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TELESCOPIC GENERALIZATIONS FOR TWO ${}_3F_2$-SERIES IDENTITIES

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ABSTRACT. By combining a telescopic summation formula with Kummer-Thomae-Whipple transformation, we prove two nonterminating ${}_3F_2(1)$-series identities with one of them confirming a conjecture by Milgram (2009) and another one extending a couple of terminating series identities due to Gessel and Stanton (1982).

For an indeterminate x and an integer n, denote the shifted-factorial by

$$(x)_0 = 1 \qquad \text{and} \qquad (x)_n = \Gamma(x+n)/\Gamma(x)$$

where Γ is Euler's gamma function, which enables us to write explicitly

$$(x)_n = \begin{cases} x(x+1)\dots(x+n-1), & n \in \mathbb{N}; \\ 1/(x-1)(x-2)\dots(x+n), & -n \in \mathbb{N}. \end{cases}$$

Following Bailey [1], the hypergeometric series, for a complex variable z and two nonnegative integers p and q, is defined by

$${}_{1+p}F_q\left[\begin{matrix} a_0, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \,\middle|\, z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \dots (a_p)_k}{k!(b_1)_k \dots (b_q)_k} z^k$$

where $\{a_i\}$ and $\{b_j\}$ are complex parameters such that no zero factors appear in the denominators of the summands on the right hand side. If one of numerator

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parameters $\{a_k\}$ is a negative integer, then the series becomes terminating, which reduces to a polynomial in z.

For the sake of brevity, the product and fraction of shifted factorials will be abbreviated respectively as

$$[A,\ B,\ \dots,\ C]_n\ = (A)_n(B)_n\dots(C)_n,$$

$$\left[\begin{matrix}\alpha,\ \beta,\ \dots,\ \gamma\\ A,\ B,\ \dots,\ C\end{matrix}\right]_n = \frac{(\alpha)_n(\beta)_n\dots(\gamma)_n}{(A)_n(B)_n\dots(C)_n}.$$

There are many hypergeometric series formulae in the literature of mathematics and physics. In 1982, Gessel and Stanton [3, equations 1.6; 1.9] found, among others, the following two interesting terminating series identities

$$ {}_3F_2\left[\begin{matrix}2a, 1-a, -n\\ 2+2a, -a-\frac{1+3n}{2}\end{matrix}\ \Big|\ 1\right] = (1+2a)\frac{\left(\frac{1+n}{2}\right)_{n+1}}{\left(\frac{1+n+2a}{2}\right)_{n+1}}, \tag{1}$$

$$ {}_3F_2\left[\begin{matrix}b-1, 1-bd+d, -n\\ b+1, -bd-nd-n\end{matrix}\ \Big|\ 1\right] = b\frac{(d+dn)_{n+1}}{(bd+dn)_{n+1}}. \tag{2}$$

As observed by Gessel and Stanton [3], the equation (1) results equivalently from the special case of (2) corresponding to $b \mapsto 1+2a$ and $d \mapsto 1/2$. For another proof of (2) by partial fraction decomposition, refer to [2, Equation 3.5e].

By applying the transformation of Kummer-Thomae-Whipple to a nonterminating telescopic summation formula, we shall establish two nonterminating ${}_3F_2(1)$-series identities, which include these two identities of Gessel and Stanton as special cases.

For the sequence τ_k given by

$$\tau_k := \left[\begin{matrix}1+\alpha,\ 1+\beta\gamma\\ 1+\alpha\gamma,\ 1+\beta\end{matrix}\right]_k$$

it is almost trivial to compute its forward difference

$$\Delta\tau_k = \tau_{k+1}-\tau_k = (1+k)\left[\begin{matrix}1+\alpha,\ 1+\beta\gamma\\ 2+\alpha\gamma,\ 2+\beta\end{matrix}\right]_k \frac{(\alpha-\beta)(1-\gamma)}{(1+\alpha\gamma)(1+\beta)}.$$

Then the telescoping method leads us to the following finite sum identity

$$\sum_{k=m}^{n-1} \Delta\tau_k = \tau_n - \tau_m$$

which can explicitly be restated as the following lemma.

Lemma 1 (**Terminating series identity**).

$$\sum_{k=m}^{n-1}(k+1)\begin{bmatrix}1+\alpha,\ 1+\beta\gamma\\ 2+\alpha\gamma,\ 2+\beta\end{bmatrix}_k$$

$$= \frac{(1+\alpha\gamma)(1+\beta)}{(\alpha-\beta)(1-\gamma)}\left\{\begin{bmatrix}1+\alpha,\ 1+\beta\gamma\\ 1+\alpha\gamma,\ 1+\beta\end{bmatrix}_n - \begin{bmatrix}1+\alpha,\ 1+\beta\gamma\\ 1+\alpha\gamma,\ 1+\beta\end{bmatrix}_m\right\}.$$

Recall the asymptotic relation of Γ-function (cf. [5, §2.11])

$$\Gamma(x+n) \approx (n-1)!\, n^x \qquad \text{when} \quad n \to \infty. \tag{3}$$

When $\Re\big\{(\alpha-\beta)(\gamma-1)\big\} > 0$, the following limit can be evaluated

$$\lim_{n\to+\infty}\begin{bmatrix}1+\alpha, 1+\beta\gamma\\ 1+\alpha\gamma, 1+\beta\end{bmatrix}_n$$

$$= \frac{\Gamma(1+\alpha\gamma)\Gamma(1+\beta)}{\Gamma(1+\alpha)\Gamma(1+\beta\gamma)}\lim_{n\to+\infty}\frac{\Gamma(1+\alpha+n)\Gamma(1+\beta\gamma+n)}{\Gamma(1+\alpha\gamma+n)\Gamma(1+\beta+n)}$$

$$= \frac{\Gamma(1+\alpha\gamma)\Gamma(1+\beta)}{\Gamma(1+\alpha)\Gamma(1+\beta\gamma)}\lim_{n\to+\infty} n^{(\alpha-\beta)(1-\gamma)} = 0.$$

Letting $m = 0$ and $n \to +\infty$ in Lemma 1, we find the following interesting identity.

Theorem 2. $(\Re\{(\alpha-\beta)(\gamma-1)\} > 0)$

$${}_3F_2\left[\begin{matrix}2,\ 1+\alpha,\ 1+\beta\gamma\\ 2+\alpha\gamma,\ 2+\beta\end{matrix}\,\middle|\, 1\right] = \frac{(1+\alpha\gamma)(1+\beta)}{(\alpha-\beta)(\gamma-1)}.$$

This formula has been conjectured by Milgram [4, Equation 24], which has been the primary motivation for the author to write this paper.

Under the same condition $\Re\big\{(\alpha-\beta)(\gamma-1)\big\}>0$, there holds analogously the limiting relation

$$\begin{aligned}\lim_{m\to-\infty}\begin{bmatrix}1+\alpha,1+\beta\gamma\\1+\alpha\gamma,1+\beta\end{bmatrix}_m &= \lim_{m\to-\infty}\begin{bmatrix}-\alpha\gamma,-\beta\\-\alpha,-\beta\gamma\end{bmatrix}_{-m}\\ &=\frac{\Gamma(-\alpha)\Gamma(-\beta\gamma)}{\Gamma(-\alpha\gamma)\Gamma(-\beta)}\lim_{m\to-\infty}\frac{\Gamma(-\alpha\gamma-m)\Gamma(-\beta-m)}{\Gamma(-\alpha-m)\Gamma(-\beta\gamma-m)}\\ &=\frac{\Gamma(-\alpha)\Gamma(-\beta\gamma)}{\Gamma(-\alpha\gamma)\Gamma(-\beta)}\lim_{m\to-\infty}(-m)^{(\alpha-\beta)(1-\gamma)} = 0.\end{aligned}$$

Then letting $m\to-\infty$ and $n\to+\infty$ in Lemma 1 results in another unusual bilateral series identity.

Theorem 3. ($\Re\{(\alpha-\beta)(\gamma-1)\}>0$)

$$\sum_{k=-\infty}^{+\infty}(k+1)\begin{bmatrix}1+\alpha,1+\beta\gamma\\2+\alpha\gamma,2+\beta\end{bmatrix}_k=0.$$

For the ${}_3F_2(1)$-series, there is useful transformation due to Kummer-Thomae-Whipple (cf. Bailey [1, §3.2]), that can be reproduced as

$${}_3F_2\left[\begin{matrix}a,\ c,\ e\\ b,\ d\end{matrix}\,\Big|\,1\right]=\Gamma\begin{bmatrix}b,\ d,\ s\\ a,\ c{+}s,e{+}s\end{bmatrix}{}_3F_2\left[\begin{matrix}b-a,d-a,\ s\\ c+s,e+s\end{matrix}\,\Big|\,1\right],\tag{4}$$

$${}_3F_2\left[\begin{matrix}a,\ c,\ e\\ b,\ d\end{matrix}\,\Big|\,1\right]=\Gamma\begin{bmatrix}d,\ s\\ d-a,a+s\end{bmatrix}{}_3F_2\left[\begin{matrix}a,\ b-c,b-e\\ b,b+d-c-e\end{matrix}\,\Big|\,1\right].\tag{5}$$

where $s=b+d-a-c-e$ and the latter follows from iterating the former to the ${}_3F_2$-series on the right hand side of (4).

Applying (4) to the ${}_3F_2$-series displayed in Theorem 2, we get the following formula.

Theorem 4 (Identity for the 2-balanced ${}_3F_2$-series).

$${}_3F_2\left[\begin{matrix}\alpha\gamma,\beta,(\alpha-\beta)(\gamma-1)\\ 1{+}\alpha\gamma{-}\alpha{+}\beta,1{+}\alpha\gamma{-}\beta\gamma{+}\beta\end{matrix}\,\Big|\,1\right]$$

$$=\frac{\Gamma(1+\alpha\gamma-\alpha+\beta)\Gamma(1+\alpha\gamma-\beta\gamma+\beta)}{\Gamma(1+\alpha\gamma)\Gamma(1+\beta)\Gamma(1+(\alpha-\beta)(\gamma-1))}.$$

Instead, applying (5) to the same ${}_3F_2$-series in Theorem 2 yields another identity.

Theorem 5. $(\Re(1-\alpha+\beta)>0)$

$${}_3F_2\left[\begin{matrix}1+\alpha,\alpha\gamma,1+\alpha\gamma-\beta\gamma\\ 2+\alpha\gamma,1+\alpha\gamma-\beta\gamma+\beta\end{matrix}\,\Big|\,1\right]=(1+\alpha\gamma)\frac{\Gamma(1-\alpha+\beta)\Gamma(1+\alpha\gamma-\beta\gamma+\beta)}{\Gamma(1+\beta)\Gamma(1+(\alpha-\beta)(\gamma-1))}.$$

It is not hard to check that the two terminating series identities (1) and (2) due to Gessel and Stanton are special cases of Theorem 5 corresponding to

$$\text{“}\alpha\mapsto -a,\qquad \beta\mapsto -a-(n+1)/2,\quad \gamma\mapsto -2\text{”}$$

$$\text{“}\alpha\mapsto d-bd,\quad \beta\mapsto -bd-dn,\qquad \gamma\mapsto -1/d\text{”},$$

respectively.

Furthermore, Milgram [4, Equation 23] has shown via WZ-method the following nontermination form of (1)

$${}_3F_2\left[\begin{matrix}2a,1-a,2n\\ 2+2a,3n-a-\frac{1}{2}\end{matrix}\,\Big|\,1\right]=(1+2a)\frac{\Gamma\big(a-n+\frac{1}{2}\big)\Gamma\big(\frac{5}{2}-3n\big)}{3\Gamma\big(\frac{3}{2}-n\big)\Gamma\big(\frac{3}{2}+a-3n\big)}$$

which is obviously a special case of Theorem 5. For other special cases and variants of Theorems 4 and 5, the reader is invited to consult Milgram [4, Appendix B].

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