_n) \in X$ for each $u = (u_0, \dots, u_n) \in \mathbb{R}^{n+1}$;
- $f(t, \cdot, \dots, \cdot)$ is continuous for all $t \in \mathbb{R}$.

Definition 2.2. By a solution of equation (1) we mean a function $x\colon \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

- $x \in X$ is differentiable in (t_k, t_{k+1}) $(k \in \mathbb{Z})$, there exist the limits $\lim_{t\to t_k^-} x'(t) = x'(t_k)$ and $\lim_{t\to t_k^+} x'(t)$ $(k \in \mathbb{Z})$;
- $x' \in X$;
- $x(t) = -x(t+T)$ for all $t \in \mathbb{R}$;
- The equations in (1) are satisfied.

Furthermore, suppose

(A2) $a|_{(t_k,t_{k+1})} \in C^0(t_k,t_{k+1})$ satisfies $a(t+T)=a(t)$ for all $t\in\mathbb{R}$ and there exist the limits $\lim\limits_{t\to t_k^-} a(t)$ and $\lim\limits_{t\to t_k^-} a(t)$ for all $k\in\mathbb{Z}$, denote $a^+(t)=\max\{0,a(t)\}$ and $a^-(t)=\max\{0,-a(t)\}$ for all $t\in\mathbb{R}$;

(A3) $\alpha_k\in C^1(\mathbb{R})$ with $\alpha_k(t+T)=T+\alpha(t)$ for all $t\in\mathbb{R}$ and $k=1,\ldots,n$;

(A4) f is an impulsive continuous function satisfying

$$f(t+T,-x_0,-x_1,\ldots,-x_n)=-f(t,x_0,x_1,\ldots,x_n)$$

for all $t\in\mathbb{R}$ and $(x_0,x_1,\ldots,x_n))\in\mathbb{R}^{n+1}$;

(A5) $I_k(k\in\mathbb{Z})$ are continuous functions satisfying $x(x+I_k(x))\geq 0$ for all $x\in\mathbb{R}$ and $k\in\mathbb{Z}$.

For $x\in X$, we define the nonlinear operator L by

$$\begin{aligned}(Lx)(t) \quad = \quad & -\frac{1}{1+\exp\left(\int\limits_0^T a(u)\,\mathrm{d}u\right)}\times\\ & \left[\int\limits_t^{t+T}\exp\left(\int\limits_t^s a(u)\,\mathrm{d}u\right)f(s,x(s),x(\alpha_1(s)),\ldots,x(\alpha_n(s)))\,\mathrm{d}s\right.\\ & \left.+\sum_{t\leq t_k<t+T}\exp\left(\int\limits_t^{t_k}a(u)\,\mathrm{d}u\right)I_k(x(t_k))\right].\end{aligned}$$

Lemma 2.1. *Suppose that* (A1)–(A5) *hold and* $x\in X$. *Then* $Lx\in X$.

P r o o f. It is easy to see that

$$\begin{aligned}&(Lx)(t+T)\\ =&-\frac{1}{1+\exp\left(\int\limits_0^T a(u)\,\mathrm{d}u\right)}\times\\ &\left[\int\limits_{t+T}^{t+2T}\exp\left(\int\limits_{t+T}^s a(u)\,\mathrm{d}u\right)f(s,x(s),x(\alpha_1(s)),\ldots,x(\alpha_n(s)))\,\mathrm{d}s\right.\\ &\left.+\sum_{t+T\leq t_k<t+2T}\exp\left(\int\limits_{t+T}^{t_k}a(u)\,\mathrm{d}u\right)I_k(x(t_k))\right]\end{aligned}$$

$$= -\frac{1}{1+e^{\int_0^T a(u)\,du}}\left[\int_t^{t+T}\exp\left(\int_{t+T}^{T+v} a(u)\,du\right)\times\right.$$

$$f(T+v, x(T+v), x(\alpha_1(T+v)), \ldots, x(\alpha_n(T+v)))\,ds\times$$

$$\left.+\sum_{t\le t_k-T<t+T}\exp\left(\int_{t+T}^{t_k} a(u)\,du\right) I_k(x(t_k))\right]$$

$$= -\frac{1}{1+\exp\left(\int_0^T a(u)\,du\right)}\left[\int_t^{t+T}\exp\left(\int_{t+T}^{T+v} a(u)\,du\right)\times\right.$$

$$f(T+v, -x(v), x(T+\alpha_1(v)), \ldots, x(T+\alpha_n(v)))\,dv\times$$

$$\left.+\sum_{t\le t_{k-l_0}<t+T}\exp\left(\int_{t+T}^{t_{k-l_0}+T} a(u)\,du\right) I_k(x(t_{k-l_0}+T))\right]$$

$$= -\frac{1}{1+\exp\left(\int_0^T a(u)\,du\right)}\left[\int_t^{t+T}\exp\left(\int_{t+T}^{T+v} a(u)\,du\right)\times\right.$$

$$f(T+v, -x(v), -x(\alpha_1(v)), \ldots, -x(\alpha_n(v)))\,dv\times$$

$$\left.+\sum_{t\le t_{k-l_0}<t+T}\exp\left(\int_{t+T}^{t_{k-l_0}+T} a(u)\,du\right) I_k(x(t_{k-l_0}+T))\right]$$

$$= -\frac{1}{1+\exp\left(\int_0^T a(u)\,du\right)}\left[\int_t^{t+T}\exp\left(\int_t^{v} a(u)\,du\right)\times\right.$$

$$[-f(v, x(v), x(\alpha_1(v)), \ldots, x(\alpha_n(v)))]\,dv\times$$

$$\left.+\sum_{t\le t_{k-l_0}<t+T}\exp\left(\int_t^{t_{k-l_0}} a(u)\,du\right) I_k(-x(t_{k-l_0}))\right]$$

$$
\begin{aligned}
&= \frac{1}{1+\exp\left(\int\limits_0^T a(u)\,\mathrm{d}u\right)}\times \\
&\quad\left[\int\limits_t^{t+T}\exp\left(\int\limits_t^v a(u)\,\mathrm{d}u\right)f(v,x(v),x(\alpha_1(v)),\ldots,x(\alpha_n(v)))\,\mathrm{d}v\right. \\
&\quad\left.-\sum_{t\le t_s<t+T}\exp\left(\int\limits_t^{t_s}a(u)\,\mathrm{d}u\right)I_k(x(t_s))\right] \\
&= -(Lx)(t).
\end{aligned}
$$

On the other hand, one can easily show that $(Lx)|_{(t_k,t_{k+1})}\in C^0(t_k,t_{k+1}), k\in\mathbb{Z}$ and there exist the limits $\lim\limits_{t\to t_k^-}(Lx)(t)=(Lx)(t_k)$ and $\lim\limits_{t\to t_k^+}(Lx)(t)$ for all $k\in\mathbb{Z}$. This completes the proof. $\square$

Lemma 2.2. *Suppose that* (A1)–(A5) *hold. Then $x\in X$ is a anti-periodic solution of equation* (1) *if and only if x is a solution of the operator equation $x=Lx$.*

P r o o f. Suppose that $x\in X$ satisfies $x=Lx$. Then

$$
\begin{aligned}
x(t) = &-\frac{1}{1+\exp\left(\int\limits_0^T a(u)\,\mathrm{d}u\right)} \\
&\left[\int\limits_t^{t+T}\exp\left(\int\limits_t^s a(u)\,\mathrm{d}u\right)f(s,x(s),x(\alpha_1(s)),\ldots,x(\alpha_n(s)))\,\mathrm{d}s\right. \\
&\left.+\sum_{t\le t_k<t+T}\exp\left(\int\limits_t^{t_k}a(u)\,\mathrm{d}u\right)I_k(x(t_k))\right].
\end{aligned}
$$

For $t\neq t_k$, since f and $x\in X$ are continuous at t, we know that x is differentiable at t and

$$
\begin{aligned}
x'(t) = &-\frac{1}{1+\exp\left(\int\limits_0^T a(u)\,\mathrm{d}u\right)}\times \\
&\left[\mathrm{e}^{\int_t^{t+T}a(u)\,\mathrm{d}u}f(t+T,x(t+T),x(\alpha_1(T+t)),\ldots,x(\alpha_n(t+T)))\right.
\end{aligned}
$$

$$
\begin{aligned}
&- a(t)\int\limits_t^{t+T}\exp\left(\int\limits_t^s a(u)\,\mathrm{d}u\right) f(s,x(s),x(\alpha_1(s)),\dots,x(\alpha_n(s)))\,\mathrm{d}s\\
&- f(t,x(t),x(\alpha_1(t)),\dots,x(\alpha_n(t)))\\
&-a(t)\sum_{t\le t_k<t+T}\exp\left(\int\limits_t^{t_k} a(u)\,\mathrm{d}u\right) I_k(x(t_k))\Bigg].
\end{aligned}
$$

Then

$$
\begin{aligned}
&x'(t)+a(t)x(t)\\
=&-\frac{1}{1+\exp\left(\int\limits_0^T a(u)\,\mathrm{d}u\right)}\times\\
&\left[\exp\left(\int\limits_t^{t+T} a(u)\,\mathrm{d}u\right) f(t+T,-x(t),-x(\alpha_1(t)),\dots,-x(\alpha_n(t)))\right.\\
&\left.- f(t,x(t),x(\alpha_1(t)),\dots,x(\alpha_n(t)))\right]\\
=&-\frac{1}{1+\exp\left(\int\limits_0^T a(u)\,\mathrm{d}u\right)}\times\\
&\left[\exp\left(\int\limits_0^{T} a(u)\,\mathrm{d}u\right) f(t+T,-x(t),-x(\alpha_1(t)),\dots,-x(\alpha_n(t)))\right.\\
&\left.- f(t,x(t),x(\alpha_1(t)),\dots,x(\alpha_n(t)))\right]\\
=&\, f(t,x(t),x(\alpha_1(t)),\dots,x(\alpha_n(t))).
\end{aligned}
$$

On the other hand, it is easy to show that $x(t+T)=-x(t)$ for all $t\in\mathbb{R}$ and $\Delta x(t_k)=\lim\limits_{t\to t_k^-}x(t)=x(t_k)$ and $\Delta x(t_k)=\lim\limits_{t\to t_k^+}x(t)=x(t_k)-\lim\limits_{t\to t_k^-}x(t)=I_k(x(t_k))$ for all $k\in\mathbb{Z}$.

Now suppose that x is a anti-periodic solution of equation (1). We get that

$$
\begin{cases}
x'(t)+a(t)x(t)=f(t,x(t),x(\alpha_1(t)),\dots,x(\alpha_n(t))), & t\in\mathbb{R},\\
\Delta x(t_k)=I_k(x(t_k)), \qquad k\in\mathbb{Z},
\end{cases}
$$

Then

$$\left(x(t)\exp\left(\int_0^t a(s)\,\mathrm{d}s\right)\right)' \\ = f(t,x(t),x(\alpha_1(t)),\dots,x(\alpha_n(t)))\exp\left(\int_0^t a(s)\,\mathrm{d}s\right). \tag{2.1}$$

Integrating (2) from t to $t+T$, one gets that

$$\begin{aligned} &x(t+T)\exp\left(\int_0^{t+T} a(u)\,\mathrm{d}u\right) - x(t)\exp\left(\int_0^t a(u)\,\mathrm{d}u\right) \\ &= \int_t^{t+T}\exp\left(\int_0^s a(u)\,\mathrm{d}u\right) f(s,x(s),x(\alpha_1(s)),\dots,x(\alpha_n(s)))\,\mathrm{d}s \\ &\quad + \sum_{t\le t_k<t+T}\exp\left(\int_0^{t_k} a(u)\,\mathrm{d}u\right) I_k(x(t_k)). \end{aligned}$$

Then $x(t+T)=-x(t)$ implies that

$$\begin{aligned} x(t) = &-\frac{1}{1+\exp\left(\int_0^T a(u)\,\mathrm{d}u\right)}\times \\ &\left[\int_t^{t+T}\exp\left(\int_t^s a(u)\,\mathrm{d}u\right) f(s,x(s),x(\alpha_1(s)),\dots,x(\alpha_n(s)))\,\mathrm{d}s\right. \\ &\left.+\sum_{t\le t_k<t+T}\exp\left(\int_t^{t_k} a(u)\,\mathrm{d}u\right) I_k(x(t_k))\right] = (Lx)(t). \end{aligned}$$

The proof is complete. □

Lemma 2.3. *Suppose that* (A1)–(A5) *hold. Then L is a completely continuous operator.*

P r o o f. It suffices to prove that L is continuous and L is compact. We divide the proof into two steps:

Step 1. Prove that L is continuous about x. Suppose $x_n \in X$ and $x_n \to x_0 \in X$. Then

$$\begin{aligned}(Lx_n)(t) &= -\frac{1}{1+\exp\left(\int\limits_0^T a(u)\,\mathrm{d}u\right)} \times \\ &\left[\int\limits_t^{t+T} \exp\left(\int\limits_t^s a(u)\,\mathrm{d}u\right) f(s, x_n(s), x_n(\alpha_1(s)), \dots, x_n(\alpha_n(s)))\,\mathrm{d}s\right. \\ &\left. + \sum_{t\le t_k<t+T} \exp\left(\int\limits_t^{t_k} a(u)\,\mathrm{d}u\right) I_k(x_n(t_k))\right], \qquad i = 0, 1, \dots .\end{aligned}$$

Since f and I_k are continuous, the continuity of L follows.

Step 2. Prove that L is compact.

Let $\Omega \subseteq X$ bet a bounded set. Suppose that $\Omega \subseteq \{x \in X : \|x\| \le M\}$. For $x \in \Omega$, we have

$$\begin{aligned}|(Lx)(t)| = &\frac{1}{1+\exp\left(\int\limits_0^T a(u)\,\mathrm{d}u\right)} \times \\ &\left|\int\limits_t^{t+T} \exp\left(\int\limits_t^s a(u)\,\mathrm{d}u\right) f(s, x(s), x(\alpha_1(s)), \dots, x(\alpha_n(s)))\,\mathrm{d}s\right. \\ &\left. + \sum_{t\le t_k<t+T} \exp\left(\int\limits_t^{t_k} a(u)\,\mathrm{d}u\right) I_k(x(t_k))\right| \\ \le &\frac{\max\limits_{t\in\mathbb{R}, |x_i|\le M, i=0,1,\dots,n} |f(s, x_0, x_1, \dots, x_n)|}{1+\exp\left(\int\limits_0^T a(u)\,\mathrm{d}u\right)} \times \\ &\left|\int\limits_t^{t+T} \exp\left(\int\limits_t^s a(u)\,\mathrm{d}u\right)\,\mathrm{d}s\right. \\ &\left. + \max_{|x|\le M} |I_k(x)| \sum_{t\le t_k<t+T} \exp\left(\int\limits_t^{t_k} a(u)\,\mathrm{d}u\right)\right|\end{aligned}$$

$$\leq \frac{\max\limits_{t\in\mathbb{R},|x_i|\leq M,i=0,1,\ldots,n} |f(s,x_0,x_1,\ldots,x_n)|}{1+\exp\left(\int\limits_0^T a(u)\,\mathrm{d}u\right)}\times$$

$$T\exp\left(\int\limits_0^T |a(u)|\,\mathrm{d}u\right)\,\mathrm{d}s + \max_{|x|\leq M}|I_k(x)|\sum_{0\leq t_k<T}\exp\left(\int\limits_0^T |a(u)|\,\mathrm{d}u\right)$$

and similarly $(Lx)'(t)$ is bounded. This shows that $(Lx)(t)$ is equi-continuous on $\mathbb{R}$. The Arzela-Askoli theorem guarantees that $L(\Omega)$ is relative compact, which means that L is compact. Hence the continuity and the compactness of L imply that L is completely continuous. □

The following abstract existence theorem, which is called Schauder's fixed point theorem, will be used in the proof of the main results of this paper. Its proof can be seen in [13].

Lemma 2.4. *Let X be a Banach space. Suppose $L\colon X\to X$ is completely continuous operator. If there exists a open bounded subset Ω such that $0\in\Omega\subset X$ and $x\neq\lambda Lx$ for all $x\in D(L)\cap\partial\Omega$ and $\lambda\in[0,1]$, then there is at least one $x\in\Omega$ such that $x=Lx$.*

Now, we establish existence results for at least one solution of equation (1).

Theorem 2.1. *Suppose that* (A1)–(A5) *hold and*

(G1) *$xI_k(x)\geq 0$ for all $x\in\mathbb{R}$ and $k\in\mathbb{Z}$;*

(G2) *there exist impulsive continuous functions $h\colon\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, $g_i\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $r\in X$ such that*

(i) *$f(t,x_0,\ldots,x_n) = h(t,x_0,\ldots,x_n) + \sum\limits_{i=0}^n g_i(t,x_i) + r(t)$ holds for all $(t,x_0,\ldots,x_n)\in\mathbb{R}\times\mathbb{R}^{n+1}$;*

(ii) *there exists $t_0\in\mathbb{R}$ and constants $m\geq 0$ and $\beta>0$ such that*

$$h(t,x_0,\ldots,x_n)x_0\geq 0$$

holds for all $(t,x_0,\ldots,x_n)\in[t_0,t_0+T]\times\mathbb{R}^{n+1}$;

(iii) *there exist the limits*

$$\lim_{|x|\to+\infty}\sup_{t\in[t_0,t_0+T]}\frac{|g_i(t,x)|}{|x|} = r_i\in[0,+\infty),\qquad i=0,\ldots,n.$$

Then equation (1) *has at least one anti-periodic solution if*

$$\frac{1}{2}\exp\left(-2\int_0^T |a(u)|\,\mathrm{d}u\right) - T\left[1+\exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)\right]\sum_{i=0}^n r_i > 0. \tag{2.2}$$

P r o o f. Let $\lambda \in [0,1]$. Consider the operator equation $x = \lambda Lx$. If $x \in X$ is a solution of $x = \lambda Lx$, we get that

$$\begin{aligned} x(t) = & -\lambda \frac{1}{1+\exp\left(\int_0^T a(u)\,\mathrm{d}u\right)} \times \\ & \left[\int_t^{t+T} \exp\left(\int_t^s a(u)\,\mathrm{d}u\right) f(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))\,\mathrm{d}s \right. \\ & \left. + \sum_{t \le t_k < t+T} \exp\left(\int_t^{t_k} a(u)\,\mathrm{d}u\right) I_k(x(t_k))\right] = (Lx)(t). \end{aligned}$$

Then

$$\begin{cases} x'(t) + a(t)x(t) = \lambda f(t, x(t), x(\alpha_1(t)), \ldots, x(\alpha_n(t))), & t \in \mathbb{R}, \\ \Delta x(t_k) = \lambda I_k(x(t_k)), \quad k \in \mathbb{Z}. \end{cases} \tag{2.3}$$

Transforming the first equation in (4) into

$$\begin{aligned} & \left(x(t)\exp\left(\int_{t_0}^t a(s)\,\mathrm{d}s\right)\right)' \\ & = \lambda f(t, x(t), x(\alpha_1(t)), \ldots, x(\alpha_n(t))) \exp\left(\int_{t_0}^t a(s)\,\mathrm{d}s\right). \end{aligned} \tag{2.4}$$

Multiplying both sides of the equation of (5) by $x(t)\mathrm{e}^{\int_{t_0}^t a(s)\,\mathrm{d}s}$, we get that

$$\begin{aligned} & \left(x(t)\exp\left(\int_{t_0}^t a(s)\,\mathrm{d}s\right)\right)\left(x(t)\exp\left(\int_{t_0}^t a(s)\,\mathrm{d}s\right)\right)' \\ & = \lambda f(t, x(t), x(\alpha_1(t)), \ldots, x(\alpha_n(t)))\left(x(t)\exp\left(2\int_{t_0}^t a(s)\,\mathrm{d}s\right)\right). \end{aligned}$$

Since (H1) implies that

$$x(t_k^+)x(t_k) = x(t_k)[x(t_k) + \lambda I_k(x(t_k))] \geq \lambda x(t_k) I_k(x(t_k)) \geq 0,$$

we get that

$$\left(x(t_k^+) \exp\left(\int_{t_0}^{t_k} a(s)\,\mathrm{d}s \right) \right) \left(x(t_k^-) \exp\left(\int_{t_0}^{t_k} a(s)\,\mathrm{d}s \right) \right) \geq 0, \qquad k \in \mathbb{Z}.$$

It follows from

$$x(t_0 + T) \exp\left(\int_{t_0}^{t_0+T} a(s)\,\mathrm{d}s \right) = -x(t_0) \exp\left(\int_{t_0}^{t_0+T} a(s)\,\mathrm{d}s \right)$$

that there exists $\xi \in [t_0, t_0 + T]$ such that $x(\xi) = 0$. For $t_0 \leq t \leq \xi$, we have

$$\begin{aligned}
&\int_t^{\xi} \left(x(s) \exp\left(\int_{t_0}^{s} a(u)\,\mathrm{d}u \right) \right) \left(x(s) \exp\left(\int_{t_0}^{s} a(u)\,\mathrm{d}u \right) \right)' \mathrm{d}s \\
&= \lambda \int_t^{\xi} f(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s))) \left(x(s) \exp\left(2 \int_{t_0}^{s} a(u)\,\mathrm{d}u \right) \right) \mathrm{d}s.
\end{aligned}$$

It follows (H2) that

$$\begin{aligned}
&- \frac{1}{2} \left(x(t) \exp\left(\int_{t_0}^{t} a(u)\,\mathrm{d}u \right) \right)^2 \\
&- \frac{1}{2} \sum_{t \leq t_k < \xi} \left[\left(x(t_k^+) \exp\left(\int_{t_0}^{t_k} a(s)\,\mathrm{d}s \right) \right)^2 - \left(x(t_k^-) \exp\left(\int_{t_0}^{t_k} a(s)\,\mathrm{d}s \right) \right)^2 \right] \\
&= \lambda \int_{\xi}^{t} f(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s))) \left(x(s) \exp\left(2 \int_{t_0}^{s} a(u)\,\mathrm{d}u \right) \right) \mathrm{d}s
\end{aligned}$$

$$= \lambda \int_{\xi}^{t} h(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s))) \left(x(s) \exp\left(2 \int_{t_0}^{s} a(u)\,\mathrm{d}u \right) \right) \mathrm{d}s$$

$$+ \lambda \sum_{i=0}^{n} \int_{\xi}^{t} g_i(s, x(\alpha_i(s))) \left(x(s) \exp\left(2 \int_{t_0}^{s} a(u)\,\mathrm{d}u \right) \right) \mathrm{d}s$$

$$+ \lambda \int_{\xi}^{t} r(s) \left(x(s) \exp\left(2 \int_{t_0}^{s} a(u)\,\mathrm{d}u \right) \right) \mathrm{d}s.$$

It follows from (H1) that

$$\begin{aligned} \left(x(t_k^+)\right)^2 - \left(x(t_k^-)\right)^2 &= \left(x(t_k^+) - x(t_k^-)\right)\left(x(t_k^+) + x(t_k^-)\right) \\ &= \Delta x(t_k^-)\left(2x(t_k^-) + \Delta x(t_k^-)\right) \\ &= \lambda I_k(x(t_k^-))\left(2x(t_k^-) + \lambda I_k(x(t_k^-))\right) \\ &\geq \lambda I_k(x(t_k^-)) x(t_k^-) \geq 0. \end{aligned}$$

Then, we get

$$\frac{1}{2}\left(x(t) \exp\left(\int_{t_0}^{t} a(u)\,\mathrm{d}u \right) \right)^2$$

$$= -\frac{1}{2} \sum_{t \leq t_k < \xi} \left[\left(x(t_k^+) \exp\left(\int_{t_0}^{t_k} a(s)\,\mathrm{d}s \right) \right)^2 - \left(x(t_k^-) \exp\left(\int_{t_0}^{t_k} a(s)\,\mathrm{d}s \right) \right)^2 \right]$$

$$- \lambda \sum_{i=0}^{n} \int_{t}^{\xi} g_i(s, x(\alpha_i(s))) \left(x(s) \exp\left(2 \int_{t_0}^{s} a(u)\,\mathrm{d}u \right) \right) \mathrm{d}s$$

$$- \lambda \int_{t}^{\xi} r(s) \left(x(s) \exp\left(2 \int_{t_0}^{s} a(u)\,\mathrm{d}u \right) \right) \mathrm{d}s$$

$$\leq \sum_{i=0}^{n} \int_{t}^{\xi} |g_i(s, x(\alpha_i(s)))| \exp\left(2 \int_{t_0}^{s} a(u)\,\mathrm{d}u \right) |x(s)|\,\mathrm{d}s$$

$$+ \|r\| \int_{t}^{\xi} |x(s)| \exp\left(2 \int_{t_0}^{s} a(u)\,\mathrm{d}u \right) \mathrm{d}s$$

$$\leq \exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)\|x\|\sum_{i=0}^n\int_t^\xi |g_i(s,x(\alpha_i(s)))|\,\mathrm{d}s$$
$$+\|r\|\exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)T\|x\|.$$

Let $\varepsilon>0$ satisfy that

$$\frac{1}{2}\exp\left(-2\int_0^T |a(u)|\,\mathrm{d}u\right)-T\left[1+\exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)\right]\sum_{i=0}^n(r_i+\varepsilon)>0. \quad (2.5)$$

For such $\varepsilon>0$, there is $\delta>0$ such that

$$|g_i(t,x)|<(r_i+\varepsilon)|x| \text{ uniformly for } t\in[0,T] \text{ and } |x|>\delta, \qquad i=0,1,\ldots,n. \quad (2.6)$$

Let, for $i=1,\ldots,n$,

$$\Delta_{1,i}=\{t:\ t\in\mathbb{R},\ |x(\alpha_i(t))|\leq\delta\}, \qquad i=1,\ldots,n,$$
$$\Delta_{2,i}=\{t:\ t\in\mathbb{R},\ |x(\alpha_i(t))|>\delta\}, \qquad i=1,\ldots,n,$$
$$g_{\delta,i}=\max_{t\in\mathbb{R},|x|\leq\delta}|g_i(t,x)|, \qquad i=1,\ldots,n.$$

Then, we get

$$\frac{1}{2}\left(x(t)\exp\left(\int_{t_0}^t a(u)\,\mathrm{d}u\right)\right)^2$$
$$\leq \exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)\|x\|\sum_{i=0}^n\left(\int_{[t,\xi]\cap\Delta_{1,i}}+\int_{[t,\xi]\cap\Delta_{2,i}}\right)|g_i(s,x(\alpha_i(s)))|\,\mathrm{d}s$$
$$+\|r\|\exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)T\|x\|$$
$$\leq \exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)\|x\|\sum_{i=0}^n\left(T\delta_{\delta,i}+(r_i+\varepsilon)\int_{[t,\xi]\cap\Delta_{2,i}}|x(\alpha_i(s))|\,\mathrm{d}s\right)$$
$$+\|r\|\exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)T\|x\|$$

$$\leq \exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)\|x\|\sum_{i=0}^n (T\delta_{\delta,i}+(r_i+\varepsilon)T\|x\|)$$
$$+\|r\|\exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)T\|x\|.$$

Especially, one sees that

$$\frac{1}{2}|x(t_0)|^2 \leq \exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)\|x\|\sum_{i=0}^n (T\delta_{\delta,i}+(r_i+\varepsilon)T\|x\|)$$
$$+\|r\|\exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)T\|x\|.$$

Then

$$\frac{1}{2}|x(t_0+T)|^2 \leq \exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)\|x\|\sum_{i=0}^n (T\delta_{\delta,i}+(r_i+\varepsilon)T\|x\|)$$
$$+\|r\|\exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)T\|x\|.$$

Now, for $\xi \leq t \leq t_0+T$, we have

$$\int_t^{t_0+T}\left(x(s)\exp\left(\int_{t_0}^s a(u)\,\mathrm{d}u\right)\right)\left(x(s)\exp\left(\int_{t_0}^s a(u)\,\mathrm{d}u\right)\right)'\mathrm{d}s$$
$$=\lambda\int_t^{t_0^T} f(s,x(s),x(\alpha_1(s)),\ldots,x(\alpha_n(s)))\left(x(s)\exp\left(2\int_{t_0}^s a(u)\,\mathrm{d}u\right)\right)\mathrm{d}s.$$

It follows (H2) that

$$\frac{1}{2}\left(x(t_0+T)\exp\left(\int_{t_0}^{t_0+T} a(u)\,\mathrm{d}u\right)\right)^2-\frac{1}{2}\left(x(t)\exp\left(\int_{t_0}^{t} a(u)\,\mathrm{d}u\right)\right)^2$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{t\le t_k<t_0+T}\left[\left(x(t_k^+)\exp\left(\int_{t_0}^{t_k}a(s)\,\mathrm{d}s\right)\right)^2-\left(x(t_k^-)\exp\left(\int_{t_0}^{t_k}a(s)\,\mathrm{d}s\right)\right)^2\right]\\
&=\lambda\int_t^{t_0+T}f(s,x(s),x(\alpha_1(s)),\dots,x(\alpha_n(s)))\left(x(s)\exp\left(2\int_{t_0}^{s}a(u)\,\mathrm{d}u\right)\right)\mathrm{d}s\\
&=\lambda\int_t^{t_0+T}h(s,x(s),x(\alpha_1(s)),\dots,x(\alpha_n(s)))\left(x(s)\exp\left(2\int_{t_0}^{s}a(u)\,\mathrm{d}u\right)\right)\mathrm{d}s\\
&\quad+\lambda\sum_{i=0}^{n}\int_t^{t_0+T}g_i(s,x(\alpha_i(s)))\left(x(s)\exp\left(2\int_{t_0}^{s}a(u)\,\mathrm{d}u\right)\right)\mathrm{d}s\\
&\quad+\lambda\int_t^{t_0+T}r(s)\left(x(s)\exp\left(2\int_{t_0}^{s}a(u)\,\mathrm{d}u\right)\right)\mathrm{d}s.
\end{aligned}
$$

Then similarly to above discussion we get

$$
\begin{aligned}
&\frac{1}{2}\left(x(t)\exp\left(\int_{t_0}^{t}a(u)\,\mathrm{d}u\right)\right)^2\\
&=\frac{1}{2}\left(x(t_0+T)\exp\left(\int_{0}^{T}a(u)\,\mathrm{d}u\right)\right)^2\\
&\quad-\frac{1}{2}\sum_{t\le t_k<t_0+T}\left[\left(x(t_k^+)\exp\left(\int_{t_0}^{t_k}a(s)\,\mathrm{d}s\right)\right)^2-\left(x(t_k^-)\exp\left(\int_{t_0}^{t_k}a(s)\,\mathrm{d}s\right)\right)^2\right]\\
&\quad-\lambda\int_t^{t_0+T}h(s,x(s),x(\alpha_1(s)),\dots,x(\alpha_n(s)))\left(x(s)\exp\left(2\int_{t_0}^{s}a(u)\,\mathrm{d}u\right)\right)\mathrm{d}s\\
&\quad-\lambda\sum_{i=0}^{n}\int_t^{t_0+T}g_i(s,x(\alpha_i(s)))\left(x(s)\exp\left(2\int_{t_0}^{s}a(u)\,\mathrm{d}u\right)\right)\mathrm{d}s\\
&\quad-\lambda\int_t^{t_0+T}r(s)\left(x(s)\exp\left(2\int_{t_0}^{s}a(u)\,\mathrm{d}u\right)\right)\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\le \left[\exp\left(2\int\limits_0^T a^+(u)\,\mathrm{d}u\right)\|x\|\sum_{i=0}^{n}(T\delta_{\delta,i}+(r_i+\varepsilon)T\|x\|)\right.\\
&\quad \left.+\|r\|\exp\left(2\int\limits_0^T a^+(u)\,\mathrm{d}u\right)T\|x\|\right]\exp\left(2\int\limits_0^T a^+(u)\,\mathrm{d}u\right)\\
&\quad +\exp\left(2\int\limits_0^T a^+(u)\,\mathrm{d}u\right)\|x\|\sum_{i=0}^{n}(T\delta_{\delta,i}+(r_i+\varepsilon)T\|x\|)\\
&\quad +\|r\|\exp\left(2\int\limits_0^T a^+(u)\,\mathrm{d}u\right)T\|x\|\\
&= \left[1+\exp\left(2\int\limits_0^T a^+(u)\,\mathrm{d}u\right)\right]\times\\
&\quad \left[\exp\left(2\int\limits_0^T a^+(u)\,\mathrm{d}u\right)\|x\|\sum_{i=0}^{n}(T\delta_{\delta,i}+(r_i+\varepsilon)T\|x\|)\right.\\
&\qquad\qquad \left.+\|r\|\exp\left(2\int\limits_0^T a^+(u)\,\mathrm{d}u\right)T\|x\|\right].
\end{aligned}
$$

Hence for all $t\in[t_0,t_0+T]$, we get

$$
\begin{aligned}
&\frac{1}{2}\left(x(t)\exp\left(\int\limits_{t_0}^t a(u)\,\mathrm{d}u\right)\right)^2\\
&\le \left[1+\exp\left(2\int\limits_0^T a^+(u)\,\mathrm{d}u\right)\right]\times\\
&\quad \left[\exp\left(2\int\limits_0^T a^+(u)\,\mathrm{d}u\right)\|x\|\sum_{i=0}^{n}(T\delta_{\delta,i}+(r_i+\varepsilon)T\|x\|)\right.\\
&\qquad\qquad \left.+\|r\|\exp\left(2\int\limits_0^T a^+(u)\,\mathrm{d}u\right)T\|x\|\right].
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{1}{2}\|x\|^2 \exp\left(-2\int_0^T |a(u)|\,\mathrm{d}u\right)\\
&\le \frac{1}{2}\|x\|^2 \exp\left(2\int_{t_0}^t a(u)\,\mathrm{d}u\right)\\
&\le \left[1+\exp\left(2\int_0^T a(u)\,\mathrm{d}u\right)\right]\times\\
&\quad\left[\exp\left(2\int_0^T |a(u)|\,\mathrm{d}u\right)\|x\|\sum_{i=0}^n (T\delta_{\delta,i}+(r_i+\varepsilon)T\|x\|)\right.\\
&\qquad\left.+\|r\|\exp\left(2\int_0^T |a(u)|\,\mathrm{d}u\right)T\|x\|\right].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\left[\frac{1}{2}\exp\left(-2\int_0^T a^-(u)\,\mathrm{d}u\right)\right.\\
&\left.-\left[1+\exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)\right]\exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)\sum_{i=0}^n (r_i+\varepsilon)T\right]\|x\|^2\\
&\le \left[1+\exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)\right]\left[\exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)T\|x\|\sum_{i=0}^n \delta_{\delta,i}\right.\\
&\qquad\left.+\|r\|\exp\left(2\int_0^T a^+(u)\,\mathrm{d}u\right)T\|x\|\right].
\end{aligned}
$$

Hence one sees from (6) that there exists a constant $M>0$ such that $\|x\|\le M$ for all $x\in\Omega=\{x\in X:\ x=\lambda Lx\ \text{ for some }\ \lambda\in[0,1]\}$.

Let $\Omega_0=\{x\in X:\ \|x\|<M+1\}$. Then $x\ne\lambda Lx$ for all $\lambda\in[0,1]$ and all $x\in D(L)\bigcap\partial\Omega_0$. Lemmas 2.1 and 2.3 implies that $L\colon X\to X$ is completely continuous. It follows from Lemma 2.4 that there is $x\in X$ such that $x=Lx$. Then Lemma 2.2 implies that equation (1) has at least one anti-periodic solution $x\in X$. The proof is complete. $\square$

Remark 1. In paper [14], the existence of solutions of the following anti-periodic boundary value problem of the form

$$\begin{cases} x'(t) = f(t, x(t), x(\alpha_1(t)), \dots, x(\alpha_n(t))), \quad t \in [0, T], \ \ t \neq t_k, \ \ k = 1, \dots, m, \\ \Delta x(t_k) = I_k(x(t_k)), \quad k = 1, \dots, m, \\ x(0) = -x(T) \end{cases} \tag{BVP}$$

was studied. It was proved (see [14, Theorem 3]) that if $x(x + I_k(x)) \geq 0$ for all $x \in \mathbb{R}$ and $k = 1, \dots, m$, $I_k(x)(2x + I_k(x) \geq 0$ for all $x \in \mathbb{R}$ and $k = 1, \dots, m$ and (G2) holds, then (BVP) has at least one solution if

$$r_0 + \sum_{k=1}^{n} r_k < \frac{1}{4T}.$$

One can see from Theorem 2.1 in this paper and [14, Theorem 3] that the existence conditions for solutions of (1) and for solutions of (BVP) are extensively different from each other.

Theorem 2.2. *Suppose that* (A1)–(A5) *hold and*

(G3) $I_k(x)(2x + I_k(x)) \leq 0$ *for all* $x \in \mathbb{R}$ *and* $k \in \mathbb{Z}$*;*

(G4) *there exist impulsive continuous functions* $h\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $g_i\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $r \in X$ *such that*

(i) $f(t, x_0, \dots, x_n) = h(t, x_0, \dots, x_n) + \sum\limits_{i=0}^{n} g_i(t, x_i) + r(t)$ *holds for all* $(t, x_0, \dots, x_n) \in \mathbb{R} \times \mathbb{R}^{n+1}$*;*

(ii) *there exist* $t_0 \in \mathbb{R}$ *and constants* $m \geq 0$ *and* $\beta > 0$ *such that*

$$h(t, x_0, \dots, x_n)x_0 \leq 0$$

holds for all $(t, x_0, \dots, x_n) \in [t_0, t_0 + T] \times \mathbb{R}^{n+1}$*;*

(iii) *there exist the limits*

$$\lim_{|x| \to +\infty} \sup_{t \in [t_0, t_0+T]} \frac{|g_i(t, x)|}{|x|} = r_i \in [0, +\infty), \qquad i = 0, \dots, n.$$

Then equation (1) *has at least one anti-periodic solution if*

$$\frac{1}{2} \exp\left(-2 \int_0^T |a(u)| \, \mathrm{d}u\right) - T \left[1 + \exp\left(2 \int_0^T a^+(u) \, \mathrm{d}u\right)\right] \sum_{i=0}^{n} r_i > 0. \tag{2.7}$$

Proof. The proof is similar to that of Theorem 2.1. We omitted it. □

Remark 2. One can see from Theorem 2.2 in this paper and [14, Theorem 2] that the existence conditions for solutions of (1) and for solutions of (BVP) are extensively different from each other.

Theorem 2.3. *Suppose that* (A1)–(A4) *hold and*

(G5) $I_k (k \in \mathbb{Z})$ *are continuous functions satisfying that there exist constants* $c_k \geq 0$ *such that* $|I_k(x)| \leq c_k|x|$ *for all* $x \in \mathbb{R}$ *and* $k \in \mathbb{Z}$.

(G6) *there exist impulsive continuous functions* $g_i \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $r \in X$ *such that*

  (i) $f(t, x_0, \ldots, x_n) = \sum\limits_{i=0}^{n} g_i(t, x_i) + r(t)$
  holds for all $(t, x_0, \ldots, x_n) \in \mathbb{R} \times \mathbb{R}^{n+1}$*;*

  (ii) *there exists* $t_0 \in \mathbb{R}$ *such that the limits*

$$\lim_{|x|\to+\infty} \sup_{t\in[t_0,t_0+T]} \frac{|g_i(t,x)|}{|x|} = r_i \in [0, +\infty), \qquad i = 0, \ldots, n.$$

Then equation (1) *has at least one anti-periodic solution if*

$$\sum_{0\leq t_k<T} c_k + T \frac{\exp\left(\int\limits_0^T a^+(u)\,\mathrm{d}u\right)}{1 + \exp\left(\int\limits_0^T a(u)\,\mathrm{d}u\right)} \sum_{i=0}^{n} r_i < 1. \tag{2.8}$$

P r o o f. The proof is similar to that of Theorem 2.1. We consider equation (5). The the definition of L implies that

$$\begin{aligned}
|x(t)| =& \lambda \Bigg| - \frac{1}{1 + \exp\left(\int\limits_0^T a(u)\,\mathrm{d}u\right)} \times \\
&\Bigg[\int\limits_t^{t+T} \exp\left(\int\limits_t^s a(u)\,\mathrm{d}u\right) f(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))\,\mathrm{d}s \\
&+ \sum_{t\leq t_k<t+T} \exp\left(\int\limits_t^{t_k} a(u)\,\mathrm{d}u\right) I_k(x(t_k))\Bigg]\Bigg| \\
\leq& \frac{\exp\left(\int\limits_0^T a^+(u)\,\mathrm{d}u\right)}{1 + \exp\left(\int\limits_0^T a(u)\,\mathrm{d}u\right)} \left| \int\limits_t^{t+T} f(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))\,\mathrm{d}s \right| \\
&+ \sum_{t\leq t_k<t+T} |I_k(x(t_k))|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\exp\left(\int_0^T a^+(u)\,\mathrm{d}u\right)}{1+\exp\left(\int_0^T a(u)\,\mathrm{d}u\right)} \int_t^{t+T} |f(s,x(s),x(\alpha_1(s)),\dots,x(\alpha_n(s)))|\,\mathrm{d}s \\
&\quad + \sum_{t\le t_k<t+T} c_k|x(t_k)| \\
&= \frac{\exp\left(\int_0^T a^+(u)\,\mathrm{d}u\right)}{1+\exp\left(\int_0^T a(u)\,\mathrm{d}u\right)} \int_0^{T} |f(s,x(s),x(\alpha_1(s)),\dots,x(\alpha_n(s)))|\,\mathrm{d}s \\
&\quad + \sum_{0\le t_k<T} c_k|x(t_k)| \\
&\le \frac{\exp\left(\int_0^T a^+(u)\,\mathrm{d}u\right)}{1+\exp\left(\int_0^T a(u)\,\mathrm{d}u\right)} \int_0^{T} \left(|g_0(s,x(s))| + \sum_{i=1}^n |g_i(t,x(\alpha_i(s))| + |r(t)|\right)\mathrm{d}s \\
&\quad + \|x\| \sum_{0\le t_k<T} c_k.
\end{aligned}
$$

Similarly to the proof of Theorem L1, we get that

$$
\begin{aligned}
|x(t)| \le{}& \frac{\exp\left(\int_0^T a^+(u)\,\mathrm{d}u\right)}{1+\exp\left(\int_0^T a(u)\,\mathrm{d}u\right)} \int_0^{T} \left(|g_0(s,x(s))| + \sum_{i=1}^n |g_i(t,x(\alpha_i(s)))| + |r(t)|\right)\mathrm{d}s \\
&+ \|x\| \sum_{0\le t_k<T} c_k.
\end{aligned}
$$

Let $\varepsilon > 0$ satisfy that

$$
\sum_{0\le t_k<T} c_k + T\frac{\exp\left(\int_0^T a^+(u)\,\mathrm{d}u\right)}{1+\exp\left(\int_0^T a(u)\,\mathrm{d}u\right)} \sum_{i=0}^n (r_i+\varepsilon) < 1. \tag{2.9}
$$

For such $\varepsilon > 0$, there is $\delta > 0$ such that

$$|g_i(t,x)| < (r_i+\varepsilon)|x| \quad \text{uniformly for } \ t \in [0,T] \ \text{ and } \ |x| > \delta, \quad i = 0,1,\dots,n. \tag{2.10}$$

Let, for $i = 1,\dots,n$,

$$\begin{aligned}
\Delta_{1,i} =& \big\{t:\ t \in \mathbb{R},\ |x(\alpha_i(t))| \le \delta\big\}, \qquad i = 1,\dots,n,\\
\Delta_{2,i} =& \big\{t:\ t \in \mathbb{R},\ |x(\alpha_i(t))| > \delta\big\}, \qquad i = 1,\dots,n,\\
g_{\delta,i} =& \max_{t\in\mathbb{R},|x|\le\delta} |g_i(t,x)|, \qquad i = 1,\dots,n,\\
\Delta_1 =& \big\{t:\ t \in \mathbb{R},\ |x(t)| \le \delta\big\},\\
\Delta_2 =& \big\{t:\ t \in \mathbb{R},\ |x(t)| > \delta\big\},\\
\delta =& \max_{t\in\mathbb{R},|x|\le\delta} |g(t,x)|.
\end{aligned}$$

Then, we get

$$\begin{aligned}
|x(t)| \le& \frac{\exp\left(\int\limits_0^T a^+(u)\,\mathrm{d}u\right)}{1+\exp\left(\int\limits_0^T a(u)\,\mathrm{d}u\right)} \left(T\delta + T\sum_{i=1}^n g_{\delta,i} + \|r\|\right)\\
&+ \frac{\exp\left(\int\limits_0^T a^+(u)\,\mathrm{d}u\right)}{1+\exp\left(\int\limits_0^T a(u)\,\mathrm{d}u\right)} \int\limits_0^T \left(r_0+\varepsilon)|x(s))| + \sum_{i=1}^n (r_i+\varepsilon)|x(\alpha_i(s)))|\right) \mathrm{d}s\\
&+ \|x\| \sum_{0\le t_k<T} c_k\\
\le& \frac{\exp\left(\int\limits_0^T a^+(u)\,\mathrm{d}u\right)}{1+\exp\left(\int\limits_0^T a(u)\,\mathrm{d}u\right)} \left(T\delta + T\sum_{i=1}^n g_{\delta,i} + \|r\|\right)\\
&+ T\frac{\exp\left(\int\limits_0^T a^+(u)\,\mathrm{d}u\right)}{1+\exp\left(\int\limits_0^T a(u)\,\mathrm{d}u\right)} \|x\| \sum_{i=1}^n (r_i+\varepsilon) + \|x\| \sum_{0\le t_k<T} c_k.
\end{aligned}$$

It follows from (10) that there exists a constant $M > 0$ such that $\|x\| \le M$ for all $x \in \Omega = \big\{x \in X : \ x = \lambda Lx \ \text{ for some } \ \lambda \in [0,1]\big\}$.

Let $\Omega_0 = \big\{x \in X : \ \|x\| < M+1\big\}$. Then $x \neq \lambda Lx$ for all $\lambda \in [0,1]$ and all $x \in D(L) \bigcap \partial\Omega_0$. Lemmas 2.1 and 2.3 implies that $L\colon X \to X$ is completely continuous. It follows from Lemma 2.4 that there is $x \in X$ such that $x = Lx$. Then Lemma 2.2 implies that equation (1) has at least one anti-periodic solution $x \in X$. The proof is complete. $\square$

Remark 3. One can see from Theorem 2.3 in this paper and [14, Theorem 1] that the existence conditions for solutions of (1) and for solutions of (BVP) are extensively different from each other.

3. Examples

Now, we present two examples, whose solutions can not be obtained by theorems in other known papers, to illustrate the main results.

Example 1. Consider the following equation

$$\begin{cases} x'(t) + (1+\sin t)x(t) = ax(t) + \sum\limits_{k=1}^{n} b_i x\,(t - \cos it) + \sin\frac{t}{2}, & t \in \mathbb{R}, \\ \Delta x(t_k) = cx(t_k), \qquad k \in \mathbb{Z}, \\ \Delta x(s_k) = dx(s_k), \qquad k \in \mathbb{Z}, \\ \Delta x(u_k) = ex(u_k), \qquad k \in \mathbb{Z}, \end{cases} \tag{3.1}$$

where $t_k = k\pi + \frac{\pi}{4}$, $s_k = k\pi + \frac{\pi}{2}$, $u_k = k\pi + \frac{3\pi}{4}$, $k \in \mathbb{Z}$, a, b_k $(k = 1,\dots,n)$, $c, d, e \ge 0$ are constants. The question is that under what conditions equation (12) has at least one anti-periodic solution with anti-period π.

P r o o f. Corresponding to equation (1), we find that $a(t) = 1+\sin t$, $I_k(x) = c_k x$ $(k \in \mathbb{Z})$ and $f(t, x_0, x_1, \dots, x_n) = ax_0 + \sum\limits_{k=1}^{n} b_i x_i + \sin\frac{t}{2}$ and $\alpha_k(t) = t - \cos kt$ $(k \in \mathbb{Z})$.

It is easy to check that (A1)–(A5) hold.

One sees that $xI_k(x) \ge 0$ if $c, d, e \ge 0$ for all $k \in \mathbb{Z}$.

Choose $g_i(t, x_i) = |b_i||x_i|$ $(i = 0, 1, \dots, n)$, $r(t) = |\sin\frac{t}{2}|$. Then

$$|f(t, x_0, x_1, \dots, x_n)| \le \sum_{i=0}^{n} g_i(t, x_i) + r(t).$$

It is easy to check that (G5) and (G6) hold with $r_i = |b_i|$ $(i = 0, 1, \dots, n)$. Then Theorem 2.3 implies that equation (12) has at least one anti-periodic solution with anti-periodic if

$$c + d + e + \pi \frac{\exp\left(\int\limits_0^\pi (a + \sin u)\,\mathrm{d}u\right)}{1 + \exp\left(\int\limits_0^\pi (1 + \sin u)\,\mathrm{d}u\right)} \sum_{i=0}^n b_i < 1. \tag{3.2}$$

□

Example 2. Consider the following equation

$$\begin{cases} x'(t) + (1 + \sin t)x(t) = a\frac{[x(t)]^3}{1+[x(t)]^6} + \sum\limits_{k=1}^n b_i[x\,(t - \cos t)] + \sin\frac{t}{2}, \\ \Delta x(t_k) = c_k[x(t_k)]^3, \qquad k \in \mathbb{Z}, \end{cases} \tag{3.3}$$

where $a \geq 0$, $t_k = k\pi + \frac{\pi}{2}$, $k \in \mathbb{Z}$, a, b_k $(k = 1, \dots, n)$ are constants. The question is that under what conditions equation (13) has at least one anti-periodic solution with anti-period π.

Proof. Corresponding to equation (1), we find that $a(t) = 1 + \sin t$, $I_k(x) = c_k x^3$ $(k \in \mathbb{Z})$ and $f(t, x_0, x_1, \dots, x_n) = a\frac{x_0^3}{1+x_0^6} + \sum\limits_{k=1}^n b_i x_i + \sin\frac{t}{2}$ and $\alpha_k(t) = t - \cos kt$ $(k \in \mathbb{Z})$.

It is easy to check that (A1)–(A5) hold.

One sees that $x(x + I_k(x)) \geq 0$ and $xI_k(x) \geq 0$ if $c_k \geq 0$ for all $k \in \mathbb{Z}$.

Choose $h(t, x_0, x_1, \dots, x_n) = a\frac{x_0^3}{1+x_0^6}$, $g_i(t, x_i) = |b_i||x_i|$ $(i = 0, 1, \dots, n)$, $r(t) = |\sin\frac{t}{2}|$. Then

$$h(t, x_0, x_1, \dots, x_n)x_0 \geq 0.$$

It is easy to check that (G1) and (G2) hold with $r_i = |b_i|$ $(i = 0, 1, \dots, n)$. Then Theorem 2.1 implies that equation (13) has at least one anti-periodic solution with anti-periodic if

$$\frac{\mathrm{e}^{-2\int\limits_0^\pi (1+\sin u)\,\mathrm{d}u}}{2} - T\left[1 + \exp\left(2\int\limits_0^\pi (1 + \sin u)\,\mathrm{d}u\right)\right] \sum_{i=0}^n |b_i| > 0. \tag{3.4}$$

□

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