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CONVERGENCE FIELD OF ABEL'S SUMMATION METHOD

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ABSTRACT. Let $F(A)$ denote the set of all bounded sequences summable by Abel's method. It is known, that $F(A)$ is a linear subspace of the linear metric space (S,ρ) of all bounded sequences endowed with the sup metric. It is shown in [KOSTYRKO, P.: *Convergence fields of regular matrix transformations 2*, Tatra Mt. Math. Publ. **40** (2008), 143–147] that the convergence field of a regular matrix transformation is a $\sigma-$porous set. We show that $F(A)$ is very porous in S.

Introduction

In the whole text the sequences are written by denotation (a_n) and the starting index is zero.

Definition A. Let (a_n) be the sequence, $a_n \in \mathbb{R}$. Let (s_n) be the corresponding sequence of partial sums $s_n = a_0 + a_1 + \cdots + a_n$. The sequence (s_n) (the series $\sum\limits_{n=0}^{\infty} a_n$) is said to by limitable by Abel (summable by Abel) to the number s (to the sum s) if there is $\lim\limits_{x\to 1^-} \sum\limits_{n=0}^{\infty} a_n x^n = s$. We will use denotation

$$A\text{-}\lim_{n\to\infty} s_n = s \qquad A\text{-}\sum_{n=0}^{\infty} a_n = s.$$

(C.f. [2, p. 216].)

Let (S,ρ) be the metric space of all bounded real sequences endowed with the metric ρ, $\rho(a,b) = \sup\{|a_n - b_n| :\ n = 0,1,\dots\}$, $a = (a_n)$, $b = (b_n)$ and let

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$F(A)$ denote the set of all $(a_n) \in S$ where $A\text{-}\sum\limits_{n=0}^{\infty} a_n$ exists and is finite. $F(A)$ is a linear subspace of (S, ρ). (See [2, p. 216].)

Let (Y, η) be a metric space, $M \subset Y$. Let $y \in Y$, $\delta > 0$ and

$$B(y, \delta) = \{x \in Y : \ \eta(x, y) < \delta\}.$$

We put

$$\gamma(y, \delta, M) = \sup\{t > 0 : \ (\exists z \in B(y, \delta))\big(B(z, t) \subseteq B(y, \delta) \ \& \ B(z, t) \cap M = \emptyset\big)\}.$$

If such $t > 0$ does not exist we put $\gamma(y, \delta, M) = 0$.

The numbers

$$\underline{p}(y, M) = \liminf_{\delta \to 0} \frac{\gamma(y, \delta, M)}{\delta} \qquad \text{and} \qquad \overline{p}(y, M) = \limsup_{\delta \to 0} \frac{\gamma(y, \delta, M)}{\delta}$$

are called lower and upper porosity of the set M at y.

If $\overline{p}(y, M) > 0$, then M is porous at y.

If for each $y \in Y$ $\overline{p}(y, M) > 0$ holds, then M is porous in Y.

If $\underline{p}(y, M) > 0$, then M is very porous at y.

If for each $y \in Y$ $\underline{p}(y, M) > 0$ holds, then M is very porous in Y. (See [3].)

1. Results

Definition 1. Let s be a sequence of real numbers; $s = (s_n)$. We put

$$A\text{-}\liminf s = \liminf_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad A\text{-}\limsup s = \limsup_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n,$$

where (a_n) is the sequence such that (s_n) is the sequence of partial sums of the sequence (a_n).

Remark 1. $A\text{-}\liminf s$ and $A\text{-}\limsup s$ are defined correctly, because the series $\sum\limits_{n=0}^{\infty} a_n x^n$ exists. (sequence (a_n) is bounded and $0 < x < 1$)

Remark 2. If $A\text{-}\liminf s = A\text{-}\limsup s = \xi$ then $A\text{-}\lim s = \xi$. (Compare with Definition 1 and Definition A.)

Lemma 1. *There is a sequence $s = (s_n)$ of zeroes and units such that*

$$A\text{-}\liminf s = 0 \qquad \textit{and} \qquad A\text{-}\limsup s = 1.$$

Proof. Let $a = (a_n)$ is the corresponding sequence of reals to the sequence $s = (s_n)$ such that (s_n) is the sequence of partial sums of the sequence (a_n). The sequence (a_n) has the values of the set $\{-1, 0, 1\}$.

Denote p_k, $k = 1, 2, \ldots$, the indices, where the sequence (a_n) is of value 1 and q_k, $k = 1, 2, \ldots$, the indices, where the sequence (a_n) is of value -1. (s_n) is a sequence of 0 and 1, hence

$$p_1 < q_1 \qquad \text{and} \qquad p_k < q_k < p_{k+1} \quad \text{for all} \ \ k \in \mathbb{N}. \tag{1}$$

It means if we have the sequence (a_n) fulfilling (1) then (s_n) is the sequence of 0 and 1.

Let us explore the set of the sequences (a_n) satisfying (1) and corresponding limits $\lim\limits_{x\to 1^-} \sum\limits_{n=0}^{\infty} a_n x^n$; where $a_n \in \{-1,0,1\}$ and let us create the sequence (a_n) satisfying (1) in the following way.

1) $x_1 = \frac{1}{2}$, $a_0 = 1$, $a_1 = -1$ $(p_1 = 0,\ q_1 = 1)$
   Note that this inequality holds:
$$\sum_{n=0}^{\infty} a_n \left(\frac{1}{2}\right)^n \geq \frac{1}{2}$$
   (Disregarding the next members of the sequence (a_n), but a_n has to be of the set $\{-1,0,1\}$ and has to satisfy (1).)
2) We will find x_2: $x_1 < x_2 < 1$ and $1 - {x_2}^{q_1} \leq \frac{1}{2}\cdot\frac{1}{3}$.
   p_2: $q_1 < p_2$ and ${x_2}^{p_2} \leq \frac{1}{2}\cdot\frac{1}{3}$ $(a_{p_2} = 1)$
   x_3: $x_2 < x_3 < 1$ and $1 - {x_3}^{p_2} \leq \frac{1}{2}\cdot\frac{1}{3}$
   q_2: $p_2 < q_2$ and ${x_3}^{q_2} \leq \frac{1}{2}\cdot\frac{1}{3}$ $(a_{q_2} = -1)$
   $(a_n = 0$ for each n, $q_1 < n < q_2$, $n \neq p_2)$.
   Note that $\sum\limits_{n=0}^{\infty} a_n {x_2}^n \leq \frac{1}{3}$ for any members of (a_n), $n > p_2$, $a_n \in \{-1,0,1\}$ and (a_n) satisfying (1).
   Note that $\sum\limits_{n=0}^{\infty} a_n {x_3}^n \geq 1 - \frac{1}{3}$ for any members of (a_n), $n > q_2$, $a_n \in \{-1,0,1\}$ and (a_n) satisfying (1).
3) Let there are p_k, q_k, x_{2k-2}, x_{2k-1}.
   We will find $x_{2(k+1)-2} = x_{2k}$: $x_{2k-1} < x_{2k} < 1$ and $1 - {x_{2k}}^{q_k} \leq \frac{1}{2}\cdot\frac{1}{k+2}$.
   p_{k+1}: $q_k < p_{k+1}$ and ${x_{2k}}^{p_{k+1}} \leq \frac{1}{2}\cdot\frac{1}{k+2}$ $(a_{p_{k+1}} = 1)$
   $x_{2(k+1)-1} = x_{2k+1}$: $x_{2k} < x_{2k+1} < 1$ and $1 - {x_{2k+1}}^{p_k} \leq \frac{1}{2}\cdot\frac{1}{k+2}$
   q_{k+1}: $p_{k+1} < q_{k+1}$ and ${x_{2k+1}}^{q_{k+1}} \leq \frac{1}{2}\cdot\frac{1}{k+2}$ $(a_{q_{k+1}} = -1)$
   $(a_n = 0$ for each n, $q_k < n < q_{k+1}$, $n \neq p_{k+1}$.)
   Then for any admissible determination of members of the sequence a_n, where $n > p_{k+1}$ holds:
$$\sum_{n=0}^{\infty} a_n {x_{2k}}^n \leq \frac{1}{k+2}$$
   and for any admissible determination of members of the sequence (a_n), $n > q_{k+1}$ holds:
$$\sum_{n=0}^{\infty} a_n {x_{2k+1}}^n \geq 1 - \frac{1}{k+2}.$$

Thus $A\text{-}\liminf s = 0$ and $A\text{-}\limsup s = 1$. □

Lemma 2. *Let $s = (s_n)$ be the sequence such that $0 \le s_n \le 1$ for each $n \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$. Then the next inequality holds: $0 \le A\text{-}\liminf s \le A\text{-}\limsup s \le 1$.*

Proof.

a) We will use the set of the sequences such that

$$\forall N \in \mathbb{N}^0 : \quad \sum_{n=0}^{N} a_n \ge 0. \tag{2}$$

(It is the same as $s_N \ge 0$.) We want to show, that $0 \le A\text{-}\liminf s$, i.e., $\liminf\limits_{x\to 1^-} \sum\limits_{n=0}^{\infty} a_n x^n \ge 0$. We will show, that for each $x \in (0,1)$ and for each $N \in \mathbb{N}^0$: $\sum\limits_{n=0}^{N} a_n x^n \ge 0$. By induction with respect to N.

0) $\sum\limits_{n=0}^{0} a_n x^n \ge 0$ (Because (a_n) fulfills (2).)

N) Let $\sum\limits_{n=0}^{N} a_n x^n \ge 0$ hold, for all (a_n) fulfilling (2).

N+1) Then $\sum\limits_{n=0}^{N+1} a_n x^n = \sum\limits_{n=0}^{N} a_n x^n + a_{N+1} x^{N+1}$

a$_1$) if $a_{N+1} \ge 0$ then $\sum\limits_{n=0}^{N+1} a_n x^n \ge 0$ (using N)).

a$_2$) if $a_{N+1} < 0$ then

$$\sum_{n=0}^{N+1} a_n x^n \ge \sum_{n=0}^{N} a_n x^n + a_{N+1} x^N = \sum_{n=0}^{N-1} a_n x^n + (a_N + a_{N+1}) x^N \ge 0.$$

The inequality holds, because $\sum\limits_{n=0}^{N+1} a_n \ge 0$, $x \in (0,1)$ and because of N).

Hence $0 \le A\text{-}\liminf s$.

b) Let (t_n) be the sequence such that $t_n = 1 - s_n$. Then $0 \le t_n \le 1$ and by a) we have $A\text{-}\limsup s = A\text{-}\limsup 1 - t = 1 - A\text{-}\liminf t \le 1$. (We used: $A\text{-}\limsup -t = -A\text{-}\liminf t$.)

Thus (using a) and b)) the Lemma 2 holds. □

Lemma 3. *Let $s = (s_n)$ be the sequence such that $a \le s_n \le b$ for each $n \in \mathbb{N}^0$. Then the next inequality holds: $a \le A\text{-}\liminf s \le A\text{-}\limsup s \le b$.*

Proof. Let (r_n) be the sequence that for each member of (r_n) holds: $r_n = \frac{s_n - a}{b - a}$. The sequence (r_n) fulfills the assumptions of the Lemma 2, thus $0 \le A\text{-}\liminf r \le A\text{-}\limsup r \le 1$. Let us multiply these inequalities by $(b - a)$ and then add a. We obtain the assertion of Lemma 3. □

Theorem. *Let (S,ρ) be the space of all bounded sequences of reals endowed with the metric ρ. Then convergence field*

$$F(A) = \big\{x = (x_n) \in S:\ A\text{-}\lim x \ \textit{exists}\}$$

is a very porous set.

P r o o f. Let us estimate the expression $\underline{p}(x, F(A)) = \liminf\limits_{\delta\to 0} \frac{\gamma(x,\delta,F(A))}{\delta}$ where

$$\gamma(x,\delta,F(A)) = \sup\big\{t>0:\ \big(\exists z \in B(x,\delta)\big)\big(B(z,t) \subseteq B(x,\delta)\ \wedge \\ B(z,t)\cap F(A) = \emptyset\big)\big\}.$$

1) Let (x_n) be such that

$$A\text{-}\lim x = a. \tag{3}$$

Let us make δ- neighbourhood of x, $B(x,\delta) = \{z \in S:\ \rho(z,x) < \delta\}$. Denote by y_0 the sequence from the Lemma 1, $y_0 \in S$. Let y be the following sequence: $y = x + \frac{\delta}{2}y_0$. This sequence belongs to $B(x,\delta)$ and $A\text{-}\lim y$ does not exist. Precisely:

$$a = A\text{-}\liminf y < A\text{-}\limsup y = a + \frac{\delta}{2}. \tag{4}$$

(See (3) and Lemma 1.)

Let us make $\frac{\delta}{5}$- neighbourhood of y, i.e. $B\big(y,\frac{\delta}{5}\big)$. Let z belong to $B\big(y,\frac{\delta}{5}\big)$, i.e. $z = y + \frac{\delta}{5}z_0$, where $z_0 \in B(\mathbb{O},1)$. $\mathbb{O} = (c_n)_{n=0}^{\infty}$, where $c_n = 0$ for each $n \in \mathbb{N}^0$. Then the next inequalities holds:

$$a - \frac{\delta}{5} \le A\text{-}\liminf z \le a + \frac{\delta}{5} \tag{5}$$

$$a + \frac{\delta}{2} - \frac{\delta}{5} \le A\text{-}\limsup z \le a + \frac{\delta}{2} + \frac{\delta}{5}. \tag{6}$$

(See (4) and Lemma 2.)

(5) and (6) imply $A\text{-}\liminf z \le a + \frac{\delta}{5} < a + \frac{\delta}{2} - \frac{\delta}{5} \le A\text{-}\limsup z$. Thus z is not A-limitable for each $z \in B\big(y,\frac{\delta}{5}\big)$. Thus $\underline{p}(x,F(A)) = \liminf\limits_{\delta\to 0}\frac{\gamma(x,\delta,F(A))}{\delta} \ge \liminf\limits_{\delta\to 0}\frac{\frac{\delta}{5}}{\delta} = \frac{1}{5}$ if $x \in F(A)$.

2) Let (x_n) be such that $A\text{-}\lim x$ does not exist. Boudedness of x and Lemma 3 imply existence of u, v such that

$$u = A\text{-}\liminf x \tag{7}$$

$$v = A\text{-}\limsup x. \tag{8}$$

Let

$$2\delta < v - u. \tag{9}$$

Let us make δ-neighbourhood of x, $B(x,\delta)$. Denote by y_0 the sequence from Lemma 1. Let y be the following sequence: $y = x + \frac{\delta}{2}y_0$.

$$u - \frac{\delta}{2} \le A\text{-}\liminf y \le u + \frac{\delta}{2} \tag{10}$$

$$v - \frac{\delta}{2} \leq A\text{-}\limsup y \leq v + \frac{\delta}{2} \tag{11}$$

(From (7), (8) and Lemma 1.)

From (9) and (10) we get

$$A\text{-}\liminf y < v - \frac{3\delta}{2}. \tag{12}$$

(11) and (12) imply:

$$A\text{-}\liminf y < v - \frac{3\delta}{2} < v - \frac{\delta}{2} \leq A\text{-}\limsup y. \tag{13}$$

Thus $y \notin F(A)$.

Let us make $\frac{\delta}{5}$-neighbourhood of y, $B\left(y, \frac{\delta}{5}\right)$. Let z belong to $B\left(y, \frac{\delta}{5}\right)$, i.e. $z = y + \frac{\delta}{5} z_0$, where $z_0 \in B(\mathbb{O}, 1)$. From (13) and Lemma 2 we obtain

$$A\text{-}\liminf z < v - \frac{3\delta}{2} + \frac{\delta}{5} = v - \frac{13\delta}{10} \tag{14}$$

and

$$A\text{-}\limsup z \geq v - \frac{\delta}{2} - \frac{\delta}{5} = v - \frac{7\delta}{10}. \tag{15}$$

(14) and (15) imply $A\text{-}\liminf z < A\text{-}\limsup z$, for each $z \in B\left(y, \frac{\delta}{5}\right)$.

Thus $B\left(y, \frac{\delta}{5}\right) \cap F(A) = 0$. Thus

$$\underline{p}(x, F(A)) = \liminf_{\delta \to 0} \frac{\gamma(x, \delta, F(A))}{\delta} \geq \liminf_{\delta \to 0} \frac{\frac{\delta}{5}}{\delta} = \frac{1}{5}$$

if $x \notin F(A)$.

Using 1.) and 2.) we get that $F(A)$ is very porous set in (S, ρ). $\square$

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