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ON PROPERTIES OF G-EXPANSIVE HOMEOMORPHISMS

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ABSTRACT. We show that LUB of the set of G-expansive constants for a G-expansive homeomorphism h on a compact metric G-space, G compact, is not a G-expansive constant for h. We obtain a result regarding projecting and lifting of G-expansive homeomorphisms having interesting applications. We also prove that the G-expansiveness is a dynamical property for homeomorphisms on compact metric G-spaces and study G-periodic points.

1. Introduction

One of the important dynamical properties studied for dynamical systems is expansiveness which was introduced by Utz [10] in 1950 for homeomorphisms on metric spaces. Expansive homeomorphisms have lots of applications in Topological Dynamics, Ergodic theory, Continuum Theory, Symbolic dynamics etc. With the intention of studying various dynamical properties of maps under the continuous action of a topological group, the notion of expansiveness termed as G-expansive homeomorphism was defined for a self-homeomorphism on a metric G-space [5]. It is observed that the notion of expansiveness and the notion of G-expansiveness under a non trivial action of G are independent of each other. Conditions under which an expansive homeomorphism on a metric G-space is G-expansive and vice-versa are also obtained. The same paper also contains a characterization of G-expansive homeomorphism and condition under which it can be extended from a subspace to the whole space. In the process of above

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study a useful notion of pseudoequivariant map is defined. Pseudoequivariancy is a weaker notion than the usual notion of equivariancy of a map [5] and gives a natural passage to orbit spaces. Recently Choi and Kim in [4] have used this concept to generalize topological decomposition theorem proved in [1] due to Aoki for compact metric G-spaces. Some interesting properties of such maps are studied in [8]. In [7], using this concept, authors have obtained a necessary and sufficient condition for the extension of G-expansive homeomorphisms. Further, in [6] the notion of generators in G-spaces termed as G-generator is defined and a characterization of G-expansive homeomorphisms is obtained using G-generator. Some interesting consequences have been obtained for example it is shown that an arc can not admit a pseudoequivariant G-expansive homeomorphism. It is well known that an arc does not admit an expansive homeomorphism. It is worth noting here that other dynamical properties like shadowing and transitivity of maps on G-spaces are also studied [4].

Throughout $H(X)$ denotes the collection of all self-homeomorphisms of a topological space X, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Z}$ denotes the set of integers and $\mathbb{N}$ denotes the set of positive integers. By a G-space we mean a triple (X, G, θ), where X is a Hausdorff space, G is a topological group and $\theta\colon G \times X \to X$ is a continuous action of G on X. Henceforth, $\theta(g, x)$ will be denoted by gx. For $x \in X$, the set $G(x) = \{gx \mid g \in G\}$, is called the *$G$-orbit* of x in X. Note that G-orbits $G(x)$ and $G(y)$ of points x, y in X are either disjoint or equal. A subset S of X is called *G-invariant* if $\theta(G \times S) \subseteq S$. Let $X/G = \{G(x) \mid x \in X\}$ and $p_X\colon X \to X/G$ be the natural quotient map taking x to $G(x)$, $x \in X$, then X/G endowed with the quotient topology is called the *orbit space* of X (with respect to G). The map p_X which is called the *orbit map*, is continuous and open and if G is compact then p_X is also a closed map. An action of G on X is called *transitive* if $X/G = \{X\}$. If X, Y are G-spaces, then a continuous map $h\colon X \to Y$ is called *equivariant* if $h(gx) = gh(x)$ for each g in G and each x in X. We call h *pseudoequivariant* if $h(G(x)) = G(h(x))$ for each x in X. An equivariant map is clearly pseudoequivariant but converse is not true [5].

By a *metric G-space*, we mean a metric space on which a topological group G acts. If X is a metric G-space with invariant metric d i.e. $d(gx, gy) = d(x, y)$ for all $g \in G$; $x, y \in X$ and G is compact then d induces a metric ρ on X/G given by $\rho(G(x), G(y)) = \inf\big\{d(gx, ky) \mid g, k \in G\big\}$. Recall that if X is a metric space with metric d and $h \in H(X)$ then h is called *expansive*, if there exists a $\delta > 0$ such that whenever $x, y \in X, x \neq y$ then there exists an integer n satisfying $d(h^n(x), h^n(y)) > \delta$; δ is then called an *expansive constant* for h. If X is a metric G-space with metric d then an $h \in H(X)$ is called *G-expansive* with *G-expansive constant* $\delta > 0$ if whenever $x, y \in X$ with $G(x) \neq G(y)$ then there exists an integer n satisfying $d(h^n(u), h^n(v)) > \delta$, for all $u \in G(x)$ and $v \in G(y)$.

Throughout our spaces are G-spaces. In Section 2, we show that the least upper bound of the set of G-expansive constants of a G-expansive homeomorphism on a compact metric G-space with G compact is not a G-expansive constant. Similar result for expansive homeomorphisms is obtained by Bryant [3]. Section 3 of this paper is about obtaining results regarding projecting and lifting of G-expansive homeomorphisms. As a corollary we can easily show that a simple closed curve does not admit any $\mathbb{Z}$-expansive homeomorphism because of the simple observation that the real line does not admit any $\mathbb{Z}$-expansive homeomorphism. That higher dimension torus admits expansive homeomorphism can also be deduced from this result. Finally in Section 4, we define the notion of topological G-conjugacy, show that G-expansiveness is a dynamical property for homeomorphisms on G-spaces and study G-periodic points under topological G-conjugacy.

2. G-expansive constant

Let X be a compact metric G-space with G-compact and metric d and $h\colon X \to X$ be a pseudoequivariant G-expansive homeomorphism with G-expansive constant δ. Since X being a compact metric space is bounded therefore the set of all G-expansive constants for h is a bounded set of positive real numbers and hence has a least upper bound. We have the following result.

Theorem 2.1. *The least upper bound of G-expansive constants for h is not a G-expansive constant for h.*

Proof. Let θ be the least upper bound of the set of G-expansive constants for h and $\varepsilon_i = \frac{1}{i}$, $i = 1, 2, \dots$ then $\theta + \varepsilon_i$ is not a G-expansive constant for h. Thus for each i, there exist $x'_i, y'_i \in X$ with $G(x'_i) \neq G(y'_i)$ such that for some $g_i, k_i \in G$ and for each $n \in \mathbb{Z}$

$$d(h^n(g_i x'_i), h^n(k_i y'_i)) \leq \theta + \varepsilon_i.$$

Also, since h is G-expansive with G-expansive constant δ, there exists $m_i \in \mathbb{Z}$ satisfying

$$d\big(h^{m_i}(gx'_i), h^{m_i}(ky'_i)\big) > \delta, \qquad \text{for all} \quad g, k \in G \tag{$*$}$$

Let $x_i = h^{m_i}(x'_i)$ and $y_i = h^{m_i}(y'_i)$. Since X is compact metric space, we can assume that $(x_i) \to x$ and $(y_i) \to y$ for some x, y in X. By $(*)$ and the fact that h^{m_i} is pseudoequivariant, we get that $G(x) \neq G(y)$. For m in $\mathbb{Z}$ and $\alpha > 0$, choose p in $\mathbb{N}$ such that $\varepsilon_p < \frac{1}{3}\alpha$ and η such that

$$d(u, v) < \eta \implies d\big(h^m(u), h^m(v)\big) < \frac{1}{3}\alpha.$$

Let q be such that $n > q$ implies

$$d(g'_n x_n, gx) < \eta \qquad \text{and} \qquad d(k'_n y_n, ky) < \eta,$$

where $h^{m_i}(g_i x'_i) = g'_i h^{m_i}(x'_i)$, $h^{m_i}(k_i y'_i) = k'_i h^{m_i}(y'_i)$, with $(g'_i) \to g$ and $(k'_i) \to k$ for some $g, k \in G$. Thus $(g'_i x_i) \to gx$ and $(k'_i y_i) \to ky$. For $t > \max\{p, q\}$, there exist g_t, k_t in G such that for each $n \in \mathbb{Z}$

$$d(h^n(g_t x'_t), h^n(k_t y'_t)) \le \theta + \varepsilon_t. \tag{$**$}$$

Since h being pseudoequivariant h^{m_t} is pseudoequivariant and therefore

$$h^{m_t}(g_t x'_t) = g'_t h^{m_t}(x'_t) \qquad \text{and} \qquad h^{m_t}(k_t y'_t) = k'_t h^{m_t}(y'_t)$$

for some $g'_t, k'_t \in G$. Now

$$\begin{aligned} &d(h^m(gx), h^m(ky)) \\ &\le d(h^m(gx), h^m(g'_t x_t)) + d(h^m(g'_t x_t, h^m k'_t y_t)) + d((h^m k'_t y_t), h^m(ky)) \\ &\le \frac{1}{3}\alpha + \Big(\theta + \frac{1}{3}\alpha\Big) + \frac{1}{3}\alpha = \theta + \alpha. \end{aligned}$$

Observe that for $t > q$

$$d(g'_t x_t, gx) < \eta \implies d(h^m(g'_t x_t), h^m(gx)) < \frac{1}{3}\alpha$$

and

$$d(k'_t y_t, ky) < \eta \implies d(h^m(k'_t y_t), h^m(ky)) < \frac{1}{3}\alpha.$$

Moreover, using $(**)$,

$$d(h^m(g'_t x_t), h^m(k'_t y_t)) = d(h^{m+m_t}(g_t x'_t), h^{m+m_t}(k_t y'_t))$$

gives

$$d(h^m(g'_t x_t), h^m(k'_t y_t)) \le \theta + \varepsilon_t \le \theta + \varepsilon_p \le \theta + \frac{1}{3}\alpha.$$

Thus for each $m \in \mathbb{Z}$ and $\alpha > 0$,

$$d(h^m(gx), h^m(ky)) \le \theta + \alpha \implies d(h^m(gx), h^m(ky)) \le \theta.$$

Hence θ is not a G-expansive constant for h. □

3. Projecting and lifting of G-expansive homeomorphisms

Definition 3.1. Let $\widetilde{X}$ and X be metric spaces with metrics $\widetilde{d}$ and d respectively, and let $\pi\colon \widetilde{X} \to X$ be a continuous onto map then π is called a *locally isometric covering map* if for each $x \in X$, there exists a neighborhood $U(x)$ of x such that

$$\pi^{-1}(U(x)) = \bigcup_{\alpha} U_\alpha,$$

where $\{U_\alpha\}$ is a pairwise disjoint family of open sets and $\pi|_{U_\alpha}\colon U_\alpha \to U(x)$ is an isometry for each α [1].

Proposition 3.2. *Let $\widetilde{X}$ and X be metric G-spaces with metrics $\widetilde{d}$ and d respectively and $\pi\colon \widetilde{X} \to X$ be a pseudoequivariant locally isometric covering map. Let $f\colon \widetilde{X} \to \widetilde{X}$ and $g\colon X \to X$ be homeomorphisms such that $\pi \circ f = g \circ \pi$. Suppose X is compact and there exists $\delta_0 > 0$ such that for each $x \in \widetilde{X}$ and $0 < \delta \le \delta_0$, the open set $U_\delta(x)$ is connected and $\pi\colon \pi^{-1}(\pi(U_\delta(x))) \to U_\delta(\pi(x))$ is an isometry. Then f is G-expansive if and only if g is G-expansive.*

Proof. Suppose g is G-expansive with G-expansive constant e and let $\gamma = \min\{e, \delta_0\}$. For $p, q \in \widetilde{X}$, if for some $k_1, k_2 \in G$ and for each $i \in \mathbb{Z}$

$$\widetilde{d}(f^i(k_1p), f^i(k_2q)) \le \gamma, \tag{$*$}$$

then

$$d(\pi(f^i(k_1p)), \pi(f^i(k_2q))) \le \gamma.$$

Using $\pi \circ f = g \circ \pi$ and psedoequivariancy of π, there exists $k_1', k_2' \in G$ such that for each $i \in \mathbb{Z}$

$$d(g^i(k_1'\pi(p)), g^i(k_2'\pi(q))) \le \gamma.$$

Now using G-expansivity of g with G-expansive constant $e \le \gamma$ we obtain $G(\pi(p)) = G(\pi(q))$ and hence for some $k' \in G$, $\pi(k_1p) = \pi(k'q)$. From ($*$), it follows that $k_2q \in U_\gamma(k_1p)$. Now $\gamma \le \delta_0$ and $G(k_2q) = G(q)$ implies

$$d(k'q, k_1p) < \gamma.$$

Since $\pi(k'q) = \pi(k_1p)$ and π is an isometry on $U_\gamma(k_1p)$ we get $k'q = k_1p$ which implies $G(p) = G(q)$. This proves f is G-expansive with G-expansive constant γ.

For the converse, suppose f is G-expansive with G-expansive constant e. Since f is uniformly continuous [1], choose δ, $0 < \delta \le \delta_0$ such that if $\widetilde{d}(p,q) < \delta$ then $\widetilde{d}(f(p), f(q)) < \delta_0$ and $\widetilde{d}(f^{-1}(p), f^{-1}(q)) < \delta_0$. Let $\gamma = \min\{e, \delta\}$. For $x, y \in X$ with $d(x,y) \le \gamma$, choose $p, q \in \widetilde{X}$ with $\pi(p) = x, \pi(q) = y$ and $\widetilde{d}(p,q) = d(x,y)$. If for some $k_1, k_2 \in G$ and for each $i \in \mathbb{Z}$

$$d(g^i(k_1x), g^i(k_2y)) \le \gamma,$$

then taking $i = 1$ we get

$$d(g(k_1x), g(k_2y)) \le \gamma$$

and hence for $k_1', k_2' \in G$ with

$$\pi(k_1'p) = k_1x, \qquad \pi(k_2'q) = k_2y$$

we obtain

$$\widetilde{d}(f(k_1'p), f(k_2'q)) = d(g(k_1x), g(k_2y)) \le \gamma.$$

Since $\widetilde{d}(f(k'_1p), f(k'_2q)) < \delta_0$ and $\pi(k'_1p) = k_1x$, $\pi(k'_2q) = k_2y$, we get

$$d(k'_1p, k'_2q) = d(k_1x, k_2y) \leq \gamma < \delta$$

which implies

$$\widetilde{d}(f(k'_1p, f(k'_2q)) < \delta_0.$$

Inductively for all $i \geq 0$, $\widetilde{d}(f^i(k'_1p, f^i(k'_2q)) \leq \gamma$. Similarly for all $i \leq 0$, $\widetilde{d}(f^i(k'_1p, f^i(k'_2q)) \leq \gamma$. By G-expansivity of f, we get $G(p) = G(q)$. Since $\pi(k'_1p) = k_1x, \pi(k'_2q) = k_2y$, $G(x) = G(y)$. This proves g is G-expansive with G-expansive constant γ. □

Remark 3.3.

(1) Since no linear map f on $\mathbb{R}$ is $\mathbb{Z}$-expansive under the usual action of $\mathbb{Z}$ on $\mathbb{R}$ defined by $nx = n + x$, $n \in \mathbb{Z}$, $x \in \mathbb{R}$ and the map $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ defined by $\pi(x) = x + \mathbb{Z}$, $x \in \mathbb{R}$ satisfies the conditions in the hypothesis of the above result therefore there exist no $\mathbb{Z}$-expansive homeomorphism g satisfying $\pi \circ f = g \circ \pi$ on $\mathbb{R}/\mathbb{Z}$ which is homeomorphic to 1-torus $\mathbb{T}$.

(2) Since the natural quotient map $\pi\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ satisfies the conditions of the hypothesis in the above result,$f\colon \mathbb{R}^n \to \mathbb{R}^n$, $x \to \alpha x$, where $\alpha \neq -1, 0, 1$ is $SL(n, \mathbb{Z})$-expansive under the usual action of $SL(n, \mathbb{Z})$ on $\mathbb{R}^n$ and the induced map $\widetilde{f}\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ satisfies $\pi \circ f = \widetilde{f} \circ \pi$ therefore $\widetilde{f}$ is also $SL(n, \mathbb{Z})$-expansive.

4. Topological conjugacy in G-spaces.

Definition 4.1. Let X and Y be G-spaces and $f\colon X \to X$, $g\colon Y \to Y$ be continuous surjections. then we say that f is *topologically G-conjugate* to g if there exists a pseudoequivariant homeomorphism $\sigma\colon X \to Y$ such that $\sigma \circ f = g \circ \sigma$.

Theorem 4.2. *Let (X, d) and (Y, ρ) be compact metric G-spaces and let f_1 in $H(X)$ be topologically G-conjugate to an f_2 in $H(Y)$. Then f_1 is G-expansive if and only if f_2 is.*

Proof. Let $\sigma\colon X \to Y$ be a pseudoequivariant homeomorphism satisfying $\sigma \circ f_1 = f_2 \circ \sigma$. We only show that if f_1 is G-expansive with G-expansive constant, say α, then f_2 is G-expansive, as the reverse implication can be obtained similarly. By the uniform continuity of σ^{-1}, for given $\alpha > 0$ there exists $\beta > 0$ such that for all y, z in Y

$$d(\sigma^{-1}(y), \sigma^{-1}(z)) \geq \alpha \implies \rho(y, z) > \beta \tag{1}$$

Suppose y, z in Y have distinct G-orbits. Since σ is onto there exist $u, v \in X$ such that $\sigma(u) = y$ and $\sigma(v) = z$. Note that u and v also have distinct G-orbits in X and therefore we can find an integer n such that for all $g, k \in G$

$$d(f_1^n(gu), f_1^n(kv)) > \alpha$$

which is equivalent to

$$d(f_1^n(g\sigma^{-1}(y)), f_1^n(k\sigma^{-1}(z))) > \alpha.$$

Since σ^{-1} is also pseudoequivariant and satisfies $f_1 \circ \sigma^{-1} = \sigma^{-1} \circ f_2$, for all $g', k' \in G$, we get

$$d\big(\sigma^{-1} f_2^n(g'y), \sigma^{-1} f_2^n(k'z)\big) > \alpha. \tag{2}$$

As (2) is similar to (1), there exists $\beta > 0$ such that for all $g', k' \in G$

$$\rho(f_2^n(g'y), f_2^n(k'z)) > \beta.$$

This proves f_2 is G-expansive with G-expansive constant β. □

Let X be a G-space and $f\colon X \to X$ is a continuous map then $x \in X$ is called a G-periodic point if for some $n \in \mathbb{N}$, $f^n(x) \in G(x)$, i.e. $f^n(x) = g.x$ for some $g \in G$. The smallest $n \in \mathbb{N}$ for which $f^n(x) \in G(x)$ is called *G-periodicity* of x [4].

Theorem 4.3. *Let X, Y be G-spaces, $f\colon X \to X$, $g\colon Y \to Y$ be continuous surjections and σ be topological G-conjugacy between f and g. If $x \in X$ is G-periodic point of f of period n then $\sigma(x)$ is a G-periodic point of g with same G-periodicity.*

Proof. Since $f^n(x) \in G(x)$ therefore $\sigma(f^n(x)) \in \sigma(G(x))$. Observe that $\sigma \circ f^n = g^n \circ \sigma$. Using pseudoequivariancy of σ we get $g^n(\sigma(x)) \in G(\sigma(x))$. □

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