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PARTIAL INFORMATION SYSTEMS AND THE SMYTH POWERDOMAIN

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ABSTRACT. The dual of the join semilattice of proper compact Scott open subsets of a domain D is called the *Smyth powerdomain* of D. The Smyth powerdomain is used in programming semantics as a model for demonic nondeterminism. In this paper, we introduce the concept of partial information systems; and, as an application, show that the Smyth powerdomain of any domain can be realized in terms of the sub partial information systems of the domain's corresponding information system.

1. Introduction

The classical denotational semantic notion of the *information system* and the entities derived from it have been employed in a range of applications in theoretical computer science and other disciplines (for recent examples, see Spreen *et. al* [9] or Xu and Mao [11]). In particular, information systems have been used to provide compelling representations of complex objects (see for example Bedregal [2] or Droste and Göbel [3]).

Order-convex subobjects play an important role in the study of many partially ordered structures (ideals in lattices and convex normal subgroups in ℓ-groups come to mind immediately). Since it is possible to view information systems as preordered structures, it therefore seems reasonable to attempt representing order theoretic entities well-known to computer science in terms of order-convex subobjects of information systems, especially when these subobjects can be given a natural semantic interpretation. In Hart and Tsinakis [5], we prove that the

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Hoare powerdomain can be represented in such a way. The current paper continues this theme by showing it is possible to identify the Smyth powerdomain with a family of order-convex subobjects in a structure closely related to information systems. Before embarking, we pause to introduce the relevant concepts.

We begin with the concept of a *domain.* A *domain* for a programming language is the underlying set of data objects for an admissible type equipped with an information-based partial ordering. (Our use of the term "domain" follows that of Davey and Priestley [1].) To make precise what what is meant, we must delve briefly into order theory.

A subset V of a poset P is *directed* if every finite nonempty subset of V has an upper bound in V. We will say a poset P is *directed-complete* (a DCPO) provided the join of every directed subset of P exists in P, and we will refer to a DCPO with least element as a *complete* poset (a CPO). Our use of these terms is common but not universal — see Davey and Priestley [1] for a discussion about nomenclature.

A subset I of a poset $(P, \leq)$ is a *lower set* (equivalently *lower set*) of P provided there exist $X \subseteq P$ such that

$$I = \downarrow X = \big\{p \in P : \ (\exists x \in X)(p \leq x)\big\}$$

A lower set I is *principal* provided $I = \downarrow\{x\}$ for some $x \in P$. It is common to write $I = \downarrow x$ in this case. (An *upper-set* $U = \uparrow X$ of a poset is defined dually.)

An *ideal* of a poset P is a directed lower set of P. The set $\mathtt{Idl}(P)$ of all ideals of P, ordered by set-inclusion, is always a DCPO (where joins are unions), and the assignment $x \mapsto \downarrow x$ provides an order embedding of P into $(\mathtt{Idl}(P), \subseteq)$. The poset $\mathtt{Idl}(P)$ is called the *ideal completion* of P.

An element x of a DCPO D is *compact* if, whenever x is below the supremum of a directed set $V \subseteq D$, then $x \in \downarrow V$. We use $K(D)$ to denote the subposet of compact elements of D. A DCPO D is *algebraic* if, for all $d \in D$, the set $K(d) = \downarrow d \cap K(D)$ is directed and $d = \bigvee K(d)$.

We note in passing that the ideal completion of any poset P is algebraic. The compact elements of $\mathtt{Idl}(P)$ are precisely the principal lower sets of P.

In this paper, we will use the term "domain" for an algebraic poset in which the meet of every non-empty subset exists. Equivalently, a "domain" is an algebraic poset in which the join of every upper bounded subset exists. Note that a domain is a CPO. (These particular objects are often called *Scott* domains.)

A lower set of a DCPO D is *Scott-closed* if it contains the join of each of its directed subsets. Let $\Gamma(D)$ denote the set of all Scott-closed subsets of a DCPO D, ordered by set-inclusion. The collection $\Gamma(D)$ is the family of closed sets for the *Scott topology* on D. The open sets of the Scott topology are those upper-sets U of D which are *inaccessible by directed joins*; that is, whenever $V \subseteq D$ is directed, $\bigvee V$ exists in D, and $\bigvee V \in U$, then $U \cap V \neq \emptyset$. We will let $\Sigma(D)$ denote the family of open sets of D under the Scott topology, partially ordered by set-inclusion.

Initially, domains were used only to model deterministic programming languages. When researchers began investigating models for languages which supported nondeterministic choice, it became necessary to enrich the theory of domains to include so-called *power domains* — domain theoretic analogs of the power set. Plotkin [6, 7], a pioneer in this field, identified three distinct ways to construct power domains for a given domain D. These constructs ultimately came to be known as the *Hoare* power domain $P_H(D)$, the *Smyth* power domain $P_S(D)$, and the *Plotkin* power domain $P_P(D)$. All three were initially defined as the ideal completion of the set $P_f[(K(D)]$ of finite, non-empty subsets of compact elements of D under various preorders derived from the ordering on the domain $(D, \leq)$, namely

(1) The *Hoare* preorder: $X \sqsubseteq Y \iff (\forall x \in X)(\exists y \in Y)(x \leq y)$

(2) The *Smyth* preorder: $X \sqsubseteq Y \iff (\forall y \in Y)(\exists x \in X)(x \leq y)$

(3) The *Plotkin* preorder:

$$X \sqsubseteq Y \iff [(\forall x \in X)(\exists y \in Y)(x \leq y)] \wedge [(\forall y \in Y)(\exists x \in X)(x \leq y)]$$

Smyth observed that these constructs could also be realized within the Scott topology on a domain D (see Plotkin [7]). In particular, $P_H(D)$ is order isomorphic to the lattice $\Gamma^*(D)$ of *non-empty* Scott-closed subsets of D, while $P_S(D)$ is order isomorphic to the meet semilattice $(\Sigma_K^*(D))^{\mathrm{op}}$ of all nonempty compact Scott-open subsets of D under reverse set-inclusion. From this perspective, the mappings $x \mapsto \downarrow x$ and $y \mapsto \uparrow y$ serve to embed the domain D into the respective power domains.

We next turn attention to information systems. Viewed from a logician's perspective, an information system for an object or a process is a triple $(S, \mathrm{Con}, \vdash)$, where S is a collection of propositions (or instructions) concerning the object or process, Con is a collection of finite subsets of S which are somehow "consistent" with one another, and $\vdash$ is a relation of interdependence between members of Con. The members of S are seen as providing simple bits of information about the object or process and are therefore called *tokens*. The set Con is called the *consistency predicate*, and $\vdash$ is known as a relation of *entailment*. An information system is assumed to obey certain common sense properties normally associated with the notions of consistency and entailment. These properties are made mathematically precise in the following definition. (In this definition and all the work that follows, we let $\mathtt{Fin}(S)$ denote the set of all finite subsets of a set S.)

Definition 1.1. An *information system* is a triple $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ consisting of

(1) a set S whose elements are called *propositions* or *tokens*;

(2) a non-empty subset Con of $\mathtt{Fin}(S)$, called the *consistency predicate*; and

(3) a binary relation $\vdash$ on Con, called the *entailment relation*.

These entities satisfy the following axioms:

- (IS1) Con is a lower set of $\mathtt{Fin}(S)$ — with respect to set-inclusion — such that $\bigcup \mathrm{Con} = S$;
- (IS2) if $A \in \mathrm{Con}$ and $B \subseteq A$, then $A \vdash B$;
- (IS3) if $A, B, C \in \mathrm{Con}$, $A \vdash B$, and $B \vdash C$, then $A \vdash C$; and
- (IS4) if $A, B, C \in \mathrm{Con}$, $A \vdash B$, and $A \vdash C$, then $B \cup C \in \mathrm{Con}$ and $A \vdash (B \cup C)$.

Note that axiom IS1 implies that every singleton subset of S is a member of Con and that whenever $A \in \mathrm{Con}$ and $B \subseteq A$, then $B \in \mathrm{Con}$. Furthermore, axioms IS2 and IS3 imply that $(\mathrm{Con}, \vdash)$ is a preordered set; that is, they imply $\vdash$ is a reflexive and transitive relation on Con.

We advise the reader that our definition of an information system is stated differently from the one commonly appearing in the literature, where entailment is defined as a relation on the set $\mathrm{Con} \times S$ (see for example Scott [8] or Davey and Priestley [1]). A comparison quickly shows our definition to be equivalent; it has the advantage of allowing us to think of $(\mathrm{Con}; \vdash)$ as a preordered set. (See Hart and Tsinakis [5] and Droste and Göbel [4] for additional examples of this approach).

We close this section by describing the aforementioned well-known correspondence between domains and information systems. Let $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ be an information system. For each $A \in \mathrm{Con}$, let

$$\overline{A} = \{B \in \mathrm{Con} : \ A \vdash B\}$$

and let $D_S = \mathtt{Idl}(\mathcal{S})$ denote the ideal completion of the family $\{\overline{A} : \ A \in \mathrm{Con}\}$. As such, D_S is a CPO with respect to set-union having $\overline{\emptyset}$ as least element; in fact, it is an algebraic poset whose compact members are precisely the sets $\overline{A}$ such that $A \in \mathrm{Con}$. It is routine to prove that D_S is closed under non-empty intersections and is therefore a domain.

On the other hand, suppose that D is a domain. Let $S_D = K(D)$, let

$$\mathrm{Con}_D = \big\{F \in \mathtt{Fin}[K(D)] : \ \big(\exists k \in K(D)\big)\big(F \subseteq {\downarrow} k\big)\big\}$$

and, for all $A, B \in \mathrm{Con}_D$, let $A \vdash_D B$ if and only if $\bigvee_D B \leq \bigvee_D A$. It is a routine matter to prove that the triple $\mathcal{S}_D = (S_D, \mathrm{Con}_D, \vdash_D)$ is an information system.

If D is any domain, then $D_{\mathcal{S}_D}$ is order isomorphic to D. The situation is more complicated for information systems. Let $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ be any information system and set

$$\theta = \big\{(A, B) \in \mathrm{Con} \times \mathrm{Con} :\ A \vdash B \ \text{ and } \ B \vdash A\big\}$$

The set θ is clearly an equivalence relation; and the quotient $(\mathrm{Con}, \vdash)/\theta$ is a poset. With this in mind, $(\mathrm{Con}, \vdash)/\theta$ is order isomorphic to $(\mathrm{Con}_{D_S}, \vdash_{D_S})$. The set S of tokens is usually quite different from the set S_{D_S} of tokens for $\mathcal{S}_{D_S}$, although a bijective correspondence exists between S_{D_S} and the equivalence classes of Con (relative to θ) which contain singletons.

The description we have outlined for the correspondence between information systems and domains differs somewhat from what is commonly found in the literature (see, for example Davey and Priestley [1]). The differences are superficial and stem from our emphasis on the pre-order structure of the consistency predicate. For the purposes of this paper, it is more convenient to use the descriptions presented above.

2. A novel representation of the Smyth Powerdomain

In this section, we introduce a concept which extends that of information systems and show that this notion provides a novel way of representing the Smyth Powerdomain of any domain. We begin with the fundamental definition, then discuss its motivation.

Definition 2.1. A *partial information system* takes the form of a quadruple $\mathbb{S} = (S, \mathrm{Con}, \beta, \vdash)$ consisting of

(1) a set S whose elements are called *propositions* or *tokens*;

(2) a non-empty subset Con of $\mathtt{Fin}(S)$, called the *consistency predicate*;

(3) a nonempty subset β of Con, called the *frontier*; and

(4) a binary relation $\vdash$ on Con, called the *entailment relation*.

These entities satisfy the following axioms:

- (PS1) $S = \bigcup \mathrm{Con}$;
- (PS2) If $A, X \in \mathrm{Con}$ and $X \subseteq A$, then $A \vdash X$;
- (PS3) if $A, B, C \in \mathrm{Con}$, $A \vdash B$, and $B \vdash C$, then $A \vdash C$;

- (PS4) if $A, B, C \in \mathrm{Con}$, $A \vdash B$, and $A \vdash C$, then $B \cup C \in \mathrm{Con}$ and $A \vdash (B \cup C)$;
- (PS5) If $B \in \beta$, then no proper subset of B is a member of Con;
- (PS6) If $A \in \mathrm{Con}$, then $A \vdash B$ for some $B \in \beta$;

Note that Axioms PS2 and PS3 together imply that $\vdash$ is a preorder on Con. Every information system $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ gives rise to a partial information system, namely $\mathcal{S}_P = (S, \mathrm{Con}, \{\emptyset\}, \vdash)$. On the other hand, suppose $\mathbb{S} = (S, \mathrm{Con}, \beta, \vdash)$ is a partial information system. If $\emptyset \in \mathrm{Con}$, then it must be the case that $\beta = \{\emptyset\}$. Consequently, if Con is a lower set in $\mathtt{Fin}(S)$ then $\mathbb{S}$ gives rise to an information system, namely $\mathbb{S}_S = (S, \mathrm{Con}, \vdash)$.

Definition 2.2. Let $\mathbb{S} = (S, \mathrm{Con}_S, \beta_S, \vdash_S)$ and $\mathbb{T} = (T, \mathrm{Con}_T, \beta_T, \vdash_T)$ be partial information systems. We say that $\mathbb{T}$ is a *sub* partial information system of $\mathbb{S}$ provided

(1) $T \subseteq S$, $\mathrm{Con}_T \subseteq \mathrm{Con}_S$, and $\vdash_T \subseteq \vdash_S$;

(2) If $A \in \beta_T$, then some subset B of A is a member of β_S, and $A \vdash B$.

We will write $\mathbb{T} \sqsubseteq \mathbb{S}$ if this is the case.

Lemma 2.3. *Let $\mathbb{S} = (S, \mathrm{Con}, \beta, \vdash)$ be a partial information system and suppose that $\mathbb{T} = (T, \mathrm{Con}_T, \beta_T, \vdash_T)$ and $\mathbb{U} = (U, \mathrm{Con}_U, \beta_U, \vdash_U)$ are sub partial information systems of $\mathbb{S}$. If $\mathbb{T} \sqsubseteq \mathbb{U}$ and $\mathbb{U} \sqsubseteq \mathbb{T}$, then $T = U$, $\mathrm{Con}_T = \mathrm{Con}_S$, $\vdash_T = \vdash_S$, and $\beta_T = \beta_S$.*

Proof. We need only prove that $\beta_T = \beta_S$. To this end, suppose that $A \in \beta_T$. By definition, there exist $B \in \beta_S$ such that $B \subseteq A$ and $A \vdash B$. Likewise, there exist $C \in \beta_T$ such that $C \subseteq B$ and $B \vdash C$. Axiom PS3 tells us that $A \vdash C$. Since $C \subseteq A$ and $A \in \beta_T$, Axiom PS5 tells us that $A = C$. Hence, we know that $A = B$ as well; and we see that $\beta_T \subseteq \beta_S$. The reverse inclusion follows similarly. □

We will let $\mathtt{SPInf}(\mathbb{S})$ denote the family of all sub partial information systems of a partial information system $\mathbb{S}$, partially ordered by the relation $\sqsubseteq$.

Having defined partial information systems, we now pause to give some motivation for this concept. The definition of an information system given in the previous section makes it possible to think of the consistency predicate as a preordered set; hence, it is natural to examine the pre-order convex substructures of this set. In Hart and Tsinakis [5], we show that the lower sets of the consistency predicate for an information system $\mathcal{S}$ correspond to so-called *full* subinformation systems of $\mathcal{S}$ and show that this family provides a concrete realization of the Hoare powerdomain of the domain corresponding to $\mathcal{S}$. Since the Smyth preorder is, in a sense, the dual of the Hoare preorder, it is natural to think that the Smyth powerdomain for the domain corresponding to $\mathcal{S}$ may be realized concretely as some family of upper-sets of the consistency predicate. This turns out to be true, but upper-sets of the consistency predicate pose a semantic problem not encountered with lower sets — they do not correspond to any type of subinformation system in $\mathcal{S}$. As we will show, partial information systems provide a way of understanding upper-sets in the consistency predicate as substructures of the information system.

There is another way to motivate the concept of partial information systems. Intuitively, we can think of partial information systems as representing "reverse engineering" processes. Suppose we wish to determine the simplest components needed to describe fully the operation of a complex machine (such as the human body). In this situation, we assume there is an information system $\mathcal{S}$ which fully determines the machine. We know some information about the more complex aspects of the machine, but have not fully identified the "tokens" of information for the machine or the entirety of its consistency predicate. In this setting, the sub partial information systems represent the processes of understanding the machine more fully. The frontiers of these systems represent the "cutting edge" of our understanding — we have developed the consistency predicate "above" the frontier (in the sense of entailment), but have yet to develop any part "below" the frontier. To say that $\mathbb{T} \sqsubseteq \mathbb{S}$ means that we know more about consistency and entailment in $\mathbb{S}$ than in $\mathbb{T}$, but the condition on the frontiers also says that the "cutting edge" of understanding has progressed, since in $\mathbb{S}$ we know more "below" the frontier of $\mathbb{T}$.

Suppose that $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ is an information system. For each $A \in \mathrm{Con}$, let

$$\overline{A} = \{B \in \mathrm{Con} : \ A \vdash B\}$$

It is well-known (and easy to show) that $\Sigma(\mathtt{Idl}(\mathcal{S}))$ is a bialgebraic (algebraic and dually algebraic), distributive lattice whose compact, join-prime elements are precisely the sets

$$\uparrow\overline{A} = \left\{v \in \mathtt{Idl}(\mathcal{S}) : \ \overline{A} \subseteq v\right\}$$

(For a novel approach to the proof, see Vickers [10].) With this fact in mind, we have the following result.

Lemma 2.4. *Suppose that $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ is an information system. If $\mathbb{T} = (T, \mathrm{Con}_T, \beta_T, \vdash_T)$ is a sub partial information system of $\mathcal{S}_P$, then the set $U_T = \bigcup\left\{\uparrow\overline{B} : \ B \in \beta_T\right\}$ is a nonempty Scott open subset of $\mathtt{Idl}(\mathcal{S})$.*

Proof. By construction, U_T is a nonempty upper set of $\mathtt{Idl}(\mathcal{S})$. Suppose that $D \subseteq \mathtt{Idl}(\mathcal{S})$ is directed and such that $\bigcup D \in U_T$. It follows that there exist

$B \in \beta_T$ such that $\overline{B} \subseteq D$. Let $B = \{b_1, \ldots, b_n\}$, then it follows that for each $1 \leq j \leq n$, there exist $D_j \in D$ such that $\{b_j\} \in D$. Since D is directed, this implies that there exist $D_B \in D$ such that $B \in D_B$. Since D_B is a lower set of Con, it follows that $\overline{B} \subseteq D$. Consequently, we know that $D_B \in \uparrow\overline{B}$; hence, we know that $D_B \in U_T$. We may conclude that U_T is Scott open. $\square$

Lemma 2.5. *Suppose that $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ is an information system, and suppose that U is a nonempty Scott open subset of $\mathtt{Idl}(\mathcal{S})$. There exists a largest family $\beta_U \subseteq \mathrm{Con}$ having the property that $A \in \beta_U$ if and only if*

(1) *$\overline{A} \in U$; and*
(2) *$\overline{B} \notin U$ for any proper $B \subset A$.*

P r o o f. Since U is Scott open and nonempty, we know that there exists a family $M_U \subseteq \mathrm{Con}$ such that $U = \bigcup\{\uparrow\overline{A} : \ A \in M_U\}$. For any $A \in M_U$, let $S_U[A] = \{B \subseteq A : \ \overline{B} \in U\}$, and let

$$\beta_U[A] = \{B \in S_A : \ B \text{ has smallest cardinality}\}$$

The set $\beta_U[A]$ is nonempty (at worst, $\beta_U[A] = \{\emptyset\}$). Clearly, the members of $\beta_U[A]$ satisfy Properties (1) and (2). A simple application of Zorn's Lemma now gives a maximal family β_U. The uniqueness of β_U is trivial. $\square$

Such a set β_U will be called the *frontier* for the Scott open set U. The frontier serves as a generating set for U, as the following result shows.

Lemma 2.6. *Suppose that $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ is an information system, and suppose that U is a nonempty Scott open subset of $\mathtt{Idl}(\mathcal{S})$. We have*

$$U = \bigcup\{\uparrow\overline{B} : \ B \in \beta_U\}.$$

P r o o f. For simplicity, let $V = \bigcup\{\uparrow\overline{B} : \ B \in \beta_U\}$. It is clear that $V \subseteq U$. To establish the reverse inclusion, first suppose that $A \in \mathrm{Con}$ is such that $\overline{A} \in U$ and let $\beta[A]$ be as as defined in the proof of Lemma 2.5. If $\beta_U[A] \not\subseteq \beta_U$, then $\beta_U \cup \beta_U[A]$ is a family of Con whose members satisfy Properties (1) and (2) of Lemma 2.5 which properly contains β_U — contradicting the maximality of β_U. It follows that $\overline{A} \in V$. Now, suppose that $v \in \mathtt{Idl}(\mathcal{S})$ is a member of U. It follows that there exists a directed family D_v of compact members of $\mathtt{Idl}(\mathcal{S})$ such that $v = \bigcup D_v$. Since U is Scott open, there exist $d \in D_v \cap U$. We know that $d = \overline{A}$ for some $A \in \mathrm{Con}$; hence, we may conclude that $v \in V$, as desired. $\square$

Suppose that $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ is an information system, and suppose that U is a nonempty Scott open subset of $\mathtt{Idl}(\mathcal{S})$. The set U induces a sub partial information system of $\mathcal{S}_P$. To see how, let

(1) β_U be the frontier of U;
(2) $\mathrm{Con}_U = \{A \in \mathrm{Con} : \overline{A} \in U\}$;
(3) $T_U = \bigcup \mathrm{Con}_U$; and
(4) $\vdash_U = \vdash \cap (\mathrm{Con}_U \times \mathrm{Con}_U)$

Finally, let $\mathbb{T}_U = (T_U, \mathrm{Con}_U, \beta_U, \vdash_U)$.

Lemma 2.7. *Let $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ be an information system. If U is a nonempty Scott open subset of $\mathtt{Idl}(\mathcal{S})$, then $\mathbb{T}_U$ is a sub partial information system of $\mathcal{S}_P$.*

Proof. Notice that Axioms PS1 and PS3 are satisfied by construction. Consider Axiom PS2. Suppose that $A, X \in \mathrm{Con}_U$ and $X \subseteq A$. We therefore know that $A \vdash X$. Since $(A, X) \in \mathrm{Con}_U \times \mathrm{Con}_U$, it follows that $A \vdash_U X$, as desired.

We turn attention to Axiom PS4. Suppose that $A, B, C \in \mathrm{Con}_U$ and suppose that $A \vdash B$ and $A \vdash C$. Since $\mathcal{S}$ is an information system, we already know that $B \cup C \in \mathrm{Con}$, and we know that $A \vdash (B \cup C)$. Hence, we need only show that $B \cup C \in \mathrm{Con}_U$. To this end, observe that there exist $D \in \beta_U$ such that $B \vdash D$. The fact that $\vdash$ is a preorder therefore implies that $(B \cup C) \vdash D$. Lemma 2.6 therefore implies that $\overline{(B \cup C)} \in U$; consequently, $B \cup C \in \mathrm{Con}_U$.

Consider Axiom PS5. Suppose that $B \in \beta_U$. By construction, $\overline{B} \in U$ and no proper subset C of B is such that $\overline{C} \in B$. Consequently, B does not entail any of its proper subsets under $\vdash_U$.

Axiom PS6 is a consequence of Lemma 2.6.

It follows that $\mathbb{T}_U$ is a partial information system. It remains to prove that $\mathbb{T}_U$ is a sub partial information system of $\mathcal{S}_P$. It is clear that $T_U \subseteq S$, $\mathrm{Con}_U \subseteq \mathrm{Con}$, and $\vdash_U \subseteq \vdash$. Since $\mathcal{S}$ is an information system, we know that $\beta = \{\emptyset\}$. Since $A \vdash \emptyset$ for all $A \in \mathrm{Con}$, it follows that for every $A \in \beta_U$, there exist $B \in \beta$ such that B is a subset of A and $A \vdash B$. Hence, $\mathbb{T}_U \sqsubseteq \mathcal{S}_P$. $\square$

Lemma 2.8. *Let $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ be an information system. If U is a nonempty Scott open subset of $\mathtt{Idl}(\mathcal{S})$, then, then $U_{T_U} = U$.*

Proof. Observe that

$$v \in U \iff (\exists A \in \beta_U)\big(\overline{A} \subseteq v\big) \iff v \in U_{T_U}$$

$\square$

Definition 2.9. Let $\mathbb{S} = (S, \mathrm{Con}, \beta, \vdash)$ be a partial information system and let $\mathbb{T} = (T, \mathrm{Con}_T, \beta_T, \vdash_T)$ be a sub partial information system of $\mathbb{S}$. We say that $\mathbb{T}$ is *saturated* provided the following conditions hold

(1) $A \in \mathrm{Con}_T$ if and only if $A \in \mathrm{Con}$ and $A \vdash_T B$ for some $B \in \beta_T$.

(2) For all $A, B \in \mathrm{Con}_T$, we have $A \vdash B$ if and only if $A \vdash_T B$.

(3) $B \in \beta_T$ if and only if $B \in \mathrm{Con}_T$ and no proper subset of B is in Con_T.

If $\mathcal{S}$ is an information system and U is any nonempty Scott open subset of $\mathtt{Idl}(\mathcal{S})$, then it is easy to see that $\mathbb{T}_U$ is a saturated sub partial information system of $\mathcal{S}_P$.

Lemma 2.10. *Let $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ be an information system. If $\mathbb{T} = (T, \mathrm{Con}_T, \beta_T, \vdash_T)$ is a saturated sub partial information system of $\mathcal{S}_P$, then $\mathbb{T}_{U_T} = \mathbb{T}$.*

Proof. We first prove that $\mathrm{Con}_T = \mathrm{Con}_{U_T}$. Since $\mathbb{T}$ is saturated, we know that

$$\begin{aligned} A \in \mathrm{Con}_T &\iff A \in \mathrm{Con} \,\wedge\, (\exists B \in \beta_T)(A \vdash_T B) \\ &\iff A \in \mathrm{Con} \,\wedge\, (\exists B \in \beta_T)\big(\overline{A} \in {\uparrow}\overline{B}\big) \\ &\iff A \in \mathrm{Con} \,\wedge\, \overline{A} \in U_T \\ &\iff A \in \mathrm{Con}_{U_T}\,. \end{aligned}$$

It now follows that $T = U_T$. We next prove that $\vdash_T = \vdash_{U_T}$. Let $A, B \in \mathrm{Con}_T$. Since $\mathbb{T}$ is saturated, we know that

$$A \vdash_T B \iff A \vdash B \iff A \vdash_{U_T} B.$$

It remains to prove that $\beta_T = \beta_{U_T}$. Since $\mathbb{T}$ is saturated, we know that

$$\begin{aligned} A \in \beta_T &\iff A \in \mathrm{Con}_T \,\wedge\, (\forall B)\big[(B \subsetneqq A) \implies B \notin \mathrm{Con}_T\big] \\ &\iff A \in \mathrm{Con}_T \,\wedge\, \overline{A} \in U_T \,\wedge\, (\forall B)\big[(B \subsetneqq A) \implies \overline{B} \notin U_T\big] \\ &\iff A \in \mathrm{Con}_{U_T} \,\wedge\, (\forall B)\big[(B \subsetneqq A) \implies B \notin \mathrm{Con}_{U_T}\big] \\ &\iff A \in \beta_{U_T}. \end{aligned}$$

□

Let $\mathcal{S}$ be an information system and let $\Sigma^*(\mathtt{Idl}(\mathcal{S}))$ denote the family of all nonempty Scott open subsets of $\mathtt{Idl}(\mathcal{S})$, partially ordered by set inclusion. Since $\mathtt{Idl}(\mathcal{S})$ has least element (namely $\overline{\emptyset}$), it follows that $\Sigma^*(\mathtt{Idl}(\mathcal{S}))$ is a bialgebraic, distributive lattice. In the work to follow, let $\mathtt{SatPInf}(\mathcal{S}_P)$ denote the family of all saturated sub partial information systems of $\mathcal{S}_P$, partially ordered by $\sqsubseteq$ (see Definition 2.2).

THEOREM 2.11. *If* $\mathcal{S} = (S, \mathrm{Con}, \beta, \vdash)$ *is an information system, then* $\Sigma^*(\mathtt{Idl}(\mathcal{S}))$ *is order isomorphic to* $\mathtt{SatPInf}(\mathcal{S}_P)$.

Proof. Consider the mappings

$$\tau\colon \mathtt{SatPInf}(\mathcal{S}_P) \longrightarrow \Sigma^*(\mathtt{Idl}(\mathcal{S})) \qquad \sigma\colon \Sigma^*(\mathtt{Idl}(\mathcal{S})) \longrightarrow \mathtt{SatPInf}(\mathcal{S}_P)$$

defined by $\tau(\mathbb{T}) = U_T$ and $\sigma(U) = \mathbb{T}_U$. In light of Lemmas 2.8 and 2.10, it is clear that τ and σ are inverses of each other. We must prove these maps are order preserving.

To this end, suppose that $U, V \in \Sigma^*(\mathtt{Idl}(\mathcal{S}))$ are such that $U \subseteq V$. It is clear that $T_U \subseteq T_V$, $\mathrm{Con}_U \subseteq \mathrm{Con}_V$, and $\vdash_U \subseteq \vdash_V$. Suppose that $A \in \beta_U$. It follows that $\overline{A} \in V$; and we know that the family $\beta_V[A] \subseteq \beta_V$ (see the proof of Lemma 2.5). Consequently, $A \vdash_V B$ for all $B \in \beta_V[A]$; and we may conclude that $\mathbb{T}_U \sqsubseteq \mathbb{T}_V$.

On the other hand, suppose that $\mathbb{T} \sqsubseteq \mathbb{V}$ in $\mathtt{SatPInf}(\mathcal{S}_P)$. For each $A \in \beta_T$, there exists a subset B of A in β_V such that $A \vdash_V B$. Consequently, $\overline{B} \subseteq \overline{A}$, and it follows that $\overline{A} \in \uparrow\overline{B}$. Hence, $U_T \subseteq U_V$, as desired. □

If $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ is an information system, then Theorem 2.11 tells us that the compact members of $\mathtt{SatPInf}(\mathcal{S}_P)$ are precisely those saturated sub partial information systems of $\mathcal{S}_P$ which have finite frontiers.

Let D be any domain and let $\mathcal{S}_D = (S_D, \mathrm{Con}_D, \vdash_D)$ be the information system induced by D as described in the introduction. As mentioned in the introduction, it is well-known that the Smyth powerdomain $P_S(D)$ of a domain D is dually order isomorphic to the join semilattice of compact members of $\Sigma^*(D)$. Consequently, we have the following result.

COROLLARY 2.12. *If D is any domain, then $P_S(D)$ is dually order isomorphic to the join semilattice of saturated sub partial information systems of $\mathcal{S}_D$ having finite frontiers.*

Proof. We know that D is order isomorphic to $\mathtt{Idl}(\mathcal{S}_D)$. Hence, Theorem 2.11 tells us that $\Sigma^*(D)$ is order isomorphic to $\mathtt{SatPInf}(\mathcal{S}_P)$. The desired result follows at once. □

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