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BOCHNER ALGEBRAS AND THEIR COMPACT MULTIPLIERS

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ABSTRACT. Let $\mathfrak{A}$ be a normed algebra with identity, Ω be a locally compact Hausdorf space and λ be a positive Radon measure on Ω with $\operatorname{supp}(\lambda) = \Omega$. In this paper, we establish a necessary and sufficient condition for $L^1(\Omega, \mathfrak{A})$ to be an algebra with pointwise multiplication. Under this condition, we then characterize compact and weakly compact left multipliers on $L^1(\Omega, \mathfrak{A})$.

1. Introduction and preliminaries

Let Ω be a locally compact Hausdorff space and λ be a positive Radon measure on Ω constructed from a fixed positive functional on $C_c(\Omega)$, the space of all continuous complex valued functions on Ω with compact support; see [4] and [11] for more details. We also assume that the support of λ coincides with Ω. For a nontrivial normed algebra $\mathfrak{A}$, define $L^1(\Omega, \mathfrak{A})$ to be the space of all Bochner integrable functions $f\colon \Omega \to \mathfrak{A}$. For each two functions f, g from Ω into $\mathfrak{A}$, the pointwise multiplication of f and g is denoted by $f \cdot g$. There has been considerable progress in the study of analysis on $L^1(\Omega, \mathfrak{A})$, and several authors have studied various aspects of the subject; see for example [3], [5], [6], [10], [12] [14] and [15].

Our purpose in this work is to study compact and weakly compact left multipliers on $L^1(\Omega, \mathfrak{A})$. To study multipliers on $L^1(\Omega, \mathfrak{A})$, we first need to know when $L^1(\Omega, \mathfrak{A})$ is an algebra with pointwise multiplication. Our first result gives

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a necessary and sufficient condition for that $L^1(\Omega,\mathfrak{A})$ to be an algebra with pointwise multiplication.

Proposition 1.1. *The space $L^1(\Omega,\mathfrak{A})$ is an algebra with pointwise multiplication if and only if Ω is discrete and*

$$\inf\big\{\lambda(O):\ O\ \textit{ is a nonempty open set in }\ \Omega\big\}>0.$$

P r o o f. Let Ω be discrete and M be a positive constant with $\lambda(\{\omega\})\geq M$ for all $\omega\in\Omega$. Then

$$\int\limits_{\Omega}\|f(\omega)\|\,\mathrm{d}\lambda(\omega)=\sum_{\omega\in\Omega}\|f(\omega)\|\,\lambda(\{\omega\})$$

for all $f\in L^1(\Omega,\mathfrak{A})$. So, for each $f,g\in L^1(\Omega,\mathfrak{A})$ we conclude that

$$\begin{aligned}\|fg\|_1 &\leq \sum_{\omega\in\Omega}\|f(\omega)\|\,\|g(\omega)\|\,\lambda(\{\omega\})\\ &\leq M^{-1}\left(\sum_{\omega\in\Omega}\|f(\omega)\|\,\lambda(\{\omega\})\right)\left(\sum_{\omega\in\Omega}\|g(\omega)\|\,\lambda(\{\omega\})\right)\\ &= M^{-1}\,\|f\|_1\,\|g\|_1<\infty.\end{aligned}$$

Conversely, toward a contradiction suppose that Ω is not discrete or

$$\inf\big\{\lambda(O):\ O\ \text{ is a nonempty open set in }\ \Omega\big\}=0.$$

First, note that there exists a sequence (O_n) of relatively compact open subsets of Ω with $\lambda(O_n)<n^{-2}$; indeed, if Ω is discrete, then

$$\inf\big\{\lambda(\{\omega\}):\ \ \omega\in\Omega\big\}=0$$

whence there is a sequence (ω_n) of disjoint elements of Ω with $\lambda(\{\omega_n\})<n^{-2}$. So, the sets $O_n:=\{\omega_n\}$ are the required sequence. Now, if Ω is non-discrete, then there exists a compact set $O_1\subseteq\Omega$ with $\lambda(O_1)<1$. Choose an open set $O_2\subseteq\Omega\setminus\overline{O_1}$ with $\lambda(O_2)<2^{-2}$. Inductively, we may find a sequence (O_n) of open subsets in Ω such that

$$O_n\subseteq\Omega\setminus(\overline{O_1}\cup\cdots\cup\overline{O_{n-1}})\qquad\text{and}\qquad\lambda(O_n)<n^{-2},$$

Next, let $a\in\mathfrak{A}$ be a fixed nonzero element and set

$$g(\omega)=\sum_{n=1}^{\infty}n^{-1-(1/4)}\,\lambda(O_n)^{-1}\chi_{O_n}(\omega)\,a\qquad\text{for all}\quad\omega\in\Omega.$$

Clearly, $g \in L^1(\Omega, \mathfrak{A})$; but

$$\begin{aligned}\int\limits_\Omega \|g(\omega)^2\| \, \mathrm{d}\lambda(\omega) &= \sum_{n=1}^{\infty} \int\limits_{O_n} n^{-2-1/2} \, \lambda(O_n)^{-2} \|a^2\| \, \mathrm{d}\lambda(\omega) \\ &= \|a^2\| \sum_{n=1}^{\infty} n^{-2-1/2} \, \lambda(O_n)^{-1} \\ &\geq \|a^2\| \sum_{n=1}^{\infty} n^{-1/2}.\end{aligned}$$

It follows that $g^2 \notin L^1(\Omega, \mathfrak{A})$; this is a contradiction to that $L^1(\Omega, \mathfrak{A})$ is an algebra with pointwise multiplication. $\square$

Note that $L^1(\Omega, \mathfrak{A})$ is a normed algebra if Ω is discrete and $\lambda \geq 1$. In the following, we show that the converse is true if $\mathfrak{A}$ has an idempotent element a with $\|a\| = 1$.

Corollary 1.2. *Let $\mathfrak{A}$ be a normed algebra with an idempotent element $a \in \mathfrak{A}$ such that $\|a\| = 1$. Then the space $L^1(\Omega, \mathfrak{A})$ is a normed algebra with pointwise multiplication if and only if Ω is discrete and $\lambda \geq 1$ on nonempty subsets of Ω*

P r o o f. Suppose that $L^1(\Omega, \mathfrak{A})$ is a normed algebra. Then Ω is discrete by Proposition 1.1. Let $\omega \in \Omega$ and define $\xi_\omega^a(\omega) \in L^1(\Omega, \mathfrak{A})$ by $\xi_\omega^a(\omega) = a$ and $\xi_\omega^a = 0$ otherwise. Then

$$\begin{aligned}\lambda(\{\omega\}) &= \sum_{\eta \in \Omega} \|(\xi_\omega^a \, \xi_\omega^a)(\eta)\| \lambda(\{\eta\}) \\ &= \|\xi_\omega^a \, \xi_\omega^a\|_1 \\ &\leq \|\xi_\omega^a\|_1 \, \|\xi_\omega^a\|_1 \\ &= \lambda(\{\omega\}) \, \lambda(\{\omega\}).\end{aligned}$$

That is, $\lambda(\{\omega\}) \geq 1$. $\square$

In the case where Ω is a discrete space I, we let $\sigma \colon I \to (0, \infty)$ be the function defined by $\sigma(\iota) = \lambda(\{\iota\})$ for all $\iota \in I$; let also $\ell^1_\sigma(I, \mathfrak{A})$ be the space of all functions $f \colon I \to \mathfrak{A}$ with

$$\|f\|_{1,\sigma} = \sum_{\iota \in I} \|f(\iota)\| \sigma(\iota) < \infty;$$

as we have seen in the proof of Proposition 1.1, if

$$M_\sigma = \inf\bigl\{\sigma(\iota) : \ \iota \in I\bigr\} > 0,$$

then $\ell^1_\sigma(I, \mathfrak{A})$ is an algebra with pointwise multiplication. If, moreover, $\mathfrak{A}$ has an idempotent element with norm one, then $\ell^1_\sigma(I, \mathfrak{A})$ is a normed algebra.

In general, it follows from Proposition 1.1 that for locally compact Hausdorff space Ω, $L^1(\Omega,\mathfrak{A})$ is an algebra with pointwise multiplication if and only if Ω is a discrete space I, $M_\sigma > 0$ and $L^1(\Omega,\mathfrak{A}) = \ell^1_\sigma(I,\mathfrak{A})$.

2. Compactness of left multipliers on $\ell^1(I,\mathfrak{A})$

Let I be a set and $\mathfrak{A}$ be a normed algebra. Let $\ell^\infty(I,\mathfrak{A})$ be the space of all functions $f\colon I \to \mathfrak{A}$ with $\|f\|_\infty := \sup_{\iota\in I}\|f(\iota)\| < \infty$. Let also $\sigma\colon I \to (0,\infty)$ with $M_\sigma > 0$. Then for each $\varphi \in \ell^\infty(I,\mathfrak{A})$, the linear operator ${}_\varphi T\colon \ell^1_\sigma(I,\mathfrak{A}) \to \ell^1_\sigma(I,\mathfrak{A})$ defined by ${}_\varphi T(f) = \varphi f$ for all $f \in \ell^1_\sigma(I,\mathfrak{A})$ is a left multiplier, where $(\varphi f)(\iota) = \varphi(\iota)f(\iota)$ for all $\iota \in I$. We commence this section with the following result which shows that any left multiplier on $\ell^1_\sigma(I,\mathfrak{A})$ is of this form; for details on left multipliers see for example [9].

Proposition 2.1. *Let $\mathfrak{A}$ be a normed algebra with identity, I be a set and $\sigma\colon I \to (0,\infty)$ with $M_\sigma > 0$. Then any left multiplier on $\ell^1_\sigma(I,\mathfrak{A})$ is of the form ${}_\varphi T$ for some $\varphi \in \ell^\infty(I,\mathfrak{A})$.*

Proof. Let $T\colon \ell^1_\sigma(I,\mathfrak{A}) \to \ell^1_\sigma(I,\mathfrak{A})$ be a left multiplier. Define the function $\varphi\colon I \to \mathfrak{A}$ by $\varphi(\iota) = T(\xi^u_\iota)(\iota)$ for all $\iota \in I$, where u is the identity of $\mathfrak{A}$. Then $\|\varphi\|_\infty \le \|T\|$; indeed for each $\iota \in I$,

$$\begin{aligned}
\|\varphi(\iota)\| &= \big\|T(\sigma(\iota)^{-1}\xi^u_\iota)(\iota)\big\|\,\sigma(\iota)\\
&\le \sum_{\eta\in I}\big\|T(\sigma(\iota)^{-1}\xi^u_\iota)(\eta)\big\|\,\sigma(\eta)\\
&= \|T(\sigma(\iota)^{-1}\xi^u_\iota)\|_{1,\sigma}\\
&\le \|T\|\,\|\sigma(\iota)^{-1}\xi^u_\iota\|_{1,\sigma}\\
&= \|T\|\|u\|.
\end{aligned}$$

It follows that $\varphi \in \ell^\infty(I,\mathfrak{A})$. Since T and ${}_\varphi T$ are bounded on $\ell^1_\sigma(I,\mathfrak{A})$, we only need to prove that $T(f) = \varphi f$ for all $f \in C_c(I,\mathfrak{A})$, the space of all functions with finite support. For this end, let $f \in C_c(I,\mathfrak{A})$. Then $f = \sum_{n=1}^{m} \xi^{f(\iota_n)}_{\iota_n}$ for some $\iota_1,\dots,\iota_m \in I$. Thus

$$T(f) = \sum_{n=1}^{m} T(\xi^u_{\iota_n})\xi^{f(\iota_n)}_{\iota_n}.$$

Hence

$$\big\{\iota :\ T(f) \ne 0\big\} \subseteq \{\iota_1,\dots,\iota_m\}$$

and $T(f)(\iota_n) = \varphi(\iota_n)f(\iota_n)$ for all $n = 1,\dots,m$. That is, $T(f) = \varphi\cdot f$. $\square$

We now are ready to give the main result of the paper. First, let us recall that $c_0(I,\mathfrak{A})$ denotes the normed subalgebra of $\ell^\infty(I,\mathfrak{A})$ of all functions vanishing at infinity.

Theorem 2.2. *Let $\mathfrak{A}$ be a normed algebra with identity, I be a set, $\sigma\colon I \to (0,\infty)$ with $M_\sigma > 0$ and $\varphi \in \ell^\infty(I,\mathfrak{A})$. Then the following assertions are equivalent.*

(a) $\varphi \in c_0(I,\mathfrak{A})$.

(b) ${}_\varphi T$ *is compact on* $\ell^1_\sigma(I,\mathfrak{A})$.

(c) ${}_\varphi T$ *is weakly compact on* $\ell^1_\sigma(I,\mathfrak{A})$.

Proof.

(a) $\Longrightarrow$ (b). Suppose that $\varphi \in c_0(I,\mathfrak{A})$. Then $\operatorname{coz}(\varphi) := \big\{\iota \in I : \ \varphi(\iota) \neq 0\big\}$ is a σ-finite subset of I; indeed, if we set $F_n = \big\{\iota \in I : \ \|\varphi(\iota)\| \geq 1/n\big\}$ for all $n \geq 1$, then (F_n) is an increasing sequence of finite subsets in I and $\operatorname{coz}(\varphi) = \bigcup\limits_{n=1}^{\infty} F_n$. Moreover,

$$\|{}_{\zeta^\varphi_{F_n}}T - {}_\varphi T\| = \|{}_{\zeta^\varphi_{I\setminus F_n}}T\| = \|\zeta^\varphi_{I\setminus F_n}\|_\infty < \frac{1}{n},$$

where for each $E \subseteq I$, $\zeta^\varphi_E\colon I \to \mathfrak{A}$ is defined by $\zeta^\varphi_E(\iota) = \varphi(\iota)$ for all $\iota \in E$ and $\zeta^\varphi_E = 0$ otherwise. Since ${}_{\zeta^\varphi_{F_n}}T$ is a compact operator on $\ell^1_\sigma(I,\mathfrak{A})$ for all $n \geq 1$, it follows that ${}_\varphi T$ is also compact.

(b) $\Longrightarrow$ (c). This is trivial.

(c) $\Longrightarrow$ (a). To complete the proof, suppose that ${}_\varphi T$ is weakly compact on $\ell^1_\sigma(I,\mathfrak{A})$ and $\varphi \notin c_0(I,\mathfrak{A})$. Then there is $\varepsilon > 0$ such that $\|\zeta^\varphi_{I\setminus F}\|_\infty \geq \varepsilon$ for all $F \in \mathcal{F}$, the collection of all finite subsets of I; that is, $\|\varphi(\iota_F)\| \geq \varepsilon$ for some $\iota_F \in I \setminus F$. Then there is $g \in \ell^1_\sigma(I,\mathfrak{A})$ and a subnet $(\iota_{F(\beta)})_{\beta\in B}$ of $(\iota_F)_{F\in\mathcal{F}}$ such that

$$\sigma(\iota_{F(\beta)})^{-1}\xi^{\varphi(\iota_{F(\beta)})}_{\iota_{F(\beta)}} \to g$$

in the weak topology of $\ell^1_\sigma(I,\mathfrak{A})$, where $\xi^{\varphi(\iota_{F(\beta)})}_{\iota_{F(\beta)}} \in \ell^1(I,\mathfrak{A})$ by $\xi^{\varphi(\iota_{F(\beta)})}_{\iota_{F(\beta)}}(\iota_{F(\beta)}) = \varphi(\iota_{F(\beta)})$ and $\xi^{\varphi(\iota_{F(\beta)})}_{\iota_{F(\beta)}} = 0$ otherwise. In particular, since $\|\varphi(\iota_{F(\beta)})\| \neq 0$, there exists $\psi_\beta \in \mathfrak{A}^*$ such that $|\langle\psi_\beta, \varphi(\iota_{F(\beta)})\rangle| = \|\varphi(\iota_{F(\beta)})\|$ and $\|\psi_\beta\| = 1$; see [13]. Recall that the dual space of $\ell^1_\sigma(I,\mathfrak{A})$ is equal to $\ell^\infty_{1/\sigma}(I,\mathfrak{A}^*)$ under the duality

$$\langle h, f\rangle = \sum_{\iota\in I}\langle h(\iota), f(\iota)\rangle$$

where $h \in \ell^\infty_{1/\sigma}(I,\mathfrak{A}^*)$, $f \in \ell^1_\sigma(I,\mathfrak{A})$ and

$$\ell^\infty_{1/\sigma}(I,\mathfrak{A}^*) = \Big\{h\colon I \to \mathfrak{A}^* : \ \|h\|_{\infty,1/\sigma} := \sup_{\iota\in I}\big\|\tfrac{h(\iota)}{\sigma(\iota)}\big\| < \infty\Big\};$$

see for example [2, Appendix A]. Now, we define θ by $\theta(\iota_{F(\beta)}) = \sigma(\iota_{F(\beta)})\psi_\beta$ and $\theta(\iota) = 0$ otherwise. It is clear that $\theta \in \ell^\infty_{1/\sigma}(I, \mathfrak{A}^*)$. Hence

$$\left\langle \theta, \sigma(\iota_{F(\beta)})^{-1}\xi^{\varphi(\iota_{F(\beta)})}_{\iota_{F(\beta)}} \right\rangle \to \langle \theta, g\rangle.$$

On the one hand, $|\langle g, \sigma\rangle| \geq \varepsilon$; indeed,

$$\begin{aligned}
\langle \theta, g\rangle &= \lim_\beta \left|\left\langle \theta, \sigma(\iota_{F(\beta)})^{-1}\xi^{\varphi(\iota_{F(\beta)})}_{\iota_{F(\beta)}}\right\rangle\right| \\
&= \lim_\beta \left|\sum_{\iota\in I}\left\langle \theta(\iota), \sigma(\iota_{F(\beta)})^{-1}\xi^{\varphi(\iota_{F(\beta)})}_{\iota_{F(\beta)}}\right\rangle\right| \\
&= \lim_\beta \left|\left\langle \theta(\iota_{F(\beta)}), \sigma(\iota_{F(\beta)})^{-1}\varphi(\iota_{F(\beta)})\right\rangle\right| \\
&= \lim_\beta |\langle \psi_\beta, \varphi(\iota_{F(\beta)})\rangle| \\
&= \lim_\beta \|\varphi(\iota_{\iota_{F(\beta)}})\|.
\end{aligned}$$

On the other hand, if $\iota_0 \in I$ and $g(\iota_0) \neq 0$, then there is $\beta_0 \in B$ such that for each $\beta \geq \beta_0$, $\iota_0 \in F(\beta)$ and hence $\iota_0 \neq \iota_{F(\beta)}$. Thus there exists $\psi \in \mathfrak{A}^*$ such that $|\langle \psi, g(\iota_0)\rangle| = \|g(\iota_0)\|$ and $\|\psi\| = 1$. Let $\theta\colon I \to \mathfrak{A}^*$ defined by $\theta(\iota) = \psi$ for $\iota = \iota_0$ and $\theta(\iota) = 0$ otherwise. Thus $\theta \in \ell^\infty_{1/\sigma}(I, \mathfrak{A}^*)$ and therefore

$$\begin{aligned}
\|g(\iota_0)\| &= |\langle \psi, g(\iota_0)\rangle| = |\langle g, \theta\rangle| \\
&= \lim_\beta \left|\left\langle \theta, \sigma(\iota_{F(\beta)})^{-1}\xi^{\varphi(\iota_{F(\beta)})}_{\iota_{F(\beta)}}\right\rangle\right| \\
&= \lim_\beta \left|\sum_{\iota\in I}\left\langle \theta(\iota), \sigma(\iota_{F(\beta)})^{-1}\xi^{\varphi(\iota_{F(\beta)})}_{\iota_{F(\beta)}}(\iota)\right\rangle\right| \\
&= \lim_\beta \langle \theta(\iota_0), \sigma(\iota_{F(\beta)})^{-1}\xi^{\varphi(\iota_{F(\beta)})}_{\iota_{F(\beta)}}(\iota_0)\rangle \\
&= 0
\end{aligned}$$

that is $g(\iota_0) = 0$. This contradiction completes the proof. □

Remark 2.3. Let $\mathfrak{A}$ be a normed algebra with identity, I be a set and $\sigma\colon I \to (0,\infty)$ with $M_\sigma > 0$. Then an argument similar to the proof of Proposition 2.1 shows that any right multiplier on $\ell^1_\sigma(I, \mathfrak{A})$ is of the form T_φ for some $\varphi \in \ell^\infty(I, \mathfrak{A})$, where $T_\varphi\colon \ell^1_\sigma(I, \mathfrak{A}) \to \ell^1_\sigma(I, \mathfrak{A})$ is defined by $T_\varphi(f) = f \cdot \varphi$ for all $f \in \ell^1_\sigma(I, \mathfrak{A})$.

Now, a modification of the proof of Theorem 2.2 implies that the following assertions are equivalent.

(a) $\varphi \in c_0(I, \mathfrak{A})$.

(b) T_φ is compact on $\ell^1_\sigma(I, \mathfrak{A})$.

(c) T_φ is weakly compact on $\ell^1_\sigma(I, \mathfrak{A})$.

We end the work by an application of our main result in this section. First, let us recall from Dales [2] that the first Arens multiplication $\odot$ on the second dual of $\ell^1_\sigma(I,\mathfrak{A})$ is defined by

$$\langle F \odot G, \varphi\rangle = \langle F, G \circ \varphi\rangle$$

for $F, G \in \ell^1_\sigma(I,\mathfrak{A})^{**}$, where $\langle G \circ \varphi, f\rangle = \langle G, \varphi.f\rangle$ for $f \in \ell^1_\sigma(I,\mathfrak{A})$ and $\varphi \in \ell^1_\sigma(I,\mathfrak{A})^* = \ell^\infty_{1/\sigma}(I,\mathfrak{A}^*)$. Then $\ell^1_\sigma(I,\mathfrak{A})^{**}$ endowed with $\odot$ is an algebra. For an element G in $\ell^1_\sigma(I,\mathfrak{A})^{**}$, the maps $G \mapsto f \odot G$ and $G \mapsto G \odot f$ are weak*-weak* continuous on $\ell^1_\sigma(I,\mathfrak{A})^{**}$ for all $f \in \ell^1_\sigma(I,\mathfrak{A})$. The Arens multiplication was first introduced and studied by Arens [1]; see also [7], [8] and [9] for more details.

Corollary 2.4. *Let $\mathfrak{A}$ be a normed algebra with identity, I be a set and $\sigma\colon I \to (0,\infty)$ with $M_\sigma > 0$. Then $\ell^1_\sigma(I,\mathfrak{A})$ is an ideal in its second dual.*

Proof. Let $f \in \ell^1_\sigma(I,\mathfrak{A})$ and $F \in \ell^1_\sigma(I,\mathfrak{A})^{**}$. Then there is a net (f_γ) in $\ell^1_\sigma(I,\mathfrak{A})$ bounded by $\|F\|$ such that $f_\gamma \to F$ in the weak* topology of $\ell^1_\sigma(I,\mathfrak{A})^{**}$. In particular, $f \cdot f_\gamma \to f \odot F$ in the weak* topology of $\ell^1_\sigma(I,\mathfrak{A})^{**}$. Since $f \cdot \sigma \in c_0(I,\mathfrak{A})$ and $M_\sigma > 0$, we have $f \in c_0(I,\mathfrak{A})$ and so ${}_fT$ is weakly compact on $\ell^1_\sigma(I,\mathfrak{A})$ by Theorem 2.2. This together with that $(f \cdot f_\gamma)$ is a bounded net in $\ell^1_\sigma(I,\mathfrak{A})$ yield that there is a subnet (f_β) of (f_γ) such $(f \cdot f_\beta)$ converges to an element of $g \in \ell^1_\sigma(I,\mathfrak{A})$ in the weak topology of $\ell^1_\sigma(I,\mathfrak{A})$. Thus $f \odot F = g \in \ell^1_\sigma(I,\mathfrak{A})$. Similarly, $F \odot f \in \ell^1_\sigma(I,\mathfrak{A})$ by Remark 2.3. $\square$

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