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FINITE GROUPS WITH WEAKLY S-QUASINORMAL SUBGROUPS

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ABSTRACT. We introduce a new subgroup embedding property in a finite group called weakly S-quasinormality. We say a subgroup H of a finite group G is weakly S-quasinormal in G if there exists a normal subgroup K such that $HK \trianglelefteq G$ and $H \cap K$ is S-quasinormally embedded in G. We use the new concept to investigate the properties of some finite groups. Some previously known results are generalized.

1. Introduction

The relationship between the subgroups of a finite group G and the structure of the group G have been extensively studied in the literature. A subgroup of a group G is said to be S-quasinormal in G if it permutes with every Sylow subgroup of G. This concept was introduced by Kegel in [13] and has been studied extensively by Deskins in [8]. They said that if a subgroup H of a finite group G is S-quasinormal in G, then H/H_G is nilpotent. More recently, Ballester-Bolinches and Pedraza-Aguilera [4] introduced the following definition: A subgroup H of a group G is said to be S-quasinormally embedded in G if for each prime p dividing the order of H, a Sylow p-subgroup of H is also a Sylow p-subgroup of some S-quasinormal subgroup of G. Using this idea, a series of elegant results on the structure of groups have obtained, see [1, 7, 15]. As a development of the above conclusions, we now introduce the following concept of weakly S-quasinormal subgroups:

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Definition 1.1. Let H be a subgroup of a group G. H is said to be weakly S-quasinormal in G if there exists a normal subgroup T of G such that $HT \trianglelefteq G$ and $H \cap T$ is S-quasinormally embedded in G.

Obviously, every S-quasinormally embedded subgroup is weakly S-quasinormal. The converse does not hold in general.

Example 1.2. We can consider the 4-letters symmetric group S_4. Obviously, $\langle (12) \rangle$ can not be S-quasinormally embedded in S_4 since $\langle (12) \rangle$ and every subgroup of S_4 with order 6 containing $\langle (12) \rangle$ can not permute with every Sylow subgroup of S_4. But clearly, $\langle (12) \rangle$ is weakly S-quasinormal in G.

On the other hand, in [23], Wang defined c-normal subgroup of a finite group: A subgroup H of a group G is said to be c-normal in G if there exists a normal subgroup T of G such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the largest normal subgroup of G contained in H. Note that the condition $H \cap T \leq H_G$ in the concepts is actually equivalent to the condition $H \cap T = H_G$ (see [25]). It is easy to see that all normal subgroups and c-normal subgroups are all weakly S-quasinormal in G. But the following examples shows that the convert is false.

Example 1.3. ([24]) Let $G = \langle x, y \mid x^{16} = y^4 = 1, x^y = x^3 \rangle$. Then $\Phi(G) = \langle x^2, y^2 \rangle = \langle x^2 \rangle \times \langle y^2 \rangle$. We can see that $H = \langle y^2 \rangle$ is weakly S-quasinormal in G but not c-normal. In fact $\langle y^2 \rangle$ is permutable in G.

All groups considered here are finite. A formation $\mathfrak{F}$ is said to be saturated if it contains every group G with $G/\Phi(G) \in \mathfrak{F}$. A formation $\mathfrak{F}$ is said to be S-closed if every subgroup of a group G belongs to $\mathfrak{F}$ whenever $G \in \mathfrak{F}$. The notion and terminologies used in this paper are standard. The reader is referred to the monograph of B. Huppert [12] or W. Guo [11] for notations and terminologies not mentioned in this paper.

2. Preliminaries

For the sake of convenience, we list the following results used for the proofs in this paper.

Lemma 2.1. ([4, Lemma 1]) *Suppose that U is S-quasinormally embedded in a group G, $H \leq G$ and K a normal subgroup of G. Then:*

(a) *If $U \leq H$, then U is S-quasinormally embedded in H.*

(b) *UK is S-quasinormally embedded in G and UK/K is S-quasinormally embedded in G/K.*

(c) *If $K \leq H$ and H/K is S-quasinormally embedded in G/K, then H is S-quasinormally embedded in G.*

Lemma 2.2. *Let G be a group. Then:*

(1) *If H is weakly S-quasinormal in G and $H \leq M \leq G$, then H is weakly S-quasinormal in M;*

(2) *Let $N \trianglelefteq G$ and $N \leq H$, H is a p-subgroup. Then H is weakly S-quasinormal in G if and only if H/N is weakly S-quasinormal in G/N;*

(3) *Let $N \trianglelefteq G$ and H a subgroup of G with $(|H|, |N|) = 1$. If H is weakly S-quasinormal in G, then HN/N is weakly S-quasinormal in G/N.*

Proof.

(1) If $HK \trianglelefteq G$ and $H \cap K$ is S-quasinormally embedded in G, then $H(M \cap K) = M \cap HK \trianglelefteq M$ and $H \cap (M \cap K) = H \cap K$ is S-quasinormally embedded in M by Lemma 2.1. Hence, H is weakly S-quasinormal in M.

(2) Suppose that H is weakly S-quasinormal in G. Then there exists a normal subgroup T of G such that $HT \trianglelefteq G$ and $H \cap T$ is S-quasinormally embedded in G. Hence, $(H/N)(TN/N) \trianglelefteq G/N$ and $H/N \cap TN/N = N(H \cap T)/N$. By Lemma 2.1, $N(H \cap T)/N$ is S-quasinormally embedded in G/N. Thus H/N is weakly S-quasinormal in G/N. Obviously, the convert is also true by Lemma 2.1(c).

(3) If H is weakly S-quasinormal in G, then there exists a normal subgroup K of G such that $HK \trianglelefteq G$ and $H \cap K$ is S-quasinormally embedded in G. Clearly $(HN/N)(K/N) = (HKN)/N \trianglelefteq G/N$. Since $(|N|, |H|) = 1$,

$$(|HN \cap K : HN \cap K \cap N|, |HN \cap K : HN \cap K \cap H|) = 1.$$

By [11, Lemma 3.8.2], $HN \cap K = (HN \cap K \cap N)(HN \cap K \cap H) = (N \cap K)(H \cap K)$. Furthermore, $(HN/N) \cap (KN/N) = (H \cap K)N/N$ is S-quasinormally embedded in G/N by Lemma 2.1. Hence HN/N is weakly S-quasinormal in G/N. □

Lemma 2.3. *Let P be a non-trival normal p-subgroup of a group G and $\Phi(P) = 1$. If every maximal subgroup of P is weakly S-quasinormal in G, then some maximal subgroup of P is normal in G.*

Proof. Since $P \lhd G$ and $P \cap \Phi(G) = 1$, P is a direct product of some abelian minimal normal subgroups of G by [11, Theorem 1.8.17]. Let L be a minimal normal subgroup of G contained in P. Suppose that $L \neq P$. By Lemma 2.2(2), the hypothesis holds on G/L. By induction some maximal subgroup P_1/L of P/L is weakly S-quasinormal in G/L, which implies that P_1 is normal in G. Now suppose that $P = L$. Since $L \nsubseteq \Phi(G)$, there exists a maximal subgroup M of G such that $G = LM$ and $L \cap M = 1$. Let M_p be a Sylow p-subgroup of M and $G_p = M_pL$. Then G_p is a Sylow p-subgroup of G. Let P_1 be a maximal subgroup of G_p containing M_p and $P_2 = P_1 \cap P$. Then $|P : P_2| = |P : P_1 \cap P| = |PP_1 : P_1| = |G_p : P_1| = p$ and so P_2 is a maximal subgroup of P. By hypothesis, P_2 is weakly S-quasinormal in G. Hence there exists a normal subgroup K_2 of G such that $P_2K_2 \trianglelefteq G$ and $P_2 \cap K_2$ is a Sylow p-subgroup of an S-quasinormal subgroup T of G. Then for every Sylow q-subgroup Q of G with $q \neq p$, $P_2 \cap K_2 = P \cap TQ \trianglelefteq TQ$ and so $Q \leq N_G(P_2 \cap K_2)$. On the other hand, $P \cap K_2 \trianglelefteq G$ and consequently $P_1 \cap K_2 = P_1 \cap P \cap K_2 \trianglelefteq P_1$. It follows that $P_2 \cap K_2 \trianglelefteq P_2 P = G_p$. The arbitrary choice of q implies that $P_2 \cap K_2 \trianglelefteq G$. We only need to consider that $K_2 \neq 1$ and $K_2 P_2 \neq P_2$. Since $P = L$ is a minimal normal subgroup of G, $P \cap K_2 = 1$ or $P \cap K_2 = P$. If $P \cap K_2 = P$, then $P \leq K_2$ and so $P_2 = P_2 \cap P = P_2 \cap K_2 \trianglelefteq G$. If $P \cap K_2 = 1$, then $P_2 = P_2(P \cap K_2) = P \cap P_2 K_2 \trianglelefteq G$. Hence the lemma holds. □

Lemma 2.4. ([1, Lemma 2.4]) *Let H be a subgroup of G. Then the following two statements are equivalent:*

(a) *H is S-quasinormal nilpotent subgroup of G.*

(b) *The Sylow subgroups of H are S-quasinormal in G.*

Lemma 2.5. ([22]) *Let G be a group.*

(1) *If A is a subnormal soluble (nilpotent) subgroup of G, then A is contained in some soluble (respectively in some nilpotent) normal subgroup of G.*

(2) *If A is a subnormal in G and A is a π-subgroup of G, then $A \leq O_\pi(G)$.*

Lemma 2.6. ([19, Lemma 2.16]) *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.*

3. Main results

Theorem 3.1. *Let $G = AB$, where A is an S-quasinormal subgroup of G and B is a Hall subgroup of G of which all Sylow subgroups are all cyclic. If every maximal subgroup of every non-cyclic Sylow p subgroup of A is weakly S-quasinormal in G, then G is supersoluble.*

P r o o f. Suppose that the assertion is false and let G be a counterexample of minimal order. We proceed our proof as follows:

(1) Each proper subgroup M of G containing A is supersoluble.

Let $A \leq M < G$. Then $M = M \cap AB = A(M \cap B)$. Obviously, $M \cap B$ is a Hall subgroup of M and every Sylow subgroup of $M \cap B$ is cyclic. By Lemma 2.2, every maximal subgroup of every non-cyclic sylow subgroup of A is weakly S-quasinormal in M. The minimal choice of G implies that M is supersoluble.

(2) Let H be a non-trivial normal subgroup of G and H is a p-group for some prime p. If either H contains the Sylow p-subgroup P of A or P is cyclic or $H \leq A$, then G/H is supersoluble.

If $A \leq H$, then $G/H = BH/H \cong B/(B \cap H)$ is supersoluble. We can assume that $A \nleq H$. Clearly, $G/H = (AH/H)(BH/H)$, where AH/H is S-quasinormal and BH/H is supersoluble. Let Q/H be a Sylow q-subgroup of AH/H and M/H a maximal subgroup of Q/H. Let Q_1 be a Sylow q-subgroup of Q such that $Q = HQ_1$. Clearly, Q_1 is a Sylow q-subgroup of AH. Thus $Q = A_qH$ for some Sylow q-subgroup A_q of A. If $H \leq A$, it is obvious by Lemma 2.2. Assume that either the Sylow p-subgroup P of A is cyclic or $P \leq H$. Then $p \neq q$. Since $M \neq Q$ and $A_qH = Q$, $M \cap A_q$ is a maximal subgroup of A_q. (Indeed, if not, then there exists a subgroup T of A_q such that $M \cap A_q < T < A_q$ and so $M = H(M \cap A_q) \leq HT \leq HA_q = Q$. Since M is a maximal subgroup of Q, $M = HT$ or $HT = HA_q$. If $M = HT$, then $M \cap A_q = T$, a contradiction. Hence we can assume that $HT = HA_q$. Hence $A_q = A_q \cap HT = T(A_q \cap H) \leq T(M \cap A_q) = T$, a contradiction.) By hypothesis, $M \cap A_q$ is weakly S-quasinormal in G. Therefore $M/H = (M \cap A_q)H/H$ is weakly S-quasinormal in G/H by Lemma 2.2. Hence the conditions of the theorem are true for G/H and thereby G/H is supersoluble by the minimal choice of G.

(3) There exists at least one Sylow subgroup of A which is non-cyclic.

Indeed, every Sylow subgroup of G is either contained in A or in B. If all Sylow subgroups of A is cyclic, then G is supersoluble by [12, VI, Theorem 10.3].

(4) G is soluble.

If $A \neq G$, then A is supersoluble by (1). Let p be a largest prime of A. Then $A_p \trianglelefteq A$. By Lemma 2.5, $A_p \leq O_p(G)$. By (2), $G/O_p(G)$ is supersoluble and so G is soluble.

We only need to consider the case that $A = G$. If not, then by (2), $O_p(G) = 1$ and $O_{p'}(G) = 1$, for any prime of $|G|$. Let P be a Sylow p-subgroup of G and N a minimal normal subgroup of G, where p is the minimal prime p of $|G|$. By Feit-Thompson Theorem, we shall consider that $p = 2$. If $NP < G$, then every maximal subgroup of P is weakly S-quasinormal in NP. Hence PN satisfies the hypothesis of the theorem and so NP is supersoluble. The solubility of N implies that $O_p(N) \neq 1$ or $O_{p'}(N) \neq 1$. Since $N \trianglelefteq G$, $O_2(G) \neq 1$ or $O_{p'}(G) \neq 1$, a contradiction. Now we can assume that $G = PN$. If $NP_1 < G$ for any maximal subgroup P_1 of P, then $(P \cap N)P_1 < P$ and so $P \cap N \leq P_1$. It follows that $P \cap N \leq \Phi(P)$ and N is 2-nilpotent by [9, IV, 4.7]. Hence $N = N_2 \times N_{2'}$. Therefore $N_2 \trianglelefteq G$ and $N_{2'} \trianglelefteq G$, a contradiction. Thus there exists a maximal subgroup P_1 of P such that $G = NP_1$. By hypothesis, there exists a normal subgroup K_1 of G such that $P_1K_1 \trianglelefteq G$, where $P_1 \cap K_1$ is a Sylow subgroup of an S-quasinormal subgroup T of G. Assume that $T_G \neq 1$. Take a minimal normal subgroup N_1 of G such that $N_1 \leq T_G$. If $N_1 \neq N$, then $N_1 \cong N_1N/N$. Since G/N is a 2-group, N_1 is also a 2-group, which contrary to the fact that $O_2(G) = 1$. Thus $N_1 = N$ and $P_1 \cap K_1 \cap N$ is a Sylow 2-subgroup of N. Since $G = NP_1$, $|G : P_1| = |NP_1 : P_1| = |N : N \cap P_1| = |N : P_1 \cap K_1 \cap N| : |(N \cap P_1) :$

$(P_1 \cap K_1 \cap N)| = |N : P_1 \cap K_1 \cap N| : |(N \cap P_1)K : K|$ is a $2'$-number and thereby P_1 is a Sylow 2-subgroup of G, a contradiction. Hence $T_G = 1$. In this case, $P_1 \cap K_1$ is S-quasinormal in G by Lemma 2.4. Therefore $P_1 \cap K_1 \leq O_p(G)$ and so $P_1 \cap K_1 = 1$. Furthermore, $|K_1|_2 \leq 2$. If $|K_1|_2 = 1$, then K_1 is a soluble normal of G and so G is soluble, a contradiction. If $|K_1|_2 = 2$, then K_1 is 2-nilpotent. Hence K_1 has a normal Hall $2'$-subgroup $K_{1_{2'}}$ and so by Lemma 2.5, we have that $K_{1_{2'}}$ is contained in some soluble normal subgroup of G, which implies that G is soluble, a contradiction. (4) holds.

(5) G has a unique minimal normal subgroup N such that $N = C_G(N) = O_p(G)$ for some prime p of $|G|$ and $G = [N]M$, where M is a maximal subgroup of G with $O_p(M) = 1$.

Since the class of all the supersoluble groups is a saturated formation, there exists a unique minimal normal subgroup N of G and $\Phi(G) = 1$. Since G is soluble, $N \subseteq O_p(G)$ for some prime p. Hence $N \subseteq O_p(G) \subseteq F(G) \subseteq C_G(N)$. Therefore we can see from $\Phi(G) = 1$ that there exists a maximal subgroup M of G such that $G = NM$. Obviously, N is abelian. Hence $N \cap M \trianglelefteq NM = G$ and so $N \cap M = 1$. Consequently, $G = [N]M$. Let $C = C_G(N)$. Similarly, we have that $C \cap M = 1$. By the Dedekind identity, $C = C \cap NM = N(C \cap M) = N$, which implies that $N = O_p(G) = C_G(N)$.

(6) A is supersoluble.

Assume that A is non-supersoluble. Let $A = G$ be a soluble group in which every maximal subgroup of every non-cyclic Sylow subgroup G_p are all weakly S-quasinormal in G. If N is a Sylow p-subgroup of G. Let N_1 be a maximal subgroup of N. Obviously, N is non-cyclic. Then by hypothesis, N_1 is weakly S-quasinormal in G. Hence there exists a normal subgroup K_1 such that $N_1K_1 \trianglelefteq G$ and $N_1 \cap K_1$ is a Sylow p-subgroup of an S-quasinormal subgroup T of G. If $T_G \neq 1$, then we can take a minimal normal subgroup R of G such that $R \leq T_G$. Since G is soluble, R must be a p-subgroup. Hence $R = N$. It is easy to see that $|T|_p = |N_1 \cap K_1| \leq |N_1| < |N| = |R| \leq |T_G|_p \leq |T|_p$, a contradiction. Hence $T_G = 1$ and so $N_1 \cap K_1$ is S-quasinormal in G. Since $K_1 \trianglelefteq G$, $N \cap K_1 = 1$ or $N \leq K_1$. If $N \cap K_1 = 1$, then $N_1 = N_1(N \cap K_1) = N \cap N_1K_1$ and thereby $N \cap N_1K_1 = 1$ or $N \cap N_1K_1 = N$ by the minimality of N. If $N \cap N_1K_1 = 1$, then $N_1 = 1$ and so N is a cyclic subgroup of order p, a contradiction. If $N \cap N_1K_1 = N$, then $N_1 = N$, a contradiction. Hence we have that $N \leq K_1$ and thereby $N_1 \cap K_1 = N_1$. Let Q be a Sylow q-subgroup of G, for an arbitrary prime of G and $q \neq p$. Since $N_1 \cap K_1$ is S-quasinormal, $N_1 = N \cap N_1Q \trianglelefteq N_1Q$ and so $Q \leq N_G(N_1)$. It follows that $N_1 \trianglelefteq G$. Then $|N| = p$. This contradiction shows that N can not be a Sylow p-subgroup of G. Let q be the largest prime of $|G|$ and Q a Sylow q-subgroup of G. Then QN/N is a Sylow q-subgroup of G/N. Since G/N is supersoluble, $QN/N \trianglelefteq G/N$ and consequently $QN \trianglelefteq G$. Let P be a Sylow p-subgroup of G. If $q = p$, then $P = Q = QN \trianglelefteq G$. Therefore,

$N = O_p(G) = P$ is a Sylow p-subgroup of G, a contradiction. Suppose that $q > p$. If $QNP = QP < G$, by Lemma 2.2, QP satisfies the hypothesis and so QP is supersoluble. Hence $Q \trianglelefteq QP$. Furthermore, $QP = Q \times P$. It follows that $Q \leq C_G(N) = N$, a contradiction. Hence we can assume that $G = PQ$. Since $N \nleq \Phi(G)$, $N \nleq \Phi(P)$ by [9, III, Lemma 3.3(a)]. Let P_1 be a maximal subgroup of P such that $N \nleq P_1$. By hypothesis, there exists a normal subgroup K_1 of G such that $P_1 \cap K_1$ is a Sylow p-subgroup of an S-quasinormal subgroup T of G. If $T_G \neq 1$, then we can take a minimal normal subgroup R of G such that $R \leq T_G$. Since G is soluble, R is a p-subgroup. So that $R \leq P_1 \cap K_1$ and indeed $R = N \leq P_1$. Furthermore, $P = P \cap NM = N(P \cap M) \leq P_1(P \cap M) = P_1$, a contradiction. Thus by Lemma 2.4, $P_1 \cap K_1$ is S-quasinormal in G, which implies that $(P_1 \cap K_1)M = M(P_1 \cap K_1)$. Obviously, $P_1 \cap K_1 \neq N$. If $P_1 \cap K_1 \neq 1$, then the maximality of M implies that $M(P_1 \cap K_1) = G$. Furthermore, $N = N \cap M(P_1 \cap K_1) = (P_1 \cap K_1)(N \cap M) = P_1 \cap K_1$, a contradiction. Hence $P_1 \cap K_1 = 1$ and so $|K_1|_p \leq p$. If $K_1 = 1$, then $P_1 \trianglelefteq G$ and so $N \leq P_1$, a contradiction. If $|K_1|_p = p$, then K_1 is p-nilpotent by [17, Theorem 10.1.9]. If $|K| = p$, then $K = N$ is a cyclic subgroup and therefore G is supersoluble by Lemma 2.6, a contradiction. Hence we have that K has a non-trivial normal Hall p'-subgroup $K_{p'}$. It follows that $K_{p'} \trianglelefteq G$, a contradiction. This contradiction shows that (6) holds.

(7) Final contradiction.

Let p be the largest prime divisor of $|A|$ and A_p a Sylow p-subgroup of A. By (6), A is supersoluble and $A_p \trianglelefteq A$. Then $A_p \leq O_p(G)$. If $p||B|$, then $O_p(G) \leq G_p$, where G_p is a cyclic subgroup of B and so $O_p(G)$ is cyclic. In view of (2), $G/O_p(G)$ is supersoluble. It follows that G is supersoluble, a contradiction. Hence $A_p \nsubseteq B^x$ for every element x of G. Therefore, A_p is a Sylow p-subgroup of G and so $A_p = O_p(G)$. Let N be a minimal normal subgroup of G. Then $N = O_p(G) = A_p = G_p$, where p is the largest prime divisor of the order of A. Hence G is supersoluble by the prove of (6). The final contradiction completes the proof. □

Theorem 3.2. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group. Then $G \in \mathfrak{F}$ if and only if there exists a soluble normal subgroup H such that $G/H \in \mathfrak{F}$ and all maximal subgroups of all Sylow subgroups of $F(H)$ are weakly S-quasinormal in G.*

Proof. The necessary part is obvious. We only need to prove the sufficient part. Assume the assertion is false and let G be a counterexample with minimal order. We proceed with our proof as follows:

(1) $P \cap \Phi(G) = 1$.

If not, then $1 \neq P \cap \Phi(G) \triangleleft G$. Let $R = P \cap \Phi(G)$. We show that G/R satisfies the hypothesis. In fact, $(G/R)/(H/R) \cong G/H \in \mathfrak{F}$. Let $F(H/R) = T/R$. Then, obviously, $F(H)/R = F(H/R)$. Let P_1/R be a maximal subgroup of P/R. Then P_1 is a maximal subgroup of P. By hypothesis, P_1 is weakly S-quasinormal in G. Hence by Lemma 2.2, P_1/R is weakly S-quasinormal in G/R. Let $\bar{Q}_1$ be a maximal subgroup of the Sylow q-subgroup $\bar{Q}$ of $F(H)/R$, where $q \neq p$. Then, clearly, there exists a Sylow q-subgroup Q of $F(H)$ such that $\bar{Q} = QR/R$ and $\bar{Q}_1 = Q_1R/R$ with Q_1 is a maximal subgroup of Q. By hypothesis, Q_1 is weakly S-quasinormal in G and so Q_1R/R is weakly S-quasinormal in G/R by Lemma 2.2(3). This shows that $(G/R, H/R)$ satisfies the hypothesis. The minimal choice of (G, H) implies that $G/R \in \mathfrak{F}$. Since $R \subseteq \Phi(G)$ and $\mathfrak{F}$ is a saturated formation, $G \in \mathfrak{F}$, a contradiction. Thus (1) holds.

(2) Final contradiction.

By (1), $P \cap \Phi(G) = 1$, then P is the direct product of some minimal normal subgroups of G by [11, Theorem 1.8.17]. Hence by hypothesis and Lemma 2.3, P has a maximal subgroup P_1 such that P_1 is normal in G. Then by [10, A, Theorem 9.13] for some minimal normal subgroup L of G contained in P we have $|L| = p$. Let $C = C_H(L)$. Clearly $G/C = G/H \cap C_G(L) \in \mathfrak{F}$. Besides evidently $F \leq C$ and $L \leq Z(F)$. Hence $F(C/L) = F/L$. Now Lemma 2.2, we see that the hypothesis is still true for $(G/L; C/L)$. Thus $G/L \in \mathfrak{F}$ and hence $G \in \mathfrak{F}$ by Lemma 2.6. The final contradiction completes our proof. □

Corollary 3.2.1. *Let G be a group. If there exists a normal subgroup H such that $G/H \in \mathfrak{F}$ and all maximal subgroups of all Sylow subgroups of H are weakly S-quasinormal in G, then $G \in \mathfrak{F}$.*

Proof. By Theorem 3.1, H is supersoluble. Let p be the largest prime of H and P a Sylow q-subgroup of H. Then $P \trianglelefteq H$ and it follows that $Q \trianglelefteq G$. It is clear that G/Q satisfies the hypothesis and by induction $G/Q \in \mathfrak{F}$. Now applying Theorem 3.2, $G \in \mathfrak{F}$. □

Theorem 3.3. *A group G is supersoluble if and only if there exists a normal subgroup H of G such that G/H is supersoluble and every cyclic subgroup of N with prime order or 4 are weakly S-quasinormal in G.*

Proof. The necessary part is clear. We only need to prove the sufficiency part. Suppose that the assertion is false and let (G, H) be a counterexample for which $|G||H|$ is minimal. Then:

(1) If T is a normal Hall subgroup of H, then the hypothesis holds for (T, T) and for $(G/T, H/T)$.

Let P be an arbitrary noncyclic Sylow subgroup of T. By hypothesis, every cyclic subgroup N of P with prime order or 4 is weakly S-quasinormal in G. Then by Lemma 2.2(1), N is weakly S-quasinormal in T. Thus (T, T) satisfies the hypothesis.

Obviously, $(G/T)/(H/T)$ is supersoluble. Let R^*/T be a Sylow r-subgroup of H/T where $r||H/T|$ and R a Sylow r-subgroup of R^* such that $R^* = RT$. Then R is a Sylow r-subgroup of H. Assume that K/T is a cyclic subgroup of R^*/T with prime order or 4. Then, obviously, $K/T = \langle x\rangle T/T$, where $\langle x\rangle$ is a subgroup of R with prime order or 4 since T is a normal Hall subgroup of H. By hypothesis, $\langle x\rangle$ is weakly S-quasinormal in G. Then by Lemma 2.2, we see that K/T is also weakly S-quasinormal in G/T. Thus $(G/T, HT)$ satisfies the hypothesis.

(2) If T is a non-identity normal Hall subgroup P of H, then $T = H$.

Since T char H, $T \trianglelefteq G$. Then by (1), the hypothesis is true for $(G/T, H/T)$. Hence G/T is supersoluble. It is easy to see that the hypothesis is still true for (G, T). The minimal choice of (G, H) implies that $T = H$.

(3) If p is the smallest prime of $|H|$ and P is a Sylow p-subgroup of H, then P is not cyclic.

Indeed, if P is cyclic, then by [12, IV, Theorem 2.8], H is p-nilpotent. Hence by (2), $H = P$ is cyclic. It follows from Lemma 2.6 that G is supersoluble, a contradiction.

(4) G is a minimal non-supersoluble group.

Let K be a proper subgroup of G. Since G/H is supersoluble, $K/(H \cap K) \cong HK/H$ is also supersoluble. By Lemma 2.2, every minimal subgroup of $K \cap H$ and every cyclic subgroup of $K \cap H$ of order 4 are weakly S-quasinormal in K. This means that H (with respect to $K \cap H$) satisfies the hypothesis. The minimal choice of G implies that K is supersoluble. This shows that G is a minimal non-supersoluble group.

(5) G has a non-cyclic normal Sylow p-subgroup $P = G^{\mathfrak{U}}$ for some prime p such that $P/\Phi(P)$ is chief factor of $G/\Phi(P)$ and the exponent of P is p or 4. Since G/H is supersoluble, $P \leq H$.

It follows directly from (3), (4) and [11, Theorem 3.4.2 and Theorem 3.11.8].

(6) Final contradiction.

Let $x \in P\backslash\Phi(P)$ and $|x|$ a prime or 4. Since $P \subseteq H$, by hypothesis, we can see that $\langle x\rangle$ is weakly S-quasinormal in G. Hence there exists a normal subgroup T of G such that $\langle x\rangle T \trianglelefteq G$ and $\langle x\rangle \cap T$ is S-quasinormally embedded in G. We claim that $\langle x\rangle\Phi(P)/\Phi(P)$ is S-quasinormally embedded in $G/\Phi(P)$. (In fact, if $P \cap T = P$, then $\langle x\rangle$ is S-quasinormally embedded in G and so $\langle x\rangle\Phi(P)/\Phi(P)$ by Lemma 2.1. Hence we consider that $(P \cap T)\Phi(P) \neq P$ and so $P \cap T \leq \Phi(P)$ since $P/\Phi(P)$ is a chief factor of G. However, $\langle x\rangle(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$, which implies that $\langle x\rangle(P \cap T)\Phi(P) = P$ or $\langle x\rangle(P \cap T)\Phi(P) = \Phi(P)$. If $\langle x\rangle(P \cap T)\Phi(P) = P$, then $\langle x\rangle = P$, a contradiction. Hence $\langle x\rangle(P \cap T)\Phi(P) = \Phi(P)$ and so $\langle x\rangle\Phi(P)/\Phi(P) = 1$. We also can see that $\langle x\rangle\Phi(P)/\Phi(P)$ is S-quasinormally embedded in G.) Therefore there exists an S-quasinormal subgroup $M/\Phi(P)$ of $G/\Phi(P)$ such that

$\langle x\rangle\Phi(P)$ is a Sylow p-subgroup of $M/\Phi(P)$. Hence $\langle x\rangle\Phi(P)/\Phi(P) = P/\Phi(P) \cap M/\Phi(P)Q\Phi(P)/\Phi(P)$ for every Sylow q-subgroup Q of G with $q \neq p$. It follows that $Q\Phi(P/\Phi(P) \leq N_{G/\Phi(P)}(\langle x\rangle\Phi(P)/\Phi(P))$. On the other hand, since $P/\Phi(P)$ is abelian, $\langle x\rangle\Phi(P))/\Phi(P) \trianglelefteq P/\Phi(P)$. This implies that $\langle x\rangle\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. Hence $\langle x\rangle\Phi(P) = P$ since $P/\Phi(P)$ is chief factor of $G/\Phi(P)$ and consequently $\langle x\rangle = P$, a contradiction. Thus the proof is completed. □

4. Applications

Our theorems have many corollaries. Now we only state such special cases of them which can be found in the literature.

Theorem 3.1 immediately implies:

Corollary 4.1. (Srinivasan [20]) *If every maximal subgroup of every Sylow subgroup of G is S-quainormal in G, then G is supersoluble.*

Corollary 4.2. (Ballester-Bolinches, Pedraza-Aguilera [4]) *Let G be a finite group. If each maximal subgroup of Sylow subgroups of G is S-quasinormally embedded in G, then G is supersoluble.*

From Theorem 3.2 and Corollary 3.2.1 we obtain:

Corollary 4.3. (Ramadan [16]) *Let G be a soluble group. If all maximal subgroups of the Sylow subgroups of $F(E)$ are normal in G, then G is supersoluble.*

Corollary 4.4. (Asaad, Ramadan, Shaalan [3]) *Let G be a group and E a soluble normal subgroup of G with supersoluble quotient G/E. Suppose that all maximal subgroups of any Sylow subgroup of $F(E)$ are S-quasinormal in G. Then G is supersoluble.*

Corollary 4.5. (Ballester-Bolinches, Pedraza-Aguilera [4]) *Let G be a soluble group with a normal subgroup H such that G/H is supersoluble. If all maximal subgroups of the Sylow subgroups of $F(H)$ are S-quasinormally embedded in G, then G is supersoluble.*

Corollary 4.6. (Li, Guo [14]) *Let G be a group and E a soluble normal subgroup of G with supersoluble quotient G/E. If all maximal subgroups of the Sylow subgroups of $F(E)$ are c-normal in G, then G is supersoluble.*

Corollary 4.7. (Wei [21] *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group with a soluble normal subgroup E such that $G/E \in \mathfrak{F}$. If all maximal subgroups of the Sylow subgroups of $F(E)$ are c-normal in G, then $G \in \mathfrak{F}$.*

Corollary 4.8. (Asaad [1]) *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group with normal subgroup E such that $G/E \in \mathfrak{F}$. If every maximal subgroup of every Sylow subgroup of E is S-quasinormal in G, then $G \in \mathfrak{F}$.*

Corollary 4.9. (Asaad, Heliel [2]) *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and let G be a group. Then the following two statements are equivalent:*

(a) $G \in \mathfrak{F}$.

(b) *There is a normal subgroup H in G such that $G/H \in \mathfrak{F}$ and the maximal subgroups of the Sylow subgroups of H are S-quasinormally embedded in G.*

Corollary 4.10. (Wang [23]) *Let G be a group and E a normal subgroup of G with supersoluble quotient G/E. If all the maximal subgroups of the Sylow subgroups of E are c-normal in G, then G is supersoluble.*

By Theorem 3.3, we have:

Corollary 4.11. (Shaalan [18]) *Let G be a group and E a normal subgroup of G with supersoluble quotient. Suppose that all minimal subgroups of E and all its cyclic subgroups with order 4 are S-quasinormal in G. Then G is supersoluble.*

Corollary 4.12. (Wang [23]) *If every subgroup of G of prime order and cyclic subgroup of order 4 are c-normal in G, then G is supersoluble.*

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