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FOOLPROOF ETERNAL DOMINATION IN THE ALL-GUARDS MOVE MODEL

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ABSTRACT. The eternal domination problem requires a graph be protected against an infinitely long sequence of attacks at vertices, by guards located at vertices, with the requirement that the configuration of guards induces a dominating set at all times. An attack is defended by sending a guard from a neighboring vertex to the attacked vertex. We allow all guards to move to neighboring vertices in response to an attack, but allow the attacked vertex to choose which neighboring guard moves to the attacked vertex. This is the all-guards move version of the "foolproof" eternal domination problem that has been previously studied. We present some results and conjectures on this problem.

1. Introduction

Let $G = (V, E)$ be a simple, finite graph with n vertices. Several recent papers have considered problems associated with using mobile guards to defend G against an infinite sequence of attacks; see for instance [1, 3, 6, 9, 10]. Most of these papers consider attacks at vertices, while [5, 11, 12] consider the variation in which attacks are at edges. In this paper, we consider a variation on the vertex protection problem which was initially motivated by a desire to compare the vertex and edge protection parameters. This variation is analogous to the "foolproof" eternal domination problem considered, and characterized completely[1] in [3], but in this paper we allow all guards to move in response to an attack rather than just one.

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[1]Exactly $n - \delta$ guards are needed to protect every connected graph in the one-guard moves foolproof model.

Denote the open and closed neighborhoods of a vertex $x \in V$ by $N(x)$ and $N[x]$, respectively. That is, $N(x) = \{v : xv \in E\}$ and $N[x] = N(x) \cup \{x\}$. A *dominating set* of G is a set $D \subseteq V$ such that for each $u \in V - D$, there exists $x \in D$ adjacent to u. The minimum cardinality amongst all dominating sets is the *domination number* $\gamma(G)$.

A *vertex cover* of G is a set $C \subseteq V$ such that for each edge $uv \in E$ at least one of u, v is in C. Let $\alpha(G)$ denote the *vertex cover number* of G, the minimum number of vertices in any vertex cover of G.

An *independent set* of G is a set $I \subseteq V$ with the property that no two vertices in I are adjacent. The maximum cardinality amongst all independent sets is the *independence number*, $\beta(G)$. For all connected graphs G, it is well-known that $n - \beta(G) = \alpha(G)$.

Let $D_i \subset V$, $1 \leq i$, be a set of vertices with one guard located on each vertex of D_i. Throughout the paper, at most one guard can be located on any vertex. The set D_i is sometimes called a *configuration* of guards. If a vertex has a guard on it, we sometimes say it is *occupied*, otherwise it is *unoccupied*. If a vertex is occupied or adjacent to an occupied vertex, we say it is *protected*.

Each of the problems we consider in this paper can be modeled as a two-player game between an *attacker* and a *defender*. The defender chooses D_1 as well as D_i, $i > 1$, while the attacker chooses the locations of the attacks $r_1, r_2, \ldots$. Note that the location of an attack can be chosen by the attacker depending on the location of the guards. Each attack is handled by the defender by choosing the next D_i subject to certain constraints. The defender wins the game if he can successfully defend any series of attacks, subject to the constraints of the game; the attacker wins otherwise.

In the *eternal dominating set problem*, D_i, $1 \leq i$, is required to be a dominating set, $r_i \in V$ (assume w.l.o.g. $r_i \notin D_i$), and D_{i+1} is obtained from D_i by moving one guard to r_i from a vertex $v \in D_i$, $v \in N(r_i)$. The size of a smallest eternal dominating set for G is denoted $\gamma^\infty(G)$. Eternal dominating sets have also been called *eternal secure sets* in the literature.

In the *m-eternal dominating set problem*, D_i, $1 \leq i$, is required to be a dominating set, $r_i \in V$ (assume w.l.o.g. $r_i \notin D_i$), and D_{i+1} is obtained from D_i by moving guards to neighboring vertices. That is, each guard in D_i may move to an adjacent vertex. It is required that $r_i \in D_{i+1}$. The size of a smallest m-eternal dominating set for G is denoted $\gamma_m^\infty(G)$.

In the *m-eternal vertex cover problem*, D_i, $1 \leq i$, is required to be a vertex cover, $r_i \in E$, and D_{i+1} is obtained from D_i by moving guards to neighboring vertices. That is, each guard in D_i may move to an adjacent vertex. It is required that in moving from D_i to D_{i+1} that at least one guard move across edge r_i. The size of a smallest m-eternal vertex cover for G is denoted $\alpha_m^\infty(G)$.

We now introduce the *m-eternal vertex protection problem.* The problem is the same as the m-eternal dominating set problem in that attacks are at (unoccupied) vertices and all guards can move in response to an attack, but in this case the attacker chooses the vertex v to be attacked and which of the guards from the neighborhood of v will move to v. One can imagine there being a victim of the attack at v and allowing the victim to choose which guard to send to its defense. For example, when a site is attacked it may want to choose which of the nearby defenders it calls in, perhaps because of particular expertise in defending certain types of attack; for defense from fire, a site with firefighting expertise could be called, whereas defense from gunfire could require that a site with a tactical unit be called.

The size of a smallest m-eternal protection set for G is denoted $\rho_m^\infty(G)$. Our objective in the paper is to compare the eternal protection number with other graph parameters.

2. Preliminaries

For a set of vertices X, the

$$\left.\begin{array}{l}\textit{private neighborhood}\ \mathrm{pn}(x,X)\\ \textit{external private neighborhood}\ \mathrm{epn}(x,X)\end{array}\right\}\ \text{of } x \in X \text{ relative to } X$$

is defined by

$$\mathrm{pn}(x,X) = N[x] - N[X - \{x\}]$$
$$\mathrm{epn}(x,X) = \mathrm{pn}(x,X) - \{x\}$$

and the vertices in these sets are called, respectively, the

$$\left.\begin{array}{l}\textit{private neighbors}\\ \textit{external private neighbors}\end{array}\right\}\ \text{of } x \text{ relative to } X.$$

Throughout the paper, let $\theta(G)$ denote the clique-covering number of G, i.e., $\theta(G) = \chi(\overline{G})$.

3. Bounds on eternal protection number

3.1. Basic bounds

Given m guards $g_1, g_2, \ldots, g_m$, a *guard configuration* is a set $\{p_1, p_2, \ldots, p_m\}$ of vertices of G such that guard g_i is located at vertex p_i, $1 \le i \le m$. For guard configurations $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_m\}$, we say there is a *guard realignment* from A to B if there is a bijection $\pi\colon A \to B$ such that

$a,\pi(a) \in E$ whenever $\pi(a) \neq a$. We then obtain the observation that $\rho_m^\infty(G) = k$ if and only if there exists a collection $\mathcal{D}$ of guard configurations of size k such that given $A \in D$ and $1 \leq i \leq m$, for any vertex $x \in N(a_i)$ there exists a guard configuration B, with $b_i = x$, such that there is a guard realignment from A to B. That is, if and only if given a guard configuration A and a vertex u adjacent to a vertex v holding a guard, there is a guard configuration B for which $\pi(v) = u$ for some guard realignment bijection $\pi\colon A \to B$.

Theorem 1.

(i) $\rho_m^\infty(P_n) = \lceil \frac{n}{2} \rceil$

(ii) $\alpha_m^\infty(P_n) = n - 1$

(iii) $\rho_m^\infty(C_n) = \lceil \frac{n}{3} \rceil$

(iv) $\alpha_m^\infty(C_n) = \lceil \frac{n}{2} \rceil$

Proof. Parts (ii) and (iv) are from [11] and part (iii) is obvious. For part (i) we can partition the path from left to right into P_4's, with up to three vertices left over and keep two guards on each P_4 and either one guard on the remaining P_1 or P_2 or two guards on the remaining P_3. The idea is to ensure that the attacker only has one guard to choose to defend each attack. On P_4, this can be done by always keeping the to guards either on the two vertices of degree two or on the two vertices of degree one. On P_8, for example, we partition into two P_4's and keep the guards in the leftmost P_4 on the pair of vertices as we do the rightmost P_4, thus there is never a vertex that does not have a guard that is adjacent to two vertices with guards. □

The following inequality chain is from [6]

$$\gamma(G) \leq \gamma_m^\infty(G) \leq \beta(G) \leq \gamma^\infty(G) \leq \theta(G).$$

It was shown in [9] that

$$\gamma^\infty(G) \leq \binom{\beta(G)+1}{2}.$$

Theorem 2. *For any graph $G = (V, E)$,*

$$\rho_m^\infty(G) \leq 2\binom{\gamma(G)}{2} + 2\gamma(G).$$

Proof. Assume without loss of generality that G has no isolated vertices. Fix a minimum dominating set $D = \{v_1, v_2, \ldots v_\gamma\}$ such that $\operatorname{epn}(v, D) \neq \emptyset$ for each $v \in D$. Such a set exists for all graphs without isolated vertices, see [2].

For each $v \in D$, define $N_v \subseteq N[v]$ in such a way that the set $\{N_v : v \in D\}$ is a partition of V. Call each N_v a *cluster* and note that $v \in N_v$ for all clusters. By the property above we are ensured that $|N_v| \geq 2$ for all $v \in D$. Further, the induced subgraph $\langle N_v \rangle$ has radius one for all $v \in D$. For cluster N_v, call v the *dominating vertex* and the other vertices in the cluster *fringe* vertices.

Given dominating set D of G and vertex partition $\mathcal{N}$ as described above, define the *dominating set graph* for D with respect to $\mathcal{N}$ to be the graph with vertex set equal to the set $\{N_v : v \in D\}$ and V_1V_2 in the edge set if and only if a vertex in V_1 is adjacent to a vertex in V_2. Initially put a guard on each end of an edge in the dominating set graph (that is, put a guard on a vertex in V_1, preferably on a vertex adjacent to a vertex in V_2). Also place a guard on v for each $v \in D$, plus place one additional guard, called the *extra guard*, in V_i if V_i contains a vertex not adjacent to a vertex in any other V_k.

Our objective is to always keep a guard on v, for each $v \in D$, as well as to keep guards on each end of the edges V_1V_2 of the dominating set graph, as described above. If a guard is pulled out of its cluster in G by the attacker, then the one guard on the other end of the edge (of the form V_1V_2) can move into the first cluster. If necessary, we move a guard from a fringe vertex to the dominating vertex in a cluster.

On the other hand, suppose an attack within cluster V_1 is defended by a guard from within the cluster. There are two possibilities. If the guard at x covers edge V_1V_2 and is moved to a vertex adjacent to cluster Y, then the guard covering Y moves to the dominating vertex of V_1 and the guard at the dominating vertex of V_1 moves to x. Secondly, if the guard at x covering V_1V_2 moves to a vertex that is not adjacent to another cluster, then we move the guard at the dominating vertex to x and the extra guard to the dominating vertex. □

A natural question is the following.

Question 1. *For which graphs G is $\rho_m^\infty(G) = \gamma(G)$?*

We give an example class of graphs for which this bound is nearly obtained.

Theorem 3. *Let G be a connected split graph. Then $\rho_m^\infty(G) \leq \gamma(G) + 1$.*

Proof. The invariant to maintain is keeping $\gamma(G)$ guards on the vertices of a dominating set located in the clique and one guard located in the independent set or elsewhere in the clique. If a guard gets moved then it is easy for the remaining $\gamma(G)$ guards to reconfigure. □

To see that $\gamma(G)$ guards do not suffice for all split graphs G, consider a split graph G in which every vertex of the clique has at least two external private neighbors (for example, stars with at least two leaves or stars with at least three leaves and least two pendant vertices attached to one of the leaves). If a guard gets relocated to one of its external private neighbors, say x, then γ guards are still required to dominate the rest of the graph. That is, $\gamma(G - x) = \gamma(G)$.

It is then easy to see that for connected split graph $G = (V, E)$, $\rho_m^\infty(G) = \gamma(G)$ if and only if every vertex of G is in a minimum dominating set. An alternate characterization is as follows:

Let $V = C \cup I$ where C is a clique and I is an independent set. Every vertex of G is in a minimum dominating set if and only if for each vertex $v \in I$, there exists a set of $\gamma(G) - 1$ vertices in C that dominate $I \setminus \{v\}$ and for each $u \in C$, there exists a set of $\gamma(G) - 1$ vertices in C that dominate $I \setminus \{N(u)\}$.

However, we would like a more structural characterization of the split graphs G having $\rho_m^\infty(G) = \gamma(G)$.

Lemma 4. *Let $G = (V, E)$ be a connected split graph with $V = C \cup I$, where C is a clique and I an independent set, such that $\rho_m^\infty(G) = \gamma(G)$. Then $\gamma(G) = \beta(G)$ and $\beta(G) = |I|$.*

Proof. We can assume $\gamma(G) > 1$, else the proof is obvious. Assume every vertex of G is in a minimum dominating set and fix a minimum dominating set D such that $D \subseteq C$. It is easy to see that $|\operatorname{epn}(v, D)| \leq 1$ for each $v \in D$, else there is a vertex not in a minimum dominating set of G. Observe that each vertex $u \in C$ has at least one neighbor in I, else u is not in a minimum dominating set of G. It follows that $|I| = \beta(G)$, since any independent set can contain at most one vertex in C and each vertex in C has at least one neighbor in I.

Partition C into $P \cup N$, where $P \subseteq C$ is the set of "clique" vertices x with a pendant vertex $p_x \in I$, and $N = C - P$. Suppose first that $P \neq \emptyset$. We know no vertex in P can have two pendants in I because, in that case, $\rho_m^\infty(G) > \gamma(G)$. Let $a \in P$. The attacker can force a guard to locate a guard at p_a. Suppose the guards have reconfigured after this happens, and there is a guard in C. Then the attacker can then force a guard to locate this guard at a. Irrespective of whether the guard that was at p_a is at a or p_a, the graph is be dominated by the vertices where the remaining $\gamma(G) - 1$ guards are located, a contradiction. Therefore there is no guard in C and I is a dominating set of size $\gamma(G)$. □

Theorem 5. *Let $G = (V, E)$ be a connected split graph with vertex partition $I \cup C$, where I is a maximum independent set. Then $\rho_m^\infty(G) = \gamma(G)$ if and only if every vertex in C has a neighbor in I, and no two vertices in I have a common neighbor in C.*

Proof. It is easy to check that a graph satisfying the conditions of the theorem has $\gamma(G) = \beta(G)$, see Lemma 4. Furthermore, the strategy for the guards is also easy: either all guards are in I, or all are in C. Suitable reconfigurations are always possible. Suppose $\rho_m^\infty(G) = \gamma(G)$. Since I is a maximum independent set, every vertex in C has a neighbor in I. Since I is a minimum dominating set, no two vertices in I have a common neighbor in C: if $i_1, i_2 \in I$ have a common neighbor $x \in C$, then $(I - \{i_1, i_2\}) \cup \{x\}$ is a dominating set of size less than $\gamma(G)$, a contradiction. □

The following is easy to prove, though it gives a good bound only for certain graphs: in some sense those graphs that resemble split graphs with large independent sets.

Proposition 6. *For a graph $G = (V, E)$ with n vertices, no isolated vertices and maximum independent set I. If $G[V - I]$ has c components, then*

$$\rho_m^\infty(G) \le n - \beta(G) + c.$$

3.2. Edge protection is an upper bound

From [11] we have the following inequality chain

$$\gamma(G) \le \alpha(G) \le \alpha_m^\infty(G) \le 2\alpha(G).$$

Proposition 7. *For any graph G,*

$$\gamma_m^\infty(G) \le \rho_m^\infty(G) \le \alpha_m^\infty(G).$$

P r o o f. The leftmost inequality is obvious. For the rightmost inequality, observe that in the m-eternal vertex cover problem, when an attack occurs on an edge with guards on either end, the two guards can swap places and no other guards need to move; hence there is no net change in the guard configuration. If there is only one guard incident to attacked edge uv, that guard must move across the edge, say from u to v, to defend the attack. Hence it is as if the attacker chose the guard that will defend the attack. Now rather than having attacks at edges, imagine the attack was at v and the attacker chose the guard at u to defend it. It follows that $\rho_m^\infty(G) \le \alpha_m^\infty(G)$. □

There are many graphs having $\rho_m^\infty(G) = \alpha_m^\infty(G)$, P_2 and P_3 being two small examples.

Problem 1. *Describe classes of graphs having $\rho_m^\infty(G) = \alpha_m^\infty(G)$ for all graphs G in the class.*

There are also many graphs having $\rho_m^\infty(G) < \alpha_m^\infty(G)$, K_n, $n > 2$ being the most obvious example.

4. Independence and clique covering numbers

Theorem 1 gives two families of graphs with $\rho_m^\infty(G) = \beta(G)$. It is easy to see that $\rho_m^\infty(K_{n,m}) = 2$, which is generally less than the independence number or the vertex cover number of $K_{m,n}$.

It is interesting to note that complements of Kneser graphs (a.k.a. Johnson graphs) were used in [7] to show there exist graphs G for which

$$\gamma^\infty(G) = \binom{\beta(G)+1}{2}.$$

Let $G(n,k)$ be the graphs whose vertices are k-element subsets of an n-set with different k-sets X and Y being adjacent if $X \cap Y \neq \emptyset$.

Theorem 8. $\rho_m^\infty(G(n,k)) = \gamma(G(n,k)) = \beta(G(n,k))$.

Proof. It is easy to see that

$$\beta(G(n,k)) = \left\lfloor \frac{n}{k} \right\rfloor \quad \text{and} \quad \gamma(G(n,k)) = \beta(G(n,k)).$$

Assume $\beta(G(n,k)) > 1$. Place guards on the vertices of an independent set. Due to the symmetry in the graph, when an attack occurs, it is a simple matter to move guards to another independent set by moving the guard chosen by the attacker and all guards on vertices whose labels share an element with the attacked vertex (so that the labels of the new locations of these guards cover the elements in the labels of the guards who must move) □

The *Kneser graph* $K(n,k)$ is the complement of $G(n,k)$. The independence number $\beta(K(n,k)) = \binom{n-1}{k-1}$. It is easy to see that

$$\rho_m^\infty(K(n,k)) \leq \beta(K(n,k)).$$

As an illustration, when $k = 2$, start with vertices on a independent set all sharing "1" as an element in their label. When an attack occurs at a vertex with, say, a "3" in its label, we can move guards to an independent set consisting of vertices all having a "3" in their label. As an exact bound on the domination number of Kneser graphs is not known, except in special cases (e.g., $K(n,2) = 3$, if $n \geq 4$ and also one can show that $\gamma_m^\infty(K(n,2)) = 3$ when $n \geq 4$ [13]), it seems difficult to get a good lower bound on the eternal protection number of Kneser graphs.

Question 2. *Is $\rho_m^\infty(K(n,k)) = \beta(K(n,k))$?*

Problem 2. *Determine the value of $\gamma_m^\infty(G(n,k))$.*

To motivate the next question, note that for the vertex-transitive Petersen graph P, one can show that $\gamma(P) = 3$ and $\rho_m^\infty(P) = 4 = \beta(P)$. To see this, number the vertices around outer cycle as 0, 1, 2, 3, 4 with inner cycle numbered as 5, 6, 7, 8, 9 where 5 is adjacent to 0, 6 is adjacent to 1, and so forth. Put the guards on 4, 6, 7, a dominating set. Attack on 9 and pull the guard from 6 to defend. Now 1 is unprotected. We can protect it in either of the two following ways:

a) Send 4 to 0, but then to protect 3 we have to send 7 to 2 and 8 is unprotected.

b) Send 7 to 2 (note that 8 and 5 are now unprotected), but then we cannot move 4, else either 0 or 8 will unprotected.

Problem 3. *Describe classes of graphs having $\rho_m^\infty(G) \le \beta(G)$ for all graphs G in the class. Are vertex transitive graphs such a class?*

We note that the simple proof from [6] applies in this context to show that $\rho_m^\infty(G) = \gamma(G)$ for all Cayley graphs G.

Theorem 9. *Let G be a connected bipartite graph. Then $\rho_m^\infty(G) \le \beta(G)$.*

Proof. Let the bipartition of G be A, B and assume without loss of generality that $|A| \ge |B|$. Consider independent set (and dominating set) A. If no vertex in A has any external private neighbors, then we can initially place guards on each vertex of A and move one guard from A to B to defend an attack in B and maintain the invariant that there is at most one guard in B.

Now suppose that at least one vertex in A has an external private neighbor. Note that if $z \in \mathrm{epn}(v, A)$, then z has degree one. Partition A into two sets A_1, A_2 where each vertex in A_1 has at least one external private neighbor and each vertex in A_2 has no external private neighbors. Partition B into three sets B_1, B_2, B_3 such that B_1 consists of degree one vertices, B_2 consists of vertices of degree greater than one that are only adjacent to vertices in A_1 and $B_3 = B \setminus (B_1 \cup B_2)$. It is easy to see that $|A_2| \ge |B_3|$ and that $|B_1 \cup B_2| \ge |A_1|$.

Initially locate a guard on each vertex of the independent set $B_1 \cup B_2 \cup A_2$. Our defense strategy is as follows. If there is an attack in A_1, move guards from $B_1 \cup B_2$ (if possible only from B_1, see $(*)$ below) so that each vertex in A_1 has a guard and move guards from A_2 to B_3 so that each vertex in B_3 has a guard. Note that if an attack in A_1 is defended by a guard from B_2, then each external private neighbor of v is occupied after the attack is defended.

$(*)$ On the other hand, if an attack in A_1 is defended by a guard from B_1, then it is simple to move guards only from B_1 so as to occupy each vertex in A_1.

From our initial configuration of guards, if there is an attack in B_3, make the same move of guards as if there was an attack in A_1, so that each vertex in $A_1 \cup B_3$ has a guard; call this configuration of guards the *secondary configuration.* Now from the secondary configuration of guards, there may be at attack in either B_1, B_2 or A_2. In any of the cases, we can return to the initial configuration of guards. This process can be repeated indefinitely. □

Theorem 10. *For any graph G, $\rho_m^\infty(G) \leq 2\theta(G)$.*

Proof. Assume without loss of generality that G is connected. Let $C_1, C_2, \dots, C_\theta$ be a minimum clique cover of G. Construct an auxiliary graph, $C(G)$, called the *clique graph* of G with vertices corresponding to cliques from the minimum clique cover and two vertices u, v in $C(G)$ adjacent if the corresponding cliques in G are joined by at least one edge.

Initially place two guards in clique C_i if C_i is a clique of size at least two and one guard in C_i otherwise. We shall maintain a *guard invariant* such that if, at some point in time, clique C_i has one less than its initial number of guards, then there is a vertex u in clique C_j with a guard such that u is adjacent to at least one vertex in C_i. We say C_i is *assisted by* C_j. In addition, no clique shall ever have two less than its initial number of guards.

The aforementioned guard invariant will be maintained as follows. Our objective is always to return each clique to its initial number of guards. We shall dynamically designate or un-designate some of the edges in $C(G)$ as *red edges* to help us maintain the invariant. By un-designate, we simply mean return the red edge to its normal, non-red, status.

If a guard moves from vertex $v \in C_i$ to C_j (thereby reducing the number of guards in C_i) and there is no red edge between C_i and C_j, designate a red edge between C_i and C_j in $C(G)$. Note that C_i is now assisted by C_j. Furthermore, it may be necessary to move a guard in C_i to v if v was assisting another clique.

On the other hand, if there were a red edge already between C_i and C_j and a guard moves from C_i to C_j, then un-designate the red edge. Now it is easy to see that at any time the red edges induce a path and thus at most one vertex within any clique C_x is assisting any other clique. Note that a clique of size one can only ever be an end-vertex on a path of red edges, hence it never needs to assist any other clique.

Whenever possible when an attack occurs, move guards along red edges (in the reverse direction of their original movement) to eliminate as many such edges as possible. In this manner maintain a configuration of guards as close to the initial configuration as possible. □

We do not know if the bound in the previous theorem is sharp. The next result shows that there exist graphs with m-eternal protection number greater than the clique covering number (and thus also greater than the independence number).

Theorem 11. *There exists a graph G with $\rho_m^\infty(G) \geq \frac{3}{2}\theta(G)$.*

Proof. We construct G as follows; it will be easy to see that $\theta(G) = 2$. Let us begin by taking two K_4's: clique X with vertices a, b, c, d and clique Y with vertices e, f, g, h. Now add the following edges: connect a to f, b to e, d to g, and b to h.

Observe that two guards in X cannot dominate the vertices in Y, nor can two guards in Y dominate the vertices in X. So we need at least one guard in X and at least one guard in Y at all times. Assume w.l.o.g., a guard is on vertex c, since there could have just been an attack there. Have the next attack on either a, b, or d, whichever has a neighbor in Y with a guard. Then force that guard from Y to move to X and note that the guard on c cannot re-locate to Y. The resulting configuration of guards is not a dominating set. Hence at least three guards are needed to protect G. □

Theorem 11 implies that $\rho_m^\infty(G)$ and $\gamma^\infty(G)$ are, in general, not comparable, since $\gamma^\infty(G) \leq \theta(G)$ and yet $\rho_m^\infty(G) < \gamma^\infty(G)$ for many graphs. $K_{1,m}$ is perhaps the most extreme example of the latter inequality.

Problem 4. *Characterize the graphs G with $\beta(G) = 2 = \rho_m^\infty(G)$.*

Theorem 12. *Let $G = (V, E)$ be a connected graph with $\beta(G) = 2$. Then*

$$\rho_m^\infty(G) \leq 3.$$

Proof. Let $X = \{u, v\}$ be a maximum independent set, let $W = N(u) \cap N(v)$, let $C_1 = \text{epn}(u, X)$, let $C_2 = \text{epn}(v, X)$. Note that C_1 and C_2 are cliques. There are two cases to consider.

Case 1. Suppose $W = \emptyset$.
This case is easy, as our invariant is that we always have at least one guard in the clique $C_1 \cup \{u\}$ and at least one guard in the clique $C_2 \cup \{v\}$. To do this, consider the clique with two guards. Simply ensure that one of these two guards is on a vertex having a neighbor in the other clique, such a vertex exists since G is connected.

Case 2. Suppose $W \neq \emptyset$.
Start with guards on u, v and some $w \in W$. Call this our *base* configuration. For an attack at another vertex in w we can easily maintain the base configuration. Suppose there is an attack in C_1 or C_2, say at $y \in C_1$. Then we can put the guards into a configuration with a guard at y, a guard in W, and a guard at v. From this *primary* type configuration, given any attack we will either be able to return to a base configuration or return to a comparable primary configuration unless (a) y has no neighbors in W and the next attack is in $C_1 \cup \{u\}$ and must be defended by the guard in W or (b) the guard at y is forced to defend an attack in C_2. In either case (a) or (b), we can move to a *secondary* configuration with a guard on u, a guard on v and a guard in either C_1 or C_2. From the secondary configuration, we can easily defend the next attack and re-configure into either a base, primary, or secondary configuration. □

Question 3.

(a) *Is* $\rho_m^\infty(G) \le c * \gamma(G)$ *for some constant* c*?*

(b) *Is* $\rho_m^\infty(G) \le c * \beta(G)$ *for some constant* c*?*

(c) *Is* $\rho_m^\infty(G) \le c * \gamma_m^\infty(G)$ *for some constant* c*?*

We suspect the answers to (a) and (b) are negative in general.

5. Further results

Theorem 13. *Let* T *be a tree. Then* $\rho_m^\infty(T) = \gamma_m^\infty(T)$.

Proof. The proof is identical to the proof from [10] that characterizes γ_m^∞ for trees, with one modification. We need to ensure that there does not exist a vertex v without a guard such that two vertices adjacent to v contain guards. Such a situation would allow the attacker to move the guards into a configuration that is not a dominating set. The inductive algorithm from [10] is easily adapted to behave like this. □

For many graphs such as $K_n, C_n, K_{m,n}$, we have that $\rho_m^\infty(G) = \gamma_m^\infty(G)$. We describe a graph with $\rho_m^\infty(G) > \gamma_m^\infty(G)$. Take C_6 and label the vertices around the cycle as $v_1, v_2, \ldots, v_6$. Add an edge between v_2 and v_4 and call this graph G. It is easy to see that $\gamma_m^\infty(G) = 2$. We claim that $\rho_m^\infty(G) = 3$. Suppose we could protect G (in the m-eternal vertex protection problem) with two guards. At some point the attacker can force a guard onto v_1 which means the second guard must be on v_4. Now the attacker attacks v_2 and forces the guard at v_4 to defend in which case the defender must move its other guard to v_6 to maintain a dominating set. The attacker next attacks v_1 and forces the guard at v_2 to defend, and no move by the defender can result in a guard being in the closed neighborhood of v_3. Hence $\rho_m^\infty(G) > 2$.

We sometimes call the Cartesian product $P_m \times P_n$ the $m \times n$ grid graph. In [8], it was shown that for any $n \ge 2$, $\gamma_m^\infty(P_2 \times P_n) = \lceil \frac{2n}{3} \rceil$.

Theorem 14. $\rho_m^\infty(P_2 \times P_n) = \lceil \frac{2n}{3} \rceil$.

Proof. It must be that $\rho_m^\infty(P_2 \times P_n) \ge \lceil \frac{2n}{3} \rceil$, since $\gamma_m^\infty(P_2 \times P_n) = \lceil \frac{2n}{3} \rceil$.

Let the vertices on the first row be $v_1, v_2, \ldots, v_n$ (from left to right) and on the second row $v_{n+1}, \ldots, v_{2n}$ from right to left. We refer to $v_1, v_2, \ldots, v_{2n}, v_1$ as the cycle of length $2n$. Assume for simplicity that $3|2n$, else the argument proceeds in a similar fashion.

We shall maintain a guard on every third vertex around the cycle of length $2n$. The preferred way of defending an attack at an unoccupied vertex v is to rotate all the guards around the cycle of length $2n$ either clockwise or counterclockwise, as needed. It may be the case that two vertices v_i, v_j with guards have a common neighbor and if the attacker attacks that vertex, they can request the attack be defended by the guard that is not the immediate clockwise (counterclockwise neighbor) of that vertex. However, in this case v_i and v_j must have a second common neighbor and by moving the the guards at v_i and v_j to both these common neighbors, the remaining guards can rotate in a clockwise (counterclockwise) direction and maintain the property that a guard is on every third vertex on the cycle of length $2n$. □

In general however, the exact value of $\gamma_m^\infty(P_m \times P_n)$ is not known. Hence the following may be difficult.

Question 4. *For all grid graphs G, is $\rho_m^\infty(G) = \gamma_m^\infty(G)$?*

It is easy to see that $\gamma^\infty(K_{1,m}) = m$ and $\rho_m^\infty(K_{1,m}) = m$, for $m > 0$. On the other hand, there are many graphs where equality holds, such as K_n, C_4 and P_3.

In [4], it was proved that $\gamma_m^\infty(G) \leq \lceil \frac{n}{2} \rceil$ for all graphs G with no isolated vertices.

Theorem 15. *For graph $G = (V, E)$ with n vertices and a cycle of length n, $\rho_m^\infty(G) \leq \lceil \frac{n}{2} \rceil$.*

Proof. Assume n is even, else the proof follows in a similar fashion. Fix a Hamiltonian cycle $C = v_0, v_2, \ldots, v_{n-1}, v_0$. Initially place a guard on $v_0, v_2, v_4, \ldots, v_{n-2}$. If an attack at v_i is defended by a guard at v_{i-1} or v_{i+1} (subscripts modulo n), then we simply rotate all the guards one position, clockwise or counterclockwise, as necessary.

On the other hand, an attack at v_i might be defended by pulling a guard across a chord of C. Note that the chord divides C into two cycles (each sharing edge C). Rotate guards as necessary around each of these two cycles to maintain the desired positioning on every other vertex. □

Corollary 16. *Let G be an outerplanar graph with n vertices and no isolated vertices. Then $\rho_m^\infty(G) \leq \lceil \frac{n}{2} \rceil$.*

Proof. Since each 2-connected outerplanar graph has a Hamiltonian cycle, we can assume the graph has a cut-vertex v. Note that the bound applies to trees due to Theorem 13.

The proof proceeds by induction by "cutting" the graph into two subgraphs G_1 and G_2 at v. We assume without loss of generality that v "cuts" G into two subgraphs, the argument is similar if it is more than two. If there are a guard at v, we shall assume it belongs to either the guard configuration of G_1 or the guard configuration of G_2. We shall generally attempt to maintain the invariant that if there is a guard at v (and the guard belongs to G_1), then there is a guard at a neighboring vertex to v in G_2.

To be precise, we consider two cases. Assume for now that we consider v to be a vertex in both G_1 and G_2. The first case assumes at least one of G_1, G_2 has an even number of vertices. Assume G_1 contains $\lceil \frac{|V(G_1)|}{2} \rceil$ guards and G_2 contains $\lceil \frac{|V(G_2)|}{2} \rceil$ guards. If possible we move guards in G_1 and G_2 independently. The only problem occurs when there is a guard, say from the guards assigned to G_1 on v. Since we have operated G_1 and G_2 independently (up until now), there must be a guard in G_2 on a vertex neighboring v. Now if the attacker pulls the guard from v into G_2, we can move the guard from the neighboring vertex onto v.

On the other hand, suppose both of G_1, G_2 have an odd number of vertices (assuming v is a vertex in both subgraphs). Now we cannot proceed as above as this would require an extra guard (because of the ceiling function). Then let us assume initially that $v \in V(G_1), v \notin V(G_2)$ and operate the guards independently in G_1 and G_2. If a guard in G_1 is on v and gets pulled into G_2 then we "swap" v to be in G_2 rather than G_1 and adjust the remaining guards in G_1 accordingly. □

Theorem 17. *Let G be a graph with n vertices and no isolated vertices. Then*

$$\rho_m^\infty(G) \le \left\lfloor \frac{5n}{6} \right\rfloor.$$

Proof. Construct the clique graph $C(G)$ as in the proof of Theorem 10. Associate cliques of size one with a neighboring clique of size greater than one. We can assume that we never have a pendant vertex in $C(G)$ that is a clique of size two whose neighboring vertex is a clique of size one, else we could move a vertex from one clique to the other.

Place guards in the cliques as follows. Place two guards in each clique of size greater than two (as in Theorem 10. Place one guard in each clique of size two (at a pendant vertex in the clique if there is one). Place a guard in each clique of size one with one exception: if there is more than one such clique adjacent to the same vertex, x, in which case we place a guard on just one of the cliques of size one and a guard on x (the guard on x may be in addition to guards already in that clique). For example, consider a K_3 on vertices x, y, z with two vertices u, v adjacent to x. Then we place guard on u, x and leave v unoccupied. In addition we make sure there are guards on two of x, y, z: this can be done for K_3 by placing only one additional guard).

The number of guards in this configuration is at most $\lfloor \frac{5n}{6} \rfloor$, as the worst case is a K_3 with a pendant vertex attached to each vertex of the K_3.

Now proceed as in the proof of Theorem 10, taking special care with cliques of size two. Let ab be a clique of size two in G with a guard on b. Suppose a guard is pulled from this clique to a neighboring clique C_i, so C_i is now assisting ab. However, it may be now that vertex a is unprotected. If vertex a is a pendant vertex, then there we will not leave it unprotected since we initially placed guards on the pendant vertices of such cliques and because in such cases we require that there always be a guard in a neighboring clique adjacent to b, if a is a pendant vertex and the guard initially located there has been pulled to b.

On the other hand, if a is not a pendant vertex, when a guard is pulled from b to another clique, if necessary, move a guard in a clique neighboring a to a vertex a' adjacent to a to protect a. If an attack occurs at a, we can move the guard from a' to a and add a red edge between these two cliques and simultaneously eliminating the red-edge path with endpoint b. □

Conjecture 5. *For a graph $G = (V, E)$ with n vertices and no isolated vertices,*

$$\rho_m^\infty(G) \leq \left\lceil \frac{n}{2} \right\rceil.$$

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