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# ON GENERALIZED $(\theta, \phi)$-DERIVATIONS IN SEMIPRIME RINGS WITH INVOLUTION

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ABSTRACT. The main purpose of this paper is to prove the following result: Let $R$ be a 2-torsion free semiprime $*$-ring. Suppose that $\theta$, $\phi$ are endomorphisms of $R$ such that $\theta$ is onto. If there exists an additive mapping $F\colon R \to R$ associated with a $(\theta, \phi)$-derivation d of $R$ such that $F(xx^*) = F(x)\theta(x^*) + \phi(x)\mathrm{d}(x^*)$ holds for all $x \in R$, then $F$ is a generalized $(\theta, \phi)$-derivation. Further, some more related results are obtained.



## 1. Introduction

Throughout the discussion, unless otherwise mentioned, $R$ will denote an associative ring having at least two elements with center $Z(R)$. However, $R$ may not have unity. For any $x, y \in R$, the symbol $[x, y]$ (resp. $(x \circ y)$) will denote the commutator $xy - yx$ (resp. the anti-commutator $xy + yx$). Recall that $R$ is *prime* if $aRb = \{0\}$ implies that $a = 0$ or $b = 0$. A ring $R$ is called *semiprime* if $aRa = \{0\}$ with $a \in R$ implies $a = 0$. An additive mapping $x \mapsto x^*$ satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$, is called an involution on $R$. A ring equipped with an involution is called a $*$-ring or ring with involution.

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An additive mapping $\mathrm{d}\colon R \to R$ is called a derivation (resp. Jordan derivation) if $\mathrm{d}(xy) = \mathrm{d}(x)y + x\mathrm{d}(y)$ (resp. $\mathrm{d}(x^2) = \mathrm{d}(x)x + x\mathrm{d}(x)$) holds for all $x, y \in R$. Following Bresar [6], an additive mapping $F\colon R \to R$ is said to be a *generalized derivation* (resp. *generalized Jordan derivation*) on $R$ if there exists a derivation $\mathrm{d}\colon R \to R$ such that $F(xy) = F(x)y + x\mathrm{d}(y)$ (resp. $F(x^2) = F(x)x + x\mathrm{d}(x)$) holds for all $x, y \in R$. It is easy to check that the notion of generalized derivation covers the concept of derivation as well as of a left multiplier i.e., an additive mapping $f$ on $R$ satisfying $f(xy) = f(x)y$ for all $x, y \in R$.

Following [11] and [16], an additive mapping $T\colon R \to R$ is called a left (resp. right) centralizer if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) holds for all $x, y \in R$. If $T$ is both left as well as right centralizer, then $T$ is called a centralizer. If $a \in R$, then $T_a(x) = ax$ is a left centralizer and $T_a(x) = xa$ is a right centralizer. Note that in a 2-torsion free ring the condition $\mathrm{d}(x^2) = \mathrm{d}(x)x + x\mathrm{d}(x)$ is equivalent to $\mathrm{d}(x \circ y) = \mathrm{d}(x) \circ y + x \circ \mathrm{d}(y)$ for all $x, y \in R$. Therefore, we can define a Jordan centralizer to be an additive mapping $T$ which satisfies $T(x \circ y) = T(x) \circ y = x \circ T(y)$ for all $x, y \in R$. Since the product $\circ$ is commutative, there is no difference between the Jordan left and right centralizers. Let $T\colon R \to R$ be an additive mapping and $\alpha$ be an endomorphism of $R$. According to [1], $T$ is called a Jordan left (resp. right) $\alpha$-centralizer if $T(x^2) = T(x)\alpha(x)$ (resp. $T(x^2) = \alpha(x)T(x)$) holds for all $x \in R$. Obviously every left (right) $\alpha$-centralizer is a Jordan left (right) $\alpha$-centralizer. The converse is not true in general (see [1, Example 1]). In [1], Albaş showed that under the condition $\alpha(Z(R)) = Z(R)$, in a 2-torsion free semiprime ring $R$, every Jordan $\alpha$-centralizer is a $\alpha$-centralizer. Further, third author together with Haetinger [2] proved that every Jordan left (resp. right) $\alpha$-centralizer on a 2-torsion free semiprime ring is a left (resp. right) $\alpha$-centralizer. Considerable work has been done on Jordan left (resp. right) centralizers in prime and semiprime rings during the last couple of decades (see for example: [1], [2], [10], [11], [12], [14] and [15] where further references can be found).

Let $\theta$ and $\phi$ be endomorphisms of $R$. An additive mapping $\mathrm{d}\colon R \to R$ is said to be a $(\theta, \phi)$-derivation (resp. Jordan $(\theta, \phi)$-derivation) if $\mathrm{d}(xy) = \mathrm{d}(x)\theta(y) + \phi(x)\mathrm{d}(y)$ (resp. $\mathrm{d}(x^2) = \mathrm{d}(x)\theta(x) + \phi(x)\mathrm{d}(x)$) holds for all $x, y \in R$. Following [4], an additive mapping $F\colon R \to R$ is called a generalized $(\theta, \phi)$-derivation (resp. generalized Jordan $(\theta, \phi)$-derivation) on $R$ if there exists a $(\theta, \phi)$-derivation $\mathrm{d}\colon R \to R$ such that $F(xy) = F(x)\theta(y) + \phi(x)\mathrm{d}(y)$ (resp. $F(x^2) = F(x)\theta(x) + \phi(x)\mathrm{d}(x)$) holds for all $x, y \in R$. Note that for $I_R$ the identity map on $R$, an $(I_R, I_R)$-generalized Jordan derivation is called simply a generalized Jordan derivation. It is obvious to see that every generalized $(\theta, \phi)$-derivation on a ring is a generalized Jordan $(\theta, \phi)$-derivation. But the converse need not be true in general (see [2, Example 3.1]). A number of authors have studied this problem in the setting of prime and semiprime rings (viz. [2], [3], [4], [5], [7], [9] and [13], where further references can be found). Very recently, third author together with Haetinger [2], proved that every generalized Jordan $(\theta, \phi)$-derivation on a 2-torsion free semiprime ring is a generalized $(\theta, \phi)$-derivation. In [8], Daif and El-Sayiad obtained the following result: Let $R$ be a 2-torsion free semiprime $*$-ring and $F\colon R \to R$ is an additive mapping associated with a derivation $\mathrm{d}\colon R \to R$ such that $F(xx^*) = F(x)x^* + x\mathrm{d}(x^*)$ holds for all $x \in R$. Then $F$ is a generalized Jordan derivation.

The main purpose of this paper is to extend the above mentioned result for generalized Jordan $(\theta, \phi)$-derivation.

## 2. Main results

The main result of the present paper states as follows:

**Theorem 2.1.** *Let $R$ be a 2-torsion free semiprime $*$-ring. Suppose that $\theta$, $\phi$ are endomorphisms of $R$ such that $\theta$ is an automorphism of $R$. If there exists an additive mapping $F\colon R \to R$ associated with a $(\theta, \phi)$-derivation* d *of $R$ such that*

$$F(xx^*) = F(x)\theta(x^*) + \phi(x)\mathrm{d}(x^*)$$

*holds for all $x \in R$, then*

$$F(xy) = F(x)\theta(y) + \phi(x)\mathrm{d}(y) \qquad \text{for all} \quad x, y \in R.$$

We begin our discussion with the following known lemmas:

**Lemma 2.1.** ([2, Theorem 2.2]) *Let $R$ be a 2-torsion free semiprime ring and $\theta$ be an automorphism of $R$. If $H\colon R \to R$ is an additive mapping such that $H(x^2) = H(x)\theta(x)$ for all $x \in R$, then $H$ is a left $\theta$-centralizer.*

**Lemma 2.2.** ([2, Theorem 3.1]) *Let $R$ be a 2-torsion free semiprime ring. Suppose that $\theta$, $\phi$ are endomorphisms of $R$ such that $\theta$ is an automorphism of $R$. If $F\colon R \to R$ is a generalized Jordan $(\theta, \phi)$-derivation on $R$, then $F$ is a generalized $(\theta, \phi)$-derivation on $R$.*

Now we prove the following:

**Lemma 2.3.** *Let $R$ be a 2-torsion free semiprime ring. Suppose that $\theta$ is an automorphism of $R$. If there exists an element $a \in R$ such that $a\theta(x^*) = a\theta(x)$ holds for all $x \in R$, then $a \in Z(R)$.*

Proof. By hypothesis, we have $a\theta(x^*) = a\theta(x)$, for all $x \in R$. Replacing $x$ by $x^*y$, we obtain $a\theta([x, y]) = 0$, for all $x, y \in R$. Replacing $y$ by $\theta^{-1}(a)$ in the latter relation we find that $a[\theta(x), a] = 0$, for all $x \in R$. Again replacing $x$ by $yx$, we find that

$$a\theta(y)[\theta(x), a] = 0 \qquad \text{for all} \quad x, y \in R. \tag{2.1}$$

Now, putting $xy$ for $y$ in (2.1), we get

$$a\theta(x)\theta(y)[\theta(x), a] = 0 \qquad \text{for all} \quad x, y \in R. \tag{2.2}$$

Left multiplication by $\theta(x)$ to the equation (2.1) yields that

$$\theta(x)a\theta(y)[\theta(x), a] = 0 \qquad \text{for all} \quad x, y \in R. \tag{2.3}$$

Combining (2.2) and (2.3), we have $[\theta(x), a]\theta(y)[\theta(x), a] = 0$ for all $x, y \in R$. Since $\theta$ is onto, the last expression implies that $[\theta(x), a]R[\theta(x), a] = \{0\}$ for all $x \in R$. Thus, the semiprimeness of $R$ forces that $[\theta(x), a] = 0$ for all $x \in R$ and hence $a \in Z(R)$. $\square$

Proof of Theorem 2.1. Suppose that the associated $(\theta, \phi)$-derivation $\mathrm{d} \neq 0$. Then, by the hypothesis, we have

$$F(xx^*) = F(x)\theta(x^*) + \phi(x)\mathrm{d}(x^*) \qquad \text{for all} \quad x \in R. \tag{2.4}$$

Replace $x$ by $x + y$ in (2.4), to get

$$\begin{gathered} F(xy^* + yx^*) = F(x)\theta(y^*) + F(y)\theta(x^*) + \phi(y)\mathrm{d}(x^*) + \phi(x)\mathrm{d}(y^*) \\ \text{for all} \quad x, y \in R. \end{gathered} \tag{2.5}$$

Replacing $y$ by $x^*$ in (2.5), we obtain

$$\begin{gathered} F(xx + x^*x^*) = F(x)\theta(x) + F(x^*)\theta(x^*) + \phi(x^*)\mathrm{d}(x^*) + \phi(x)\mathrm{d}(x) \\ \text{for all} \quad x \in R. \end{gathered} \tag{2.6}$$

This can be rewritten as

$$\delta(x) + \delta(x^*) = 0 \qquad \text{for all} \quad x \in R \tag{2.7}$$

where $\delta(x) = F(x^2) - F(x)\theta(x) - \phi(x)\mathrm{d}(x)$.

Now, taking $xy^* + yx^*$ for $y$ in (2.5), we find that

$$F(x(y^* + y)x^*) = -\delta(x)\theta(y^*) + F(x)\theta((y + y^*)x^*) + \phi(x)\mathrm{d}((y + y^*)x^*). \tag{2.8}$$

Replacing $y$ by $y - y^*$ in (2.8) we get

$$\delta(x)\theta(y) = \delta(x)\theta(y^*) \qquad \text{for all} \quad x, y \in R. \tag{2.9}$$

In view of Lemma 2.3, the above expression implies that $\delta(x) \in Z(R)$ for all $x \in R$. Further, replace $y$ by $y^*$ in (2.5), to get

$$F(xy + y^*x^*) = F(x)\theta(y) + F(y^*)\theta(x^*) + \phi(y^*)\mathrm{d}(x^*) + \phi(x)\mathrm{d}(y) \quad \text{for all} \quad x, y \in R. \tag{2.10}$$

Now, replacing $y$ by $xy$ in (2.10), we obtain

$$F(x^2y + y^*{x^*}^2) = F(x)\theta(xy) + F(y^*x^*)\theta(x^*) + \phi(y^*x^*)\mathrm{d}(x^*) + \phi(x)\mathrm{d}(xy) \quad \text{for all} \quad x, y \in R. \tag{2.11}$$

On the other hand, replacing $x$ by $x^2$ in (2.10), we find that

$$F(x^2y + y^*{x^*}^2) = F(x^2)\theta(y) + F(y^*)\theta({x^*}^2) + \phi(y^*)\mathrm{d}({x^*}^2) + \phi(x^2)\mathrm{d}(y) \quad \text{for all} \quad x, y \in R. \tag{2.12}$$

Combining (2.11) and (2.12), we obtain

$$\delta(x)\theta(y) + (F(y^*)\theta(x^*) - F(y^*x^*) + \phi(y^*)\mathrm{d}(x^*))\theta(x^*) = 0 \qquad \text{for all} \quad x, y \in R. \tag{2.13}$$

In particular, the above relation reduces to

$$\delta(x)\theta(x) - \delta(x^*)\theta(x^*) = 0 \qquad \text{for all} \quad x \in R. \tag{2.14}$$

According to (2.7), one can obtain

$$\delta(x)\theta(x + x^*) = 0 \qquad \text{for all} \quad x \in R. \tag{2.15}$$

Replacing $y$ by $x$ in (2.9) we get

$$\delta(x)\theta(x - x^*) = 0 \qquad \text{for all} \quad x \in R. \tag{2.16}$$

On combining last two equations, we find that

$$\delta(x)\theta(x) = 0 \qquad \text{for all} \quad x \in R. \tag{2.17}$$

Since $\delta(x) \in Z(R)$ for all $x \in R$, the last equation implies that $\theta(x)\delta(x) = 0$ for all $x \in R$. Linearization of (2.17) yields that

$$\delta(x)\theta(y) + \delta(y)\theta(x) + A(x, y)\theta(x) + A(x, y)\theta(y) = 0 \qquad \text{for all} \quad x, y \in R, \tag{2.18}$$

where $A(x, y) = F(xy + yx) - F(x)\theta(y) - F(y)\theta(x) - \phi(x)\mathrm{d}(y) - \phi(y)\mathrm{d}(x)$. Taking $-x$ in place of $x$ in the last expression, we get

$$\delta(x)\theta(y) - \delta(y)\theta(x) + A(x, y)\theta(x) - A(x, y)\theta(y) = 0 \qquad \text{for all} \quad x, y \in R. \tag{2.19}$$

On adding (2.18) and (2.19) and using the fact that $R$ is 2-torsion free, we find that $\delta(x)\theta(y) + A(x,y)\theta(x) = 0$ for all $x, y \in R$. On right multiplication by $\delta(x)$ gives that $\delta(x)\theta(y)\delta(x) = 0$ for all $x, y \in R$. Since $\theta$ is onto, we have $\delta(x)R\delta(x) = \{0\}$ for all $x \in R$. Semiprimeness of $R$ implies that $\delta(x) = 0$, i.e., $F(x^2) = F(x)\theta(x) + \phi(x)\mathrm{d}(x)$ for all $x \in R$. Therefore, $F$ is a generalized Jordan $(\theta, \phi)$-derivation on $R$ and in view of Lemma 2.2, $F$ is a generalized $(\theta, \phi)$-derivation.

On the other hand we assume that the associated $(\theta, \phi)$-derivation $\mathrm{d} = 0$. Then, we have $F(xx^*) = F(x)\theta(x^*)$ for all $x \in R$. Following similar proof as above, we find that $F(x^2) = F(x)\theta(x)$ for all $x \in R$, i.e., $F$ is a Jordan left $\theta$-centralizer on $R$. Thus, $F$ is a left $\theta$-centralizer on $R$ by Lemma 2.1. Hence, $F$ is a generalized $(\theta, \phi)$-derivation for $\mathrm{d} = 0$. This completes the proof of the theorem. □

Following are the immediate consequences of Theorem 2.1.

**Corollary 2.1.** ([8, Theorem 2.1]) *Let $R$ be a 2-torsion free semiprime $*$-ring. If there exists an additive mapping $F\colon R \to R$ associated with a derivation $\mathrm{d}$ of $R$ such that $F(xx^*) = F(x)x^* + x\mathrm{d}(x^*)$ holds for all $x \in R$, then $F$ is a generalized Jordan derivation.*

**Corollary 2.2.** *Let $R$ be a 2-torsion free semiprime $*$-ring. If there exists an additive mapping $F\colon R \to R$ associated with a derivation $\mathrm{d}$ of $R$ such that $F(xx^*) = F(x)x^* + x\mathrm{d}(x^*)$ holds for all $x \in R$, then $F$ is a generalized derivation.*

**Corollary 2.3.** ([15, Theorem 1]) *Let $R$ be a 2-torsion free prime $*$-ring and let $T\colon R \to R$ be an additive mapping such that $T(xx^*) = T(x)x^*$ (resp. $T(xx^*) = x^*T(x)$) holds for all $x \in R$. In this case $T$ is a left (resp. right) centralizer.*

We now prove another result in the spirit of Theorem 2.1, that is:

**Theorem 2.2.** *Let $R$ be a 2-torsion free semiprime $*$-ring. Suppose that $\theta$, $\phi$ are endomorphisms of $R$ such that $\theta$ is one-one and onto. If there exists an additive mapping $F\colon R \to R$ associated with a $(\theta, \phi)$-derivation $\mathrm{d}$ of $R$ such that*

$$F(xy^*x) = F(x)\theta(y^*x) + \phi(x)\mathrm{d}(y^*)\theta(x) + \phi(xy^*)\mathrm{d}(x)$$

*holds for all $x, y \in R$, then $F$ is a generalized $(\theta, \phi)$-derivation.*

Proof. By the hypothesis, we have

$$F(xy^*x) = F(x)\theta(y^*x) + \phi(x)\mathrm{d}(y^*)\theta(x) + \phi(xy^*)\mathrm{d}(x) \qquad \text{for any} \quad x, y \in R. \tag{2.20}$$

Linearizing (2.20) on $x$, we get

$$\begin{aligned} F((x+z)y^*(x+z)) = {} & F(x)\theta(y^*x)+F(x)\theta(y^*z)+F(z)\theta(y^*x) \\ & +F(z)\theta(y^*z)+\phi(x)\mathrm{d}(y^*)\theta(x)+\phi(x)\mathrm{d}(y^*)\theta(z) \\ & +\phi(z)\mathrm{d}(y^*)\theta(x)+\phi(z)\mathrm{d}(y^*)\theta(z)+\phi(xy^*)\mathrm{d}(x) \\ & +\phi(xy^*)\mathrm{d}(z)+\phi(zy^*)\mathrm{d}(x)+\phi(zy^*)\mathrm{d}(z) \\ & \text{for all} \quad x,y,z\in R. \end{aligned} \tag{2.21}$$

On the other hand, we obtain

$$\begin{aligned} F((x+z)y^*(x+z)) = {} & F(xy^*z+zy^*x)+F(x)\theta(y^*x)+\phi(x)\mathrm{d}(y^*)\theta(x) \\ & +\phi(xy^*)\mathrm{d}(x)+F(z)\theta(y^*z)+\phi(z)\mathrm{d}(y^*)\theta(z) \\ & +\phi(zy^*)\mathrm{d}(z) \qquad \text{for all} \quad x,y,z\in R. \end{aligned} \tag{2.22}$$

Combining (2.21) and (2.22), we get

$$\begin{aligned} F(xy^*z+zy^*x) = {} & F(x)\theta(y^*z)+F(z)\theta(y^*x)+\phi(x)\mathrm{d}(y^*)\theta(z) \\ & +\phi(z)\mathrm{d}(y^*)\theta(x)+\phi(xy^*)\mathrm{d}(z)+\phi(zy^*)\mathrm{d}(x) \\ & \text{for all} \quad x,y,z\in R. \end{aligned} \tag{2.23}$$

Replacing $z$ by $x^2$ in (2.23) we find that

$$\begin{aligned} F(xy^*x^2+x^2y^*x) = {} & F(x)\theta(y^*x^2)+F(x^2)\theta(y^*x)+\phi(x)\mathrm{d}(y^*)\theta(x^2) \\ & +\phi(x^2)\mathrm{d}(y^*)\theta(x)+\phi(xy^*)\mathrm{d}(x^2)+\phi(x^2y^*)\mathrm{d}(x) \\ & \text{for all} \quad x,y\in R. \end{aligned} \tag{2.24}$$

Further, replacing $x^*y+yx^*$ for $y$ in (2.20), we get

$$\begin{aligned} F(xy^*x^2+x^2y^*x) = {} & F(x)\theta(xy^*x)+F(x)\theta(y^*x^2)+\phi(x)\mathrm{d}(xy^*)\theta(x) \\ & +\phi(x)\mathrm{d}(y^*x)\theta(x)+\phi(x^2y^*)\mathrm{d}(x)+\phi(xy^*x)\mathrm{d}(x) \\ & \text{for all} \quad x,y\in R. \end{aligned} \tag{2.25}$$

Combining the last two equations, we obtain

$$F(x^2)\theta(y^*x)-F(x)\theta(x)\theta(y^*x)-\phi(x)\mathrm{d}(x)\theta(y^*x)=0 \qquad \text{for all} \quad x,y\in R. \tag{2.26}$$

Now, we put $H(x)=F(x^2)-F(x)\theta(x)-\phi(x)\mathrm{d}(x)$. Then, the equation (2.26) reduces to

$$H(x)\theta(y^*x)=0 \qquad \text{for all} \quad x,y\in R. \tag{2.27}$$

Since $\theta$ is one-one and onto, (2.27) implies that

$$\theta^{-1}(H(x))y^*x=0 \qquad \text{for all} \quad x,y\in R. \tag{2.28}$$

Replacing $y$ by $y^*x^*$ in (2.28), we find that

$$\theta^{-1}(H(x))xyx=0 \qquad \text{for all} \quad x,y\in R. \tag{2.29}$$

Now, replace $y$ by $\theta^{-1}(z)\theta^{-1}(H(x))$ in (2.29), to get

$$\theta^{-1}(H(x))x\theta^{-1}(z)\theta^{-1}(H(x))x = 0 \qquad \text{for all} \quad x, z \in R. \tag{2.30}$$

This implies that

$$H(x)\theta(x)zH(x)\theta(x) = 0 \quad \text{i.e.,} \quad H(x)\theta(x)RH(x)\theta(x) = \{0\} \qquad \text{for all} \quad x \in R.$$

Semiprimeness of $R$ yields that

$$H(x)\theta(x) = 0 \qquad \text{for all} \quad x \in R. \tag{2.31}$$

Linearize (2.31) to get

$$H(x+y)\theta(x) + H(x+y)\theta(y) = 0 \qquad \text{for all} \quad x, y \in R. \tag{2.32}$$

Now, we find

$$\begin{aligned} H(x+y) &= \big\{F(xy+yx) - F(x)\theta(y) - F(y)\theta(x) - \phi(x)\mathrm{d}(y) - \phi(y)\mathrm{d}(x)\big\} \\ &\quad + \big\{F(x^2) - F(x)\theta(x) - \phi(x)\mathrm{d}(x)\big\} \\ &\quad + \big\{F(y^2) - F(y)\theta(y) - \phi(y)\mathrm{d}(y)\big\} \\ &= \beta(x,y) + H(x) + H(y) \qquad \text{for all} \quad x, y \in R, \end{aligned} \tag{2.33}$$

where $\beta(x,y)$ stands for $F(xy+yx) - F(x)\theta(y) - F(y)\theta(x) - \phi(x)\mathrm{d}(y) - \phi(y)\mathrm{d}(x)$. Thus, in view of (2.33) and (2.32) we have

$$H(x)\theta(y) + \beta(x,y)\theta(x) + H(y)\theta(x) + \beta(x,y)\theta(y) = 0 \qquad \text{for all} \quad x, y \in R. \tag{2.34}$$

Again, replace $x$ by $-x$ in the last equation to get

$$H(x)\theta(y) + \beta(x,y)\theta(x) - H(y)\theta(x) - \beta(x,y)\theta(y) = 0 \qquad \text{for all} \quad x, y \in R. \tag{2.35}$$

Adding (2.34) and (2.35) and using the fact that $R$ is 2-torsion free, we find that

$$H(x)\theta(y) + \beta(x,y)\theta(x) = 0 \qquad \text{for all} \quad x, y \in R. \tag{2.36}$$

On right multiplication of equation (2.36) by $H(x)$, we obtain

$$H(x)\theta(y)H(x) + \beta(x,y)\theta(x)H(x) = 0 \qquad \text{for all} \quad x, y \in R. \tag{2.37}$$

Now replacing $y$ by $y^*$ in (2.28), we find that $\theta(x)H(x)\theta(y)\theta(x)H(x) = 0$. That is,

$$\theta(x)H(x)R\theta(x)H(x) = \{0\} \qquad \text{for all} \quad x \in R. \tag{2.38}$$

Thus the semiprimeness of $R$ forces that

$$\theta(x)H(x) = 0 \qquad \text{for all} \quad x \in R. \tag{2.39}$$

Combining (2.37) and (2.39), we have $H(x)\theta(y)H(x)=0$, i.e., $H(x)RH(x)=\{0\}$ for all $x \in R$, and hence $H(x) = 0$, i.e., $F(x^2) - F(x)\theta(x) - \phi(x)\mathrm{d}(x) = 0$ for all $x \in R$. Thus $F$ is a generalized Jordan $(\theta,\phi)$-derivation of $R$. Therefore by Lemma 2.2, we get the required result. □

If we take $\mathrm{d} = 0$ in Theorem 2.2, then we get the following result.

**Theorem 2.3.** *Let $R$ be a 2-torsion free semiprime $*$-ring. Suppose that $\theta$ is an automorphism of $R$. If there exists an additive mapping $F\colon R \to R$ such that*

$$F(xy^*x) = F(x)\theta(y^*x)$$

*holds for all $x, y \in R$, then $F$ is a left $\theta$-centralizer.*

**Corollary 2.4.** *Let $R$ be a 2-torsion free semiprime $*$-ring. Let $F\colon R \to R$ be an additive mapping associated with a derivation* d *such that*

$$F(xy^*x) = F(x)y^*x + x\mathrm{d}(y^*)x + xy^*\mathrm{d}(x)$$

*holds for all $x, y \in R$. Then $F$ is a generalized derivation on $R$.*

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