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OSCILLATION CRITERIA FOR QUASI-LINEAR FUNCTIONAL DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. This paper is concerned with oscillation of the second-order quasi-linear functional dynamic equation

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)x^{\beta}(\tau(t)) = 0,$$

on a time scale $\mathbb{T}$ where γ and β are quotient of odd positive integers, r, p, and τ are positive rd-continuous functions defined on $\mathbb{T}$, $\tau\colon \mathbb{T} \to \mathbb{T}$ and $\lim\limits_{t\to\infty} \tau(t) = \infty$. We establish some new sufficient conditions which ensure that every solution oscillates or converges to zero. Our results improve the oscillation results in the literature when $\gamma = \beta$, and $\tau(t) \le t$ and when $\tau(t) > t$ the results are essentially new. Some examples are considered to illustrate the main results.

1. Introduction

In this paper, we are concerned with oscillation of the second-order quasi-linear functional dynamic equation

$$(r(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} + p(t)x^{\beta}(\tau(t)) = 0, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$. Throughout this paper, we will assume the following hypotheses:

(h_1) r and p are real valued rd-continuous positive functions defined on $\mathbb{T}$,

(h_2) γ and $\beta > 0$ are ratios of odd positive integers, $\tau\colon \mathbb{T} \to \mathbb{T}$, and $\lim\limits_{t\to\infty} \tau(t) = \infty$.

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We shall also consider the two cases:

$$\int_{t_0}^{\infty} \left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}} \Delta t = \infty, \tag{1.2}$$

and

$$\int_{t_0}^{\infty} \left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}} \Delta t < \infty. \tag{1.3}$$

Equation (1.1) is called a delay dynamic equation if $\tau(t) < t$ and is called an advance dynamic equation if $\tau(t) > t$ and ordinary if $\tau(t) = t$. Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. Throughout this paper this assumption will be supposed to hold: Let $T_0 = \min\{\tau(t) : t \geq 0\}$ and $\tau_{-1}(t)$ is the inverse of $\tau(t)$ when the latter exists. By a solution of (1.1) we mean a nontrivial real-valued function $x(t)$ which has the properties $x(t) \in C^1_{rd}[\tau_{-1}(t_0), \infty)$, and $x^{[1]} \in C^1_{rd}[\tau_{-1}(t_0), \infty)$, where C_r is the space of rd-continuous functions and

$$x^{[1]} := r\big(x^{\Delta}\big)^{\gamma}, \qquad \text{and} \qquad x^{[2]} := \big(x^{[1]}\big)^{\Delta}. \tag{1.4}$$

Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t > t_1\} > 0$ for any $t_1 \geq t_x$. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution $x(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

Much recent attention has been given to dynamic equations on time scales (or measure chains), and we refer the reader to the landmark paper of Hilger [23] for a comprehensive treatment of the subject. Since then several authors have expounded on various aspects of this new theory [11]. A book on the subject of time scales, by Bohner and Peterson [10], summarizes and organizes much of time scale calculus. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [26]), i.e, when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}} = \{t : t = q^k, \ k \in \mathbb{N}, \ q > 1\}$.

Dynamic equations on a time scale have an enormous potential for applications such as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population (see [10]). There are applications of dynamic equations on time scales to *quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics.* A recent cover story article in New

Scientist [35] discusses several possible applications. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f\colon \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$.

In the last few years, there has been increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of dynamic equations on time scales. For contribution, we refer the reader to the papers [1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 18, 19, 20, 21, 28, 29, 30, 31, 32, 33, 34], and the references cited therein.

Note that if $\mathbb{T} = \mathbb{R}$, then

$$\sigma(t) = t, \quad \mu(t) = 0, \quad f^\Delta(t) = f'(t), \quad \int_a^b f(t)\Delta t = \int_a^b f(t)\,\mathrm{d}t,$$

and (1.1) becomes the quasi-linear functional differential equation

$$\bigl(r(t)\,(x'(t))^\gamma\bigr) + p(t)x^\beta(\tau(t)) = 0. \tag{1.5}$$

If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t+1$, $\mu(t) = 1$, $f^\Delta(t) = \Delta f(t)$, $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$, and (1.1) becomes the quasi-linear difference equation

$$\Delta(r(t)\,(\Delta x(t))^\gamma) + p(t)x^\beta(\tau(t)) = 0. \tag{1.6}$$

When $\alpha = \beta$, the equation (1.1) becomes the half-linear dynamic equation which have been considered by some authors and some oscillation and nonoscillation results are obtained. As a special case of (1.1) Agarwal et al [1] considered the second-order delay dynamic equations on time scales

$$x^{\Delta\Delta}(t) + p(t)x(\tau(t)) = 0, \tag{1.7}$$

and established some sufficient conditions for oscillation of (1.7) when

$$\int_{t_0}^{\infty} \tau(t)p(t)\Delta t = \infty. \tag{1.8}$$

In [8], Akın-Bohner and Hoffacker considered the equation

$$x^{\Delta\Delta}(t) + p(t)(x^\sigma)^\gamma = 0, \tag{1.9}$$

and established some necessary and sufficient conditions for oscillation of all solutions when $\gamma > 1$ and $0 < \gamma < 1$. Their results cannot be applied in the case when $\gamma = 1$ and applied only on discrete time scales.

Saker [29] examines oscillation for half-linear dynamic equation

$$(r(t)(x^\Delta(t))^\gamma)^\Delta + p(t)x^\gamma(t) = 0, \tag{1.10}$$

on time scales, where $\gamma > 1$ is an odd positive integer which cannot be applied when $0 < \gamma \leq 1$. Erbe et al [19] considered (1.10) and the equation

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)x^{\gamma}(\sigma(t)) = 0,$$

when

$$r^{\Delta}(t) \geq 0, \qquad \int_{t_0}^{\infty} \sigma^{\gamma}(t)p(t)\Delta t = \infty, \tag{1.11}$$

and established some necessary and sufficient conditions for nonoscillation of Hille-Kneser type. Erbe et al [14] considered the half-linear delay dynamic equations on time scales

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)x^{\gamma}(\tau(t)) = 0, \tag{1.12}$$

where $\gamma > 1$ is the quotient of odd positive integers and

$$r^{\Delta}(t) \geq 0, \qquad \text{and} \qquad \int_{t_0}^{\infty} \tau^{\gamma}(t)p(t)\Delta t = \infty. \tag{1.13}$$

and utilized a Riccati transformation technique and established some oscillation criteria for (1.12). Erbe et al [15] considered the half-linear delay dynamic equation (1.12) on time scales, where $0 < \gamma \leq 1$ is the quotient of odd positive integers and established some sufficient conditions for oscillation when (1.13) holds. Han et al [22] considered (1.12) and followed the proof that has been used in [29] and established some sufficient conditions for oscillation when $r^{\Delta}(t) \geq 0$. For oscillation of quasi-linear dynamic equations, Grace et al [21] considered the equation

$$(r(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} + p(t)x^{\beta}(t) = 0, \tag{1.14}$$

where γ and $\beta > 0$ are ratios of odd positive integers, r and p are positive rd-continuous functions on $\mathbb{T}$ and established several sufficient conditions for oscillation.

Following this trend in this paper, we establish some sufficient conditions for oscillation of (1.1). The main results are proved in Section 2, which is organized as follows: In the subsection 2.1 we consider the case when $\tau(t) > t$ and in the subsection 2.2, we consider the case when $\tau(t) \leq t$. The results in this paper are different from the results established in [21] even in the case when $\tau(t) = t$ and can be applied to the equation (1.1) when $0 < \gamma < 1$ and $\tau(t) > t$. The results improve the results established in [1, 14, 15, 19, 29, 22], in the sense that the results do not require the conditions (1.8), (1.11), (1.13) and $r^{\Delta}(t) \geq 0$. The results also can applied on any time scale not only on discrete time scales when $\mu(t) \neq 0$, which is the case considered in [8]. Also our results include and improve some oscillation results for differential and difference equations obtained in [24, 36].

2. Main results

In this section, we state and prove the main oscillation results. We note that if x is a solution of (1.1) then $z(t) = -x(t)$ is also solution of (1.1). Thus for nonoscillatory solutions of (1.1) we can restrict our attention to the positive ones. We start with the following Lemma which plays an important role in the proofs of the main results.

Lemma 2.1. *Assume that* (h_1), (h_2), (1.2) *hold and* (1.1) *has a nonoscillatory solution* x *on* $[t_0, \infty)$. *Then there exists* $T \geq t_0$ *such that* $x(t)x^{[1]}(t) > 0$ *for* $t \geq T$.

P r o o f. Let x be as in the statement of this lemma. Pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ so that $t_1 > t_0$ and so that $x(\tau(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Since $x(\tau(t)) > 0$, we have from (1.1) that $(x^{[1]}(t))^{\Delta} = -p(t)x^{\beta}(\tau(t)) < 0$, for $t \geq t_1$. Then $x^{[1]}(t)$ is strictly decreasing for $t \geq t_1$ and of one sign. We claim that $x^{[1]}(t) > 0$ for $t_1 \geq t_0$. Assume not, then there is a $t_2 \geq t_1$ such that $x^{[1]}(t) =: c < 0$. Therefore, $x^{[1]}(t) \leq x^{[1]}(t_2) = c$, for $t \geq t_2$ (note $x^{[1]}$ is decreasing). This implies from the definition $x^{[1]}(t)$ that $x^{\Delta}(t) \leq ar^{\frac{-1}{\gamma}}(t)$, for $t \geq t_2$ where $a := c^{\frac{1}{\gamma}} < 0$. Integrating, we find that

$$x(t) = x(t_2) + \int_{t_2}^{t} x^{\Delta}(s)\Delta s \leq x(t_2) + a\int_{t_2}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} \to -\infty \qquad \text{as} \quad t \to \infty,$$

which implies that $x(t)$ is eventually negative. This is a contradiction. Hence $r(t)\left(x^{\Delta}(t)\right)^{\gamma} > 0$ for $t \geq t_1$ and so $x^{\Delta}(t) > 0$ for $t \geq T = t_1$. The proof is complete. □

2.1. The case when $\delta(t) > t$

In this subsection, we establish some sufficient conditions for oscillation of (1.1) when $\delta(t) > t$. We introduce the following notations:

$$Q(t) := p(t)\left(\frac{r^{1/\gamma}(t)P(t,T)}{r^{1/\gamma}(t)P(t,T) + \sigma(t) - t}\right)^{\beta} \eta^{\sigma}(t), \qquad P(t,T) := \int_{T}^{t} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} \Delta s,$$

and

$$\eta^{\sigma}(t) := \begin{cases} c_1 \text{ is any positive constant,} & \text{if } \beta > \gamma, \\ 1, & \text{if } \beta = \gamma, \\ c_2\left(\int_{T}^{\sigma(t)} \frac{1}{r^{\frac{1}{\gamma}}(s)}\Delta s\right)^{\beta-\gamma}, \ c_2 \text{ is any positive constant,} & \text{if } \beta < \gamma, \end{cases} \tag{2.1}$$

where $T \geq t_0$ is chosen sufficiently large.

Theorem 2.1. *Assume that* (h_1), (h_2) *and* (1.2) *hold. Let* x *be a nonoscillatory solution of* (1.1) *and make the Riccati substitution*

$$w(t) := \frac{x^{[1]}(t)}{x^{\gamma}(t)}. \tag{2.2}$$

Then there exists $T > t_0$ *such that* $w(t) > 0$ *for* $t > T$, *and*

$$w^{\Delta}(t) + Q(t) + \frac{\gamma}{r^{\frac{1}{\gamma}}(t)} \left(w^{\sigma}(t)\right)^{1+\frac{1}{\gamma}} \leq 0, \qquad \text{for} \quad t \geq T. \tag{2.3}$$

P r o o f. Let x be as above and without loss of generality, we assume that there is a $t_1 > t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ for $t \geq t_1$. Then from Lemma 2.1, since (1.2) holds, there exists $T > t_1$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, and $x^{[2]}(t) < 0$, for $t \geq T$. By the quotient rule [10, Theorem 1.20] and the definition of $w(t)$, we have

$$\begin{aligned} w^{\Delta}(t) &= \frac{x^{\gamma}(t)x^{[2]}(t) - \left(x^{\gamma}(t)\right)^{\Delta} x^{[1]}(t)}{x^{\gamma}(t)\left(x^{\sigma}(t)\right)^{\gamma}} = \frac{x^{[2]}(t)}{\left(x^{\sigma}(t)\right)^{\gamma}} - \frac{\left(x^{\gamma}(t)\right)^{\Delta} x^{[1]}(t)}{x^{\gamma}(t)\left(x^{\sigma}(t)\right)^{\gamma}} \\ &= \frac{x^{[2]}(t)}{\left(x^{\tau}(t)\right)^{\beta}} \frac{\left(x^{\tau}(t)\right)^{\beta}}{\left(x^{\sigma}(t)\right)^{\gamma}} - \frac{\left(x^{\gamma}(t)\right)^{\Delta} x^{[1]}(t)}{x^{\gamma}(t)\left(x^{\sigma}(t)\right)^{\gamma}}. \end{aligned}$$

From (1.1), we see that

$$w^{\Delta}(t) = -p(t)\frac{\left(x^{\tau}(t)\right)^{\beta}}{\left(x^{\sigma}(t)\right)^{\gamma}} - \frac{\left(x^{\gamma}(t)\right)^{\Delta} x^{[1]}(t)}{x^{\gamma}(t)\left(x^{\sigma}(t)\right)^{\gamma}}, \qquad \text{for} \quad t \geq T. \tag{2.4}$$

By the Pötzsche chain rule ([10, Theorem 1.90]), if $f^{\Delta}(t) > 0$ and $\gamma > 1$ (note $f^{\sigma} \geq f$) we obtain

$$\begin{aligned} \left(f^{\gamma}(t)\right)^{\Delta} &= \gamma \int_0^1 \left[f(t) + \mu h f^{\Delta}(t)\right]^{\gamma-1} f^{\Delta}(t)\,\mathrm{d}h \\ &= \gamma \int_0^1 \left[(1-h)f(t) + h f^{\sigma}(t)\right]^{\gamma-1} f^{\Delta}(t)\,\mathrm{d}h \\ &\geq \gamma \int_0^1 \left(f(t)\right)^{\gamma-1} f^{\Delta}(t)\,\mathrm{d}h = \gamma\left(f(t)\right)^{\gamma-1} f^{\Delta}(t). \end{aligned} \tag{2.5}$$

Also by the Pötzsche chain rule ([10, Theorem 1.90]), if $f^{\Delta}(t) > 0$ and $0 < \gamma \leq 1$, we obtain

$$\begin{aligned} \left(f^{\gamma}(t)\right)^{\Delta} &= \gamma \int_0^1 \left[f(t) + h\mu(t) f^{\Delta}(t)\right]^{\gamma-1}\,\mathrm{d}h\; f^{\Delta}(t) \\ &\geq \gamma \int_0^1 \left(f^{\sigma}(t)\right)^{\gamma-1}\,\mathrm{d}h\; f^{\Delta}(t) = \gamma\left(f^{\sigma}(t)\right)^{\gamma-1} f^{\Delta}(t). \end{aligned} \tag{2.6}$$

Now from (2.5) and (2.6), using $f(t) = x(t)$, and using the fact that $x(t)$ is increasing and $x^{[1]}(t)$ is decreasing, we have for $\gamma > 1$, that

$$\begin{aligned}
\frac{\left((x(t))^{\gamma}\right)^{\Delta} x^{[1]}(t)}{(x(t))^{\gamma}\,(x^{\sigma}(t))^{\gamma}} &\geq \frac{\gamma x^{[1]}(t)(x^{[1]})^{\frac{1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}x(t)\,(x^{\sigma}(t))^{\gamma}} \\
&\geq \frac{\gamma\left(x^{[1]}(t)\right)^{\sigma}\left(\left(x^{[1]}(t)\right)^{\sigma}\right)^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}x^{\sigma}(t)\,(x^{\sigma}(t))^{\gamma}} = \gamma\frac{1}{r^{\frac{1}{\gamma}}(t)}\,(w^{\sigma}(t))^{\frac{1}{\gamma}+1},
\end{aligned}$$

and for $0 < \gamma \leq 1$, we have

$$\begin{aligned}
\frac{\left(x^{\gamma}(t)\right)^{\Delta} x^{[1]}(t)}{x^{\gamma}(t)\,(x^{\sigma}(t))^{\gamma}} &\geq \frac{\gamma x^{[1]}(t)\,(x^{\sigma}(t))^{\gamma-1}\,(x^{[1]}(t))^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)x^{\gamma}(t)\,(x^{\sigma}(t))^{\gamma}} \\
&\geq \frac{\gamma\left(x^{[1]}(t)\right)^{\sigma}\left(\left(x^{[1]}\right)^{\sigma}(t)\right)^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)\,(x^{\sigma}(t))^{\gamma}\,x^{\sigma}(t)} = \gamma\frac{1}{r^{\frac{1}{\gamma}}(t)}\,(w^{\sigma}(t))^{1+\frac{1}{\gamma}}.
\end{aligned}$$

Thus for $\gamma > 0$, we have

$$\frac{\left(x^{\gamma}(t)\right)^{\Delta} x^{[1]}(t)}{x^{\gamma}(t)\,(x^{\sigma}(t))^{\gamma}} \geq \gamma\frac{1}{r^{\frac{1}{\gamma}}}\,(w^{\sigma}(t))^{1+\frac{1}{\gamma}}.$$

Substituting in (2.4), we obtain

$$w^{\Delta}(t) \leq -p(t)\frac{(x^{\tau}(t))^{\beta}}{(x^{\sigma}(t))^{\gamma}} - \gamma\frac{1}{r^{\frac{1}{\gamma}}(t)}\,(w^{\sigma})^{1+\frac{1}{\gamma}}, \qquad \text{for} \quad t \geq T. \tag{2.7}$$

Next consider the coefficient of p in (2.7). Since $x^{\sigma} = x + \mu x^{\Delta}$, we have

$$\frac{x^{\sigma}(t)}{x(t)} = 1 + \mu(t)\frac{x^{\Delta}}{x(t)} = 1 + \frac{\mu(t)}{r^{\frac{1}{\gamma}}(t)}\frac{\left(x^{[1]}(t)\right)^{\frac{1}{\gamma}}}{x(t)}.$$

Also since $x^{[1]}(t)$ is decreasing, we have

$$\begin{aligned}
x(t) &= x(T) + \int_T^t \left(x^{[1]}(s)\right)^{\frac{1}{\gamma}}\left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}\Delta s \\
&\geq x(T) + \left(x^{[1]}(t)\right)^{\frac{1}{\gamma}}\int_T^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}\Delta s > \left(x^{[1]}(t)\right)^{\frac{1}{\gamma}}\int_T^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}\Delta s.
\end{aligned}$$

It follows that

$$\frac{x(t)}{\left(x^{[1]}(t)\right)^{\frac{1}{\gamma}}} \geq \int_T^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}\Delta s = P(t,T), \qquad \text{for} \quad t \geq T. \tag{2.8}$$

Hence

$$\frac{x^{\sigma}(t)}{x(t)} = 1 + \mu(t)\frac{x^{\Delta}(t)}{x(t)} = 1 + \frac{\mu(t)}{r^{\frac{1}{\gamma}}(t)}\frac{\left(x^{[1]}(t)\right)^{\frac{1}{\gamma}}}{x(t)} \leq \frac{r^{\frac{1}{\gamma}}(t)P(t,T) + \mu(t)}{r^{\frac{1}{\gamma}}(t)P(t,T)}, \quad \text{for} \quad t \geq T.$$

Hence, we have

$$\frac{x(t)}{x^{\sigma}(t)} \geq \frac{r^{\frac{1}{\gamma}}(t)P(t,T)}{r^{\frac{1}{\gamma}}(t)P(t,T) + \sigma(t) - t}, \qquad \text{for} \quad t \geq T.$$

Thus

$$\frac{x^{\tau}(t)}{x^{\sigma}(t)} = \frac{x^{\tau}(t)}{x(t)}\frac{x(t)}{x^{\sigma}(t)} \geq \left(\frac{x^{\tau}(t)}{x(t)}\right)\frac{r^{\frac{1}{\gamma}}(t)P(t,T)}{r^{\frac{1}{\gamma}}(t)P(t,T) + \sigma(t) - t}, \qquad \text{for} \quad 3t \geq T. \tag{2.9}$$

Now, since $\tau(t) > t$ and $x(t)$ is increasing, we have $x^{\tau}(t) > x(t)$. This and (2.9) guarantees that

$$\frac{(x^{\tau}(t))^{\beta}}{(x^{\sigma}(t))^{\gamma}} \geq \left(\frac{r^{\frac{1}{\gamma}}(t)P(t,T)}{r^{\frac{1}{\gamma}}(t)P(t,T) + \sigma(t) - t}\right)^{\beta}(x^{\sigma}(t))^{\beta-\gamma}, \qquad \text{for} \quad t \geq T. \tag{2.10}$$

We consider the following three cases:
Case (i). $\beta > \gamma$.
In this case, since $x^{\Delta}(t) > 0$, then there exists $t_2 \geq t_1$ such that $x^{\sigma}(t) > x(t) > c > 0$. This implies that $(x^{\sigma}(t))^{\beta-\gamma} > c_1$, where $c_1 = c^{\beta-\gamma}$.
Case (ii). $\beta = \gamma$.
In this case, we see that $(x^{\sigma}(t))^{\beta-\gamma} = 1$.
Case (iii). $\beta < \gamma$.
From Lemma 2.1, since $x^{[1]}(t)$ is positive and decreasing, we see that $x^{[1]}(t) \leq x^{[1]}(t_2) = c$, for $t \geq t_2$. This implies that

$$x(\sigma(t)) \leq x(t_2) + c^{\frac{1}{\gamma}}\left(\int_{t_2}^{\sigma(t)} \frac{1}{r^{\frac{1}{\gamma}}(s)}\Delta s\right).$$

Thus

$$x^{\beta-\gamma}(\sigma(t)) > (c_2)^{\beta}\left(\int_{t_2}^{\sigma(t)} \frac{1}{r^{\frac{1}{\gamma}}(s)}\Delta s\right)^{\beta-\gamma}, \qquad \text{where} \quad c_2 = \left(\frac{1}{c}\right)^{\beta}. \tag{2.11}$$

Using these three cases in (2.10) and using the definition of $\eta^{\sigma}(t)$, we get

$$\frac{(x^{\tau}(t))^{\beta}}{(x^{\sigma}(t))^{\gamma}} \geq \left(\frac{r^{\frac{1}{\gamma}}(t)P(t,T)}{r^{\frac{1}{\gamma}}(t)P(t,T)+\sigma(t)-t}\right)^{\beta}\eta^{\sigma}(t), \qquad \text{for} \quad t \geq T. \tag{2.12}$$

Put (2.12) into (2.7), we obtain the inequality (2.3) and this completes the proof. □

Theorem 2.2 (**Leighton-Wintner type**). *Assume that* (h_1), (h_2) *and* (1.2) *hold. Furthermore, assume that*

$$\int_{t_0}^{\infty} Q(s)\Delta s = \infty. \tag{2.13}$$

Then every solution of (1.1) *oscillates.*

P r o o f. Suppose to the contrary and assume that x is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $x(t) > 0$, and $x(\tau(t)) > 0$ for $t \geq T$ where T is chosen large enough. We consider only this case, because the proof when $x(t) < 0$ is similar. Let w be as defined in Theorem 2.1. Then from Theorem 2.1, we see that $w(t) > 0$ and satisfies the inequality

$$-w^{\Delta}(t) \geq Q(t) + \frac{\gamma}{r^{\frac{1}{\gamma}}(t)}\left(w^{\sigma}(t)\right)^{1+\frac{1}{\gamma}} > Q(t), \qquad \text{for} \quad t \geq T. \tag{2.14}$$

From the definition of $x^{[1]}(t)$, we see that $x^{\Delta}(t) = \left(x^{[1]}(t)/r(t)\right)^{\frac{1}{\gamma}}$. Integrating from T to t, we obtain

$$x(t) = x(T) + \int_{T}^{t}\left(\frac{1}{r(s)}x^{[1]}(s)\right)^{\frac{1}{\gamma}}\Delta s, \qquad \text{for} \quad t \geq T.$$

Taking into account that $x^{[1]}(t)$ is positive and decreasing, we get

$$x(t) \geq x(T) + \left(x^{[1]}(t)\right)^{\frac{1}{\gamma}}\int_{T}^{t}\left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}\Delta s, \qquad \text{for} \quad t \geq T.$$

It follows that

$$w(t) = \frac{x^{[1]}(t)}{x^{\gamma}(t)} \leq \left(\int_{t_0}^{t}\left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}\Delta s\right)^{-\gamma}, \qquad \text{for} \quad t \in [T,\infty)_{\mathbb{T}},$$

which implies using (1.2) that $\lim_{t\to\infty} w(t) = 0$. Integrating (2.14) from T to ∞ and using $\lim_{t\to\infty} w(t) = 0$, we obtain $w(T) \geq \int_T^\infty Q(s)\Delta s$, which contradicts (2.13). The proof is complete. □

In the following, we consider the case when

$$\int_{t_0}^{\infty} Q(s)\Delta s < \infty. \tag{2.15}$$

Theorem 2.3. *Assume that* (h_1), (h_2) *and* (1.2) *hold. Furthermore assume that there exists a positive rd-continuous* Δ*-differentiable function* $\phi(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\phi(s)Q(s) - \frac{r(s)((\phi^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)}\right] \Delta s = \infty, \tag{2.16}$$

then every solution of (1.1) *oscillates.*

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.1) and let $t_1 \geq t_0$ be such that $x(t) \neq 0$ for all $t \geq t_1$. Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (1.1) with $x(t) > 0$, and $x(\tau(t)) > 0$ for all $t \geq T > t_0$ sufficiently large. Define the function $w(t)$ by the Riccati substitution (2.2) and proceeding as in the proof of Theorem 2.1 to get (2.3). From (2.3), we have

$$w^{\Delta}(t) \leq -Q(t) - \frac{\gamma}{r^{\frac{1}{\gamma}}(t)} \left(w^{\sigma}\right)^{\frac{\gamma+1}{\gamma}}, \qquad \text{for} \quad t \geq T. \tag{2.17}$$

Multiplying (2.17) by $\phi(s)$ and integrating from T to t $(t \geq T)$, we have

$$\int_T^t \phi(s)Q(s)\Delta s \leq -\int_T^t \phi(s)w^{\Delta}(s)\Delta s - \int_T^t \frac{\gamma\phi(s)}{r^{\frac{1}{\gamma}}(s)} \left(w^{\sigma}\right)^{\frac{\gamma+1}{\gamma}} \Delta s.$$

Using integration by parts, we get

$$\int_T^t \phi(s)Q(s)\Delta s \leq w(T)\phi(T) + \int_T^t \phi^{\Delta}(s)w^{\sigma}(s)\Delta s - \int_{t_1}^t \frac{\gamma\phi(s)}{r^{\frac{1}{\gamma}}(s)}\left(w^{\sigma}\right)^{\frac{\gamma+1}{\gamma}} \Delta s.$$

Setting $B = \phi^{\Delta}(s)$ and $A = \gamma\phi(s)r^{-1/\gamma}(s)$ and $u = w^{\sigma}$, and applying the inequality

$$Bu - Au^{\lambda} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^{\gamma}}, \tag{2.18}$$

we have

$$\int_T^t \phi(s)Q(s)\Delta s \le w(t_2)\phi(T) + \int_{t_1}^t \frac{r(s)(\phi^\Delta(s))^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1}\phi^\gamma(s)}\Delta s,$$

i. e.,

$$\int_{t_2}^t \left[\phi(s)Q(s) - \frac{r(s)(\phi^\Delta(s))^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1}\phi^\gamma(s)}\right]\Delta s < \phi(T)w(T),$$

which contradicts condition (2.16). Then every solution of (1.1) oscillates. The proof is complete. □

From Theorem 2.3, we can obtain different conditions for the oscillation of (1.1) by using different choices of $\phi(t)$. For instance, if $\phi(t) = t$, we have the following result.

Corollary 2.4. *Assume that* (h_1), (h_2) *and* (1.2) *holds. Furthermore, assume that*

$$\limsup_{t\to\infty} \int_{t_0}^t \left[sQ(s) - \frac{r(s)}{(\gamma+1)^{\gamma+1}s^\gamma}\right]\Delta s = \infty, \tag{2.19}$$

then every solution of (1.1) *oscillates.*

Example 1. Consider the following second-order advanced dynamic equation

$$x^{\Delta\Delta}(t) + \frac{\lambda\,(\sigma(t)-1)}{t^3}x(\tau(t)) = 0, \qquad \text{for} \quad t \in [2,\infty)_{\mathbb{T}}, \tag{2.20}$$

where $\mathbb{T}$ is a time scale such that $\int_1^\infty (\sigma(s)/s^3)\Delta s < \infty$. Here $\gamma = \beta = 1$, $\tau(t) > t$, $\lim_{t\to\infty} \tau(t) = \infty$, $r(t) = 1$, and

$$p(t) = \frac{\lambda\,(\sigma(t)-1)}{t^3},$$

where $\lambda > 0$ is a constant. Take any $T \ge 2$, and since $r(t) = 1$, we have $P(t,T) = P(t,T) = t - T$. This gives

$$Q(t) := p(t)\frac{P(t,T)}{P(t,T) + \sigma(t) - t} = \frac{\lambda\,(\sigma(t)-1)}{t^3}\frac{t-T}{\sigma(t)-T}.$$

It is easy to see that assumption (1.2) holds and also (2.15) is satisfied, since

$$\begin{aligned}\int_{t_0}^{\infty} Q(s)\Delta s &= \lambda \int_{t_0}^{\infty} \frac{(\sigma(s)-1)}{s^3}\frac{s-T}{\sigma(s)-T}\Delta s \leq \lambda \int_{2}^{\infty} \frac{(\sigma(s)-1)}{s^3}\Delta s \\ &< \lambda \int_{2}^{\infty} \frac{\sigma(s)}{s^3}\Delta s < \infty.\end{aligned}$$

To apply Corollary 2.4, it remains to discuss condition (2.19). Note

$$\begin{aligned}&\limsup_{t\to\infty} \int_{t_0}^{t} \left[sQ(s) - \frac{r(s)}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right]\Delta s \\ &= \limsup_{t\to\infty} \int_{2}^{t} \left(\frac{\lambda s\,(\sigma(s)-1)}{s^3}\frac{s-T}{\sigma(s)-T} - \frac{1}{4s}\right)\Delta s \\ &> \limsup_{t\to\infty} \int_{t}^{t} \left(\frac{\lambda s^2}{s^3} - \frac{T}{2s^2\,(s-1)} - \frac{1}{4s}\right)\Delta s = \infty,\end{aligned}$$

provided that $\lambda > 1/4$. Hence, by Corollary 2.4 every solution of (2.20) oscillates if $\lambda > 1/4$.

Another method of choosing test functions can be developed by considering the function class $\Re$ which consists of kernels of two variables. Following Saker [31], we say that the function $H \in \Re$ provided H is defined for $t_0 \leq s \leq t$, $t, s \in [t_0,\infty)_{\mathbb{T}}$ $H(t,s) \geq 0$, $H(t,t) = 0$ for $t \geq s \geq t_0$, and for each fixed t, $H^{\Delta_i}(t,s)$ is delta integrable with respect to variable i ($i = 1,2$). Important examples of H when $\mathbb{T} = \mathbb{R}$ are $H(t,s) = (t-s)^m$ for $m \geq 1$. When $\mathbb{T} = \mathbb{Z}$, $H(t,s) = (t-s)^{\underline{k}}$, $k \in \mathbb{N}$, where $t^{\underline{k}} = t(t-1)\dots(t-k+1)$.

The following theorem gives new oscillation criteria for (1.1) which can be considered as the extension of Kamenev-type oscillation criterion. The proof is similar to that of the proof in [31, Theorem 3.3], if one uses the inequality (2.3) and hence is omitted.

Theorem 2.5. *Assume that* (h_1), (h_2) *and* (1.2) *hold. Let* $\phi(t)$ *be defined as in* Theorem 2.3, $H \in \Re$, *and for* $t > s$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\phi(s)Q(s) - \frac{r(s)(\phi^{\Delta}(s))^{\gamma+1}(H^{\Delta_s}(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)H^{\gamma}(t,s)}\right]\Delta s = \infty. \tag{2.21}$$

Then every solution of (1.1) *oscillates.*

Remark 1. When $\mathbb{T} = \mathbb{R}$, Theorem 2.5 reduces to the result established in [36, Theorem 2.1].

With appropriate choices of the functions H one can present a number of oscillation criteria for (1.1) on different types of time scales. For instance if there exists a function $h(t,s) \in \Re$ such that

$$H^{\Delta_s}(t,s) := -h(t,s)H^{\frac{\gamma}{1+\gamma}}(t,s),$$

we have from Theorem 2.5 the following oscillation result.

Corollary 2.6. *Assume that* (h_1), (h_2) *and* (1.2) *hold. Let* $\phi(t)$ *be defined as in* Theorem 2.3, $H \in \Re$, *and for* $t > s$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\phi(s)Q(s) - \frac{r(s)((\phi^{\Delta}(s))^{\gamma+1}(h(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)} \right] \Delta s = \infty,$$

then every solution of equation (1.1) *is oscillatory.*

As a special case by choosing $H(t,s) = (t-s)^m$ for $m \geq 1$, we have from Corollary 2.6 the following Kamenev-type oscillation criterion.

Corollary 2.7. *Assume that* (h_1), (h_2) *and* (1.2) *hold. If for* $m > 1$

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} \left[(t-s)^m Q(s) - \frac{m^{\gamma+1}r(s)((t-s)^{m-1})^{\gamma+1}}{(\gamma+1)^{\gamma+1}(t-s)^{m\gamma}} \right] \Delta s = \infty,$$

then every solution of (1.1) *oscillates.*

For oscillation of the second order differential equation

$$x''(t) + p(t)x(t) = 0, \tag{2.22}$$

Hille [25] proved that every solution of (2.22) oscillates if

$$\liminf_{t\to\infty} t \int_{t}^{\infty} p(s)\,\mathrm{d}s > \frac{1}{4}. \tag{2.23}$$

Nehari [27] by a different approach proved that if

$$\liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} s^2 p(s)\,\mathrm{d}s > \frac{1}{4}, \tag{2.24}$$

then every solution of (2.22).

In the following, we extend these results and establish new oscillation criteria of Hille and Nehari types for (1.1). We will use the following notations:

$$\begin{aligned} p_* &:= \liminf_{t\to\infty} \frac{t^\gamma}{r(t)} \int_{\sigma(t)}^{\infty} Q(s)\Delta s, q_* := \liminf_{t\to\infty} \frac{1}{t} \int_{T}^{t} \frac{s^{\gamma+1}}{r(t)} Q(s)\Delta s, \\ l &:= \liminf_{t\to\infty} \frac{t}{\sigma(t)}. \end{aligned}$$

Theorem 2.8. *Assume that* (h_1), (h_2), (1.2) *hold and* $r^\Delta \geq 0$. *Let* x *be a positive solution of* (1.1). *Define*

$$r_* := \liminf_{t\to\infty} \frac{t^\gamma w^\sigma(t)}{r(t)}, \qquad R := \limsup_{t\to\infty} \frac{t^\gamma w^\sigma(t)}{r(t)},$$

where w *is defined as in* (2.2). *Then*

$$p_* \leq r_* - l^\gamma r_*^{1+\frac{1}{\gamma}}, \tag{2.25}$$

and

$$p_* + q_* \leq \frac{1}{l^{\gamma(\gamma+1)}}. \tag{2.26}$$

P r o o f. Let x be as above and without loss of generality, we assume that there is $T > t_0$ such that $x(t) > 0$, and $x(\tau(t)) > 0$ for $t \geq T$ where T is chosen large enough. From Lemma 2.1, we know that x satisfies $x^{[1]}(t) > 0$ and $x^{[2]}(t) < 0$, for $t \geq T$. From Theorem 2.1, we get from (2.3) that

$$-w^\Delta(t) \geq Q(t) + \frac{\gamma}{r^{\frac{1}{\gamma}}(t)} (w^\sigma(t))^{\frac{\gamma+1}{\gamma}}, \qquad \text{for} \quad t \geq T. \tag{2.27}$$

First, we prove (2.25). Integrating (2.27) from $\sigma(t)$ to ∞ and using $\lim\limits_{t\to\infty} w(t) = 0$, (see Theorem 2.2) we obtain

$$w^\sigma(t) \geq \int_{\sigma(t)}^{\infty} Q(s)\Delta s + \gamma \int_{\sigma(t)}^{\infty} \frac{(w^\sigma(s))^{\frac{1}{\gamma}} w^\sigma(s)\Delta s}{r^{\frac{1}{\gamma}}(s)}, \qquad \text{for} \quad t \geq T. \tag{2.28}$$

It follows from (2.28) that

$$\frac{t^\gamma w^\sigma(t)}{r(t)} \geq \frac{t^\gamma}{r(t)} \int_{\sigma(t)}^{\infty} Q(s)\Delta s + \frac{\gamma t^\gamma}{r(t)} \int_{\sigma(t)}^{\infty} \frac{(w^\sigma(s))^{\frac{1}{\gamma}} w^\sigma(s)\Delta s}{r^{\frac{1}{\gamma}}(s)}, \qquad \text{for} \quad t \geq T. \tag{2.29}$$

Let ε be a sufficiently small positive quantity, then by the definition of r_* and r_* we can pick $T_1 \in [T,\infty)_{\mathbb{T}}$, sufficiently large, so that

$$\frac{t^\gamma}{r(t)}\int_{\sigma(t)}^{\infty} Q(s)\Delta s \geq p_* - \varepsilon, \quad \text{and} \quad \frac{t^\gamma w^\sigma(t)}{r(t)} \geq r_* - \varepsilon, \qquad \text{for} \quad t \geq T_1. \tag{2.30}$$

From (2.29) and (2.30) and using the fact that $r^\Delta \geq 0$, it follows that

$$\begin{aligned}
\frac{t^\gamma w^\sigma(t)}{r(t)} &\geq (p_* - \varepsilon) + \gamma \frac{t^\gamma}{r(t)} \int_{\sigma(t)}^{\infty} \frac{s\left(w^\sigma(s)\right)^{\frac{1}{\gamma}} s^\gamma w^\sigma(s)}{r^{\frac{1}{\gamma}}(s) s^{\gamma+1}} \Delta s \\
&\geq (p_* - \varepsilon) + (r_* - \varepsilon)^{1+\frac{1}{\gamma}} \frac{t^\gamma}{r(t)} \int_{\sigma(t)}^{\infty} \frac{\gamma r(s)}{s^{\gamma+1}} \Delta s \\
&\geq (p_* - \varepsilon) + (r_* - \varepsilon)^{1+\frac{1}{\gamma}}\, t^\gamma \int_{\sigma(t)}^{\infty} \frac{\gamma \Delta s}{s^{\gamma+1}}, \qquad \text{for} \quad t \geq T_1.
\end{aligned} \tag{2.31}$$

Using the Pötzsche chain rule ([10, Theorem 1.90]), we see that

$$\left(\frac{-1}{s^\gamma}\right)^{\Delta} = \gamma \int_0^1 \frac{1}{[s + h\mu(s)]^{\gamma+1}}\, \mathrm{d}h \leq \int_0^1 \left(\frac{\gamma}{s^{\gamma+1}}\right) \mathrm{d}h = \frac{\gamma}{s^{\gamma+1}}.$$

This implies that

$$\int_{\sigma(t)}^{\infty} \frac{\gamma}{s^{\gamma+1}} \Delta s \geq \int_{\sigma(t)}^{\infty} \left(\frac{-1}{s^\gamma}\right)^{\Delta} \Delta s = \frac{1}{\sigma^\gamma(t)}. \tag{2.32}$$

Then from (2.31) and (2.32), we have

$$\frac{t^\gamma w^\sigma(t)}{r(t)} \geq (p_* - \varepsilon) + (r_* - \varepsilon)^{1+\frac{1}{\gamma}} \left(\frac{t}{\sigma(t)}\right)^{\gamma}.$$

Taking the lim inf of both sides as $t \to \infty$, we have $r_* \geq p_* - \varepsilon + (r_* - \varepsilon)^{1+\frac{1}{\gamma}}\, l^\gamma$. Since $\varepsilon > 0$ is arbitrary, we get

$$p_* \leq r_* - r_*^{1+\frac{1}{\gamma}} l^\gamma, \tag{2.33}$$

and this completes the proof of (2.25). Next, we prove (2.26). Multiplying both sides of (2.27) by $t^{\gamma+1}/r(t)$, and integrating from T to t $(t \geq T)$, we get

$$\int_T^t \frac{s^{\gamma+1}}{r(s)} w^\Delta(s)\Delta s \leq -\int_T^t \frac{s^{\gamma+1}}{r(s)} Q(s)\Delta s - \gamma \int_T^t \left(\frac{s^\gamma w^\sigma(s)}{r(s)}\right)^{\frac{\gamma+1}{\gamma}} \Delta s.$$

Using integration by parts, we obtain

$$\begin{aligned}\frac{t^{\gamma+1}w(t)}{r(t)} &\leq \frac{T^{\gamma+1}w(T)}{r(T)}+\int_T^t\left(\frac{s^{\gamma+1}}{r(s)}\right)^{\Delta}w^{\sigma}(s)\Delta s\\ &\quad -\int_T^t\frac{s^{\gamma+1}Q(s)\Delta s}{r(s)}-\gamma\int_T^t\left(\frac{s^{\gamma}w^{\sigma}(s)}{r(s)}\right)^{\frac{\gamma+1}{\gamma}}\Delta s.\end{aligned}$$

By the quotient rule and applying the Pötzsche chain rule, we see that

$$\left(\frac{s^{\gamma+1}}{r(s)}\right)^{\Delta}=\frac{(s^{\gamma+1})^{\Delta}}{r^{\sigma}(s)}-\frac{s^{\gamma+1}r^{\Delta}(s)}{r(s)r^{\sigma}(s)}\leq\frac{(\gamma+1)\sigma^{\gamma}(s)}{r^{\sigma}(s)}\leq\frac{(\gamma+1)\sigma^{\gamma}(s)}{r(s)}, \tag{2.34}$$

since $r^{\Delta}(t)\geq 0$. This leads to

$$\begin{aligned}\frac{t^{\gamma+1}w(t)}{r(t)} &\leq \frac{T^{\gamma+1}w(T)}{r(T)}-\int_T^t\frac{s^{\gamma+1}}{r(s)}Q(s)\Delta s\\ &\quad +\int_T^t(\gamma+1)\left(\frac{\sigma^{\gamma}(s)w^{\sigma}(s)}{r(s)}\right)\Delta s-\gamma\int_T^t\left(\frac{s^{\gamma}w^{\sigma}(s)}{r(s)}\right)^{\frac{\gamma+1}{\gamma}}\Delta s,\end{aligned}$$

for $t\geq T$. Let $\varepsilon>0$ be given, then using the definition of l, we can assume, without loss of generality, that T is sufficiently large so that $\frac{s}{\sigma(s)}>l-\varepsilon$, $s\geq T$. It follows that $\sigma(s)\leq Ks$, $s\geq T$ where $K:=\frac{1}{l-\varepsilon}>1$. Then we get that

$$\begin{aligned}&\frac{t^{\gamma+1}w(t)}{r(t)}-\frac{T^{\gamma+1}w(T)}{r(T)}\\ &\leq -\int_T^t\frac{s^{\gamma+1}}{r(s)}Q(s)\Delta s+\int_T^t\left\{(\gamma+1)K^{\gamma}\frac{s^{\gamma}w^{\sigma}(s)}{r(s)}-\gamma\left(\frac{s^{\gamma}w^{\sigma}(s)}{r(s)}\right)^{\frac{\gamma+1}{\gamma}}\right\}\Delta s,\\ &\qquad\text{for}\quad t\geq T.\end{aligned}$$

Let $u(s):=s^{\gamma}w^{\sigma}(s)/r(s)$, so $u^{\frac{\gamma+1}{\gamma}}(s)=(s^{\gamma}w^{\sigma}(s)/r(s))^{\frac{\gamma+1}{\gamma}}$. It follows that

$$\frac{t^{\gamma+1}w(t)}{r(t)}\leq\frac{T^{\gamma+1}w(T)}{r(T)}-\int_T^t\frac{s^{\gamma+1}Q(s)\Delta s}{r(s)}+\int_T^t\left\{(\gamma+1)K^{\gamma}u(s)-\gamma u^{\lambda}(s)\right\}\Delta s.$$

Using the inequality

$$Bu-Au^{\frac{\gamma+1}{\gamma}}\leq\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^{\gamma}},$$

where A, B are positive constants, we get

$$\begin{aligned}\frac{t^{\gamma+1}w(t)}{r(t)} &\le \frac{T^{\gamma+1}w(T)}{r(T)} - \int\limits_T^t \frac{s^{\gamma+1}}{r(s)}Q(s)\Delta s + \int\limits_T^t \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\frac{[(\gamma+1)K^\gamma]^{\gamma+1}}{\gamma^\gamma}\Delta s \\ &\le \frac{T^{\gamma+1}w(T)}{r(T)} - \int\limits_T^t \frac{s^{\gamma+1}}{r(s)}Q(s)\Delta s + K^{\gamma(\gamma+1)}(t-T), \qquad \text{for} \quad t \ge T.\end{aligned}$$

It follows from this that

$$\frac{t^\gamma w(t)}{r(t)} \le \frac{T^{\gamma+1}w(T)}{tr(T)} - \frac{1}{t}\int\limits_T^t \frac{s^{\gamma+1}Q(s)\Delta s}{r(s)} + K^{\gamma(\gamma+1)}\left(1-\frac{T}{t}\right), \qquad \text{for} \quad t \ge T.$$

From (2.3), we see that $w(t)$ is nonincreasing and this implies that $w^\sigma \le w$, where $\sigma(t) \ge t$. This gives us that

$$\frac{t^\gamma w^\sigma(t)}{r(t)} \le \frac{T^{\gamma+1}w(T)}{tr(T)} - \frac{1}{t}\int\limits_T^t \frac{s^{\gamma+1}Q(s)\Delta s}{r(s)} + K^{\gamma(\gamma+1)}\left(1-\frac{T}{t}\right), \qquad \text{for} \quad t \ge T.$$

Taking the lim sup of both sides as $t \to \infty$, we obtain

$$R \le -q_* + K^{\gamma(\gamma+1)} = -q_* + \frac{1}{(l-\varepsilon)^{\gamma(\gamma+1)}}.$$

Since $\varepsilon > 0$ is arbitrary, we get that $R \le -q_* + \left(1/l^{\gamma(\gamma+1)}\right)$. Using this and inequality (2.33), we get

$$p_* \le r_* - l^\gamma r_*^{1+\frac{1}{\gamma}} \le r_* \le R \le -q_* + \frac{1}{l^{\gamma(\gamma+1)}}.$$

Therefore $p_* + q_* \le 1/l^{\gamma(\gamma+1)}$ and this completes the proof of (2.26). The proof is complete. □

From Theorem 2.8, we have the following result immediately.

Theorem 2.9. *Assume that* (h_1), (h_2), (1.2) *hold and* $r^\Delta \ge 0$. *Furthermore, assume that*

$$r_* = \liminf_{t\to\infty} \frac{t^\gamma}{r(t)} \int\limits_{\sigma(t)}^\infty Q(s)\Delta s > \frac{\gamma^\gamma}{l^{\gamma^2}(\gamma+1)^{\gamma+1}}. \tag{2.35}$$

Then every solution of (1.1) *oscillates.*

P r o o f. Suppose to the contrary and assume that x is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $x(t) > 0$, and $x(\tau(t)) > 0$ for $t \ge T$ where T is chosen large enough. We consider only this case, because the proof when $x(t) < 0$ is similar. Let w and r_* be as defined in Theorem 2.8. Then from Theorem 2.8, we see that r_*,satisfies the inequality

$$r_* \leq r_* - l^\gamma r_*^{\frac{\gamma+1}{\gamma}}.$$

Applying (2.18) on the last inequality, we have

$$r_* \leq \frac{\gamma^\gamma}{l^{\gamma^2}(\gamma+1)^{\gamma+1}},$$

which contradicts (2.35). This completes the proof. □

We also have as a consequence of Theorem 2.8 the following oscillation result.

Theorem 2.10. *Assume that* (h_1), (h_2), (1.2) *hold and* $r^\Delta \geq 0$. *Furthermore, assume that*

$$\liminf_{t\to\infty} \frac{t^\gamma}{r(t)} \int_{\sigma(t)}^{\infty} Q(s)\Delta s + \liminf_{t\to\infty} \frac{1}{t} \int_T^t \frac{s^{\gamma+1}}{r(t)} Q(s)\Delta s > \frac{1}{l^{\gamma(\gamma+1)}}. \tag{2.36}$$

Then every solution of (1.1) *oscillates.*

Now, we assume that (1.3) holds and present some sufficient conditions which ensure that every solution $x(t)$ of (1.1) oscillates or converges to zero when $\tau(t) > t$. The proof is similar to the proof of [31, Theorem 3.3] and hence is omitted.

Theorem 2.11. *Assume that* (h_1), (h_2), (1.3) *hold and* $\tau(t) > t$. *Furthermore assume that*

$$\int_{t_0}^{\infty} \left[\frac{1}{r(t)} \int_{t_0}^{t} p(s)\Delta s\right]^{\frac{1}{\gamma}} \Delta t = \infty. \tag{2.37}$$

hold. If one of the conditions (2.13), (2.16), (2.21), (2.35) *and* (2.36) *holds, then every solution of* (1.1) *oscillates or converges to zero.*

2.2. The case when $\tau(t) \leq t$

In this subsection, we establish some sufficient conditions for oscillation of (1.1) when $\tau(t) \leq t$. For the delay case we will use the following notation:

$$A(t) := p(t)\delta^\beta(t)\eta(\sigma(t)),$$

where $\eta^\sigma(t)$ is defined as in (2.1), and

$$\delta(t) := \frac{r^{\frac{1}{\gamma}}(t) P(\tau(t), T)}{r^{\frac{1}{\gamma}}(t) P(t, T) + \mu(t)},$$

$$P(t,T) := \int_{\vartheta}^{u} \frac{1}{r^{\frac{1}{\gamma}}(s)} \Delta s, \quad P(\tau(t), T) := \int_T^{\tau(t)} \frac{1}{r^{\frac{1}{\gamma}}(s)} \Delta s.$$

Theorem 2.12. *Assume that* (h_1), (h_2) *and* (1.2) *hold. Let* x *be a nonoscillatory solution of* (1.1) *and* w *is defined as in* (2.2). *Then there exists* $T > t_0$ *such that* $w(t) > 0$ *for* $t > T$ *and*

$$w^{\Delta}(t) + A(t) + \gamma \frac{1}{r^{\frac{1}{\gamma}}(t)} \left(w^{\sigma}\right)^{1+\frac{1}{\gamma}}(t) \leq 0, \qquad \textit{for} \quad t \geq T. \tag{2.38}$$

P r o o f. Let x be as above and without loss of generality we assume that there is $T > t_0$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq T$. From the definition of $w(t)$, and as in the proof of Theorem 2.1, we get

$$w^{\Delta}(t) \leq -p(t) \frac{\left(x^{\tau}(t)\right)^{\beta}}{\left(x^{\sigma}(t)\right)^{\gamma}} - \gamma \frac{1}{r^{\frac{1}{\gamma}}(t)} \left(w^{\sigma}(t)\right)^{1+\frac{1}{\gamma}}, \qquad \text{for} \quad t \geq T. \tag{2.39}$$

Now, we consider the coefficient of $p(t)$ in (2.39). Since $x^{[1]}(t) = r\left(x^{\Delta}\right)^{\gamma}(t)$ is decreasing for $t \geq T$, then we have

$$x^{\sigma}(t) - x(\tau(t)) = \int\limits_{\tau(t)}^{\sigma(t)} \frac{x^{[1]}(s)}{r^{\frac{1}{\gamma}}(s)} \Delta s \leq x^{[1]}(\tau(t)) \int\limits_{\tau(t)}^{\sigma(t)} \frac{1}{r^{\frac{1}{\gamma}}(s)} \Delta s,$$

and this implies that

$$\frac{x^{\sigma}(t)}{x(\tau(t))} \leq 1 + \frac{x^{[1]}(\tau(t))}{x(\tau(t))} \int\limits_{\tau(t)}^{\sigma(t)} \frac{1}{r^{\frac{1}{\gamma}}(s)} \Delta s. \tag{2.40}$$

On the other hand, we have that

$$x(\tau(t)) > x(\tau(t)) - x(T) = \int\limits_{T}^{\tau(t)} \frac{x^{[1]}(s)}{r^{\frac{1}{\gamma}}(s)} \Delta s \geq (x^{[1]})(\tau(t)) \int\limits_{T}^{\tau(t)} \frac{1}{r^{\frac{1}{\gamma}}(s)} \Delta s,$$

which leads to

$$\frac{x^{[1]}(\tau(t))}{x(\tau(t))} < \left(\int\limits_{T}^{\tau(t)} \frac{1}{r^{\frac{1}{\gamma}}(s)} \Delta s \right)^{-1},$$

Using this inequality and (2.40) we get that

$$\frac{x^{\sigma}(t)}{x(\tau(t))} \quad < \quad 1 + \frac{\int_{\tau(t)}^{\sigma(t)} r^{-\frac{1}{\gamma}}(s) \Delta s}{\int_{T}^{\tau(t)} r^{-\frac{1}{\gamma}}(s) \Delta s} = \frac{\int_{T}^{\sigma(t)} r^{-\frac{1}{\gamma}}(s) \Delta s}{\int_{T}^{\tau(t)} r^{-\frac{1}{\gamma}}(s) \Delta s}$$

$$= \frac{\int_T^t r^{-\frac{1}{\gamma}}(s)\Delta s + \int_t^{\sigma(t)} r^{-\frac{1}{\gamma}}(s)\Delta s}{\int_T^{\tau(t)} r^{-\frac{1}{\gamma}}(s)\Delta s}$$

$$= \frac{\int_T^t r^{-\frac{1}{\gamma}}(s)\Delta s + \mu(t) r^{-\frac{1}{\gamma}}(t)}{\int_T^{\tau(t)} r^{-\frac{1}{\gamma}}(s)\Delta s} = \frac{1}{\delta(t)}, \qquad \text{for} \quad t \geq T,$$

where we used the fact that, $\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t)$. Hence, we get $x(\tau(t)) \geq \delta(t)x^{\sigma}(t)$, for $t \geq T$. This implies that

$$\frac{(x^{\tau}(t))^{\beta}}{(x^{\sigma}(t))^{\gamma}} \geq (\delta(t))^{\beta} (x^{\sigma}(t))^{\beta-\gamma}, \qquad \text{for} \quad t \geq T.$$

As in the proof of Theorem 2.1, we can obtain

$$\frac{(x^{\tau}(t))^{\beta}}{(x^{\sigma}(t))^{\gamma}} \geq (\delta(t))^{\beta} \eta^{\sigma}(t), \qquad \text{for} \quad t \geq T. \tag{2.41}$$

Substituting (2.41) into (2.39), we have the inequality (2.38) and this completes the proof. □

The proofs of the following theorems are similar to the proofs of theorems in the subsection 2.1, by using the inequality (2.38) and hence are omitted.

Theorem 2.13 (**Leighton-Wintner type**). *Assume that* (h_1), (h_2) *and* (1.2) *hold. Furthermore, assume that*

$$\int_{t_0}^{\infty} A(s)\Delta s = \infty. \tag{2.42}$$

Then every solution of (1.1) *oscillates.*

In the following, we consider the case when

$$\int_{t_0}^{\infty} A(s)\Delta s < \infty. \tag{2.43}$$

Theorem 2.14. *Assume that* (h_1), (h_2) *and* (1.2) *hold. Furthermore assume that there exist positive rd-continuous* Δ*-differentiable function* $\phi(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\phi(s)A(s) - \frac{r(s)((\phi^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)}\right] \Delta s = \infty, \tag{2.44}$$

then every solution of (1.1) *oscillates.*

Remark 2. When $\mathbb{T} = \mathbb{Z}$, Theorem 2.14 improve Theorem 4 that has been established by Cheng [24], in the sense that our results do require the condition $\Delta r_n \geq 0$ which is the case considered in [24].

Theorem 2.15. *Assume that* (h_1), (h_2) *and* (1.2) *hold. Let* $\phi(t)$ *be defined as in* Theorem 2.3, $H \in \Re$, *and for* $t > s$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\phi(s)A(s) - \frac{r(s)(\phi^{\Delta}(s))^{\gamma+1}(H^{\Delta_s}(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)H^{\gamma}(t,s)}\right] \Delta s = \infty. \tag{2.45}$$

Then every solution of (1.1) *oscillates.*

Remark 3. When $\mathbb{T} = \mathbb{R}$, Theorem 2.15 reduces to the result that has been established in [36, Theorem 2.1] when $\tau(t) = t$.

Corollary 2.16. *Assume that* (h_1), (h_2) *and* (1.2) *hold. Let* $\phi(t)$ *be defined as in* Theorem 2.3, $H \in \Re$, *and for* $t > s$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\phi(s)A(s) - \frac{r(s)((\phi^{\Delta}(s))^{\gamma+1}(h(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)}\right] \Delta s = \infty,$$

then every solution of equation (1.1) *oscillates.*

Corollary 2.17. *Assume that* (h_1), (h_2) *and* (1.2) *hold. If for* $m > 1$

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} \left[(t-s)^m A(s) - \frac{m^{\gamma+1}r(s)((t-s)^{m-1})^{\gamma+1}}{(\gamma+1)^{\gamma+1}(t-s)^{m\gamma}}\right] \Delta s = \infty,$$

then every solution of (1.1) *oscillates.*

In the following, we present some oscillation criteria of Hille and Nehari types for the delay case of (1.1). The proofs are similar to the proofs of Theorems 2.8, 2.9 and 2.10 and hence are omitted.

Theorem 2.18. *Assume that* (h_1), (h_2), (1.2) *hold and* $r^{\Delta} \geq 0$. *Furthermore, assume that*

$$\liminf_{t\to\infty} \frac{t^{\gamma}}{r(t)} \int_{\sigma(t)}^{\infty} A(s)\Delta s > \frac{\gamma^{\gamma}}{l^{\gamma^2}(\gamma+1)^{\gamma+1}}. \tag{2.46}$$

Then every solution of (1.1) *oscillates.*

Theorem 2.19. *Assume that* (h_1), (h_2), (1.2) *hold and* $r^{\Delta} \geq 0$. *Furthermore, assume that*

$$\liminf_{t\to\infty} \frac{t^{\gamma}}{r(t)} \int_{\sigma(t)}^{\infty} A(s)\Delta s + \liminf_{t\to\infty} \frac{1}{t} \int_{T}^{t} \frac{s^{\gamma+1}}{r(s)} A(s)\Delta s > \frac{1}{l^{\gamma(\gamma+1)}}. \tag{2.47}$$

Then every solution of (1.1) *oscillates.*

In the following, we assume that (1.3) holds and present some sufficient conditions which ensure that every solution $x(t)$ of (1.1) oscillates or converges to zero when $\tau(t) \leq t$.

Theorem 2.20. *Assume that* (1.3) *holds and* $\tau(t) \leq t$. *Furthermore assume that* (2.37) *holds. If one of the conditions* (2.42), (2.44), (2.45), (2.46) *and* (2.47) *holds, then every solution of* (1.1) *oscillates or converges to zero.*

Example 2. Consider the differential equation

$$\left(\frac{1}{t^3}\left(x'(t)\right)^3\right)' + \frac{\beta}{t^3} x^3(t) = 0, \qquad \text{for} \quad t \in [1,\infty)_{\mathbb{R}} \tag{2.48}$$

where $\mathbb{R}$ is the real line. Here $r(t) = 1/t^3$ and $p(t) = \lambda/t^3$ and $\gamma = \beta = 3$, $\tau(t) = t$ and $\lambda > 0$ is a constant. Note that $\int_1^{\infty} (1/r^{\frac{1}{\gamma}}(s))\Delta s = \int_1^{\infty} s \,\mathrm{d}s = \infty$. Then (1.2) is satisfied. Since $r(t) = t^3$, and $\tau(t) = t$, we have $\delta(t) = 1$ and then $A(t) = p(t) = \lambda/t^3$. It is clear that (2.43) is satisfied, since

$$\int_{t_0}^{\infty} A(s)\Delta s = \lambda \int_{t_0}^{\infty} \frac{1}{s^3} \,\mathrm{d}s < \infty.$$

To apply Theorem 2.14, it remains to discuss condition (2.44). Note, if we choose $\phi(t) = t^2$, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \int_{t_0}^{t} \left[\phi(s)A(s) - \frac{r(s)((\phi^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)}\right] \Delta s \\
&= \limsup_{t\to\infty} \int_{t_0}^{t} \left[s^2 \frac{\lambda}{s^3} - \frac{(2s)^4}{(4)^4 s^6}\right] \mathrm{d}s = \infty.
\end{aligned}$$

Then by Theorem 2.14, every solution of (2.48) is oscillatory if $\lambda > 0$.

Remark 4. We note that the results obtained for half-linear dynamic equations in the literature, see the introduction, cannot be applied on (2.48) since $r^{\Delta}(t) < 0$.

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