VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-012-0020-2
Math. Slovaca **62** (2012), No. 3, 439–450

ON FACTORIZATION OF THE FIBONACCI AND LUCAS NUMBERS USING TRIDIAGONAL DETERMINANTS

Jaroslav Seibert — Pavel Trojovský

(*Communicated by Stanislav Jakubec*)

ABSTRACT. The aim of this paper is to give new results about factorizations of the Fibonacci numbers F_n and the Lucas numbers L_n. These numbers are defined by the second order recurrence relation $a_{n+2} = a_{n+1} + a_n$ with the initial terms $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$, respectively. Proofs of theorems are done with the help of connections between determinants of tridiagonal matrices and the Fibonacci and the Lucas numbers using the Chebyshev polynomials. This method extends the approach used in [CAHILL, N. D.—D'ERRICO, J. R.—SPENCE, J. P.: *Complex factorizations of the Fibonacci and Lucas numbers*, Fibonacci Quart. **41** (2003), 13–19], and CAHILL, N. D.—NARAYAN, D. A.: *Fibonacci and Lucas numbers as tridiagonal matrix determinants*, Fibonacci Quart. **42** (2004), 216–221].

1. Introduction

Interesting connections were found between the determinants of tridiagonal matrices and the Fibonacci or Lucas numbers. For example, Strang [6] presented a family of the $n \times n$ tridiagonal matrices given by

$$\mathbb{M}(n) = \begin{pmatrix} 3 & 1 & & & & \\ 1 & 3 & 1 & & & \\ & 1 & 3 & 1 & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & 3 & 1 \\ & & & & 1 & 3 \end{pmatrix},$$

2010 Mathematics Subject Classification: Primary 11B39, 15A36; Secondary 15A15.
Keywords: Fibonacci and Lucas numbers, tridiagonal matrix, Chebyshev polynomials.
This work was supported by Specific research 2111 UHK.

where determinants $|\mathbb{M}(n)|$ are the Fibonacci numbers F_{2n+2}. Cahill et al. [1, 2] applied a family of the tridiagonal matrices

$$\mathbb{H}(n) = \begin{pmatrix} 1 & \mathrm{i} & & & & \\ \mathrm{i} & 1 & \mathrm{i} & & & \\ & \mathrm{i} & 1 & \mathrm{i} & & \\ & & \mathrm{i} & \ddots & \ddots & \\ & & & \ddots & 1 & \mathrm{i} \\ & & & & \mathrm{i} & 1 \end{pmatrix},$$

whose determinants $|\mathbb{H}(n)|$ are the Fibonacci numbers F_{n+1}.

They derived a general recurrence for the determinants of a sequence of symmetric tridiagonal matrices (see [2, Lemma 1]) and used some sequences of this type for searching of the following complex factorizations

$$F_{n+1} = \prod_{k=1}^{n} \left(1 - 2\mathrm{i}\cos\frac{k\pi}{n+1}\right), \qquad L_n = \prod_{k=1}^{n} \left(1 - 2\mathrm{i}\cos\frac{(2k-1)\pi}{2n}\right), \tag{1}$$

which are valid for $n \geq 1$. Their method is based on certain properties of the Chebyshev polynomials. The Chebyshev polynomials $T_n(x)$ of the first kind and the Chebyshev polynomials $U_n(x)$ of the second kind, respectively, are defined by the recurrences

$$\begin{aligned} T_1(x) &= x, & U_1(x) &= 2x, \\ T_2(x) &= 2x^2 - 1, & U_2(x) &= 4x^2 - 1, \\ T_n(x) &= 2x\,T_{n-1}(x) - T_{n-2}(x), & U_n(x) &= 2x\,U_{n-1}(x) - U_{n-2}(x). \end{aligned}$$

These polynomials can be generated by determinants of successive matrices of the form

$$\begin{pmatrix} x & 1 & & & & \\ 1 & 2x & 1 & & & \\ & 1 & 2x & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 2x & 1 \\ & & & & 1 & 2x \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2x & 1 & & & & \\ 1 & 2x & 1 & & & \\ & 1 & 2x & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 2x & 1 \\ & & & & 1 & 2x \end{pmatrix},$$

respectively. Setting $x = \cos\theta$, where θ is a complex number, then the Chebyshev polynomials can be written as

$$T_n(x) = \cos(n\theta) \qquad \text{and} \qquad U_n(x) = \frac{\sin((n+1)\theta)}{\sin\theta}. \tag{2}$$

The authors remarked that a similar method can be used for deriving the formula

$$F_{2n+2} = \prod_{k=1}^{n} \left(3 - 2\cos\frac{k\pi}{n+1}\right), \qquad n \geq 1. \tag{3}$$

Cahill and Narayan [3] found generalizations of matrices $\mathbb{H}(n)$, by construction of families of the symmetric tridiagonal matrices $\mathbb{M}_{\alpha,\beta}(k)$ and $\mathbb{V}_{\alpha,\beta}(k)$, $k = 1, 2, 3, \ldots$, by the following way.

Theorem 1. ([3, Theorem 2]) *The symmetric tridiagonal family of matrices $\mathbb{M}_{\alpha,\beta}(k)$, $k = 1, 2, 3, \ldots$, whose elements are given by*

$$\begin{aligned}
m_{1,1} &= F_{\alpha+\beta}, & m_{2,2} &= \left\lceil \frac{F_{2\alpha+\beta}}{F_{\alpha+\beta}} \right\rceil, \\
m_{1,2} = m_{2,1} &= \sqrt{m_{2,2}F_{\alpha+\beta} - F_{2\alpha+\beta}}, & m_{j,j} &= L_\alpha, \quad 3 \leq j \leq k\,, \\
m_{j,j+1} = m_{j+1,j} &= \sqrt{(-1)^\alpha}, \quad 2 \leq j < k,
\end{aligned}$$

with $\alpha \in \mathbb{Z}^+$ and $\beta \in \mathbb{N}$, has successive determinants $|\mathbb{M}_{\alpha,\beta}(k)| = F_{\alpha k+\beta}$.

Theorem 2. ([3, Theorem 3]) *The symmetric tridiagonal family of matrices $\mathbb{V}_{\alpha,\beta}(k)$, $k = 1, 2, 3, \ldots$, whose elements are given by*

$$\begin{aligned}
v_{1,1} &= L_{\alpha+\beta}\,, & v_{2,2} &= \left\lceil \frac{L_{2\alpha+\beta}}{L_{\alpha+\beta}} \right\rceil, \\
v_{1,2} = v_{2,1} &= \sqrt{v_{2,2}L_{\alpha+\beta} - L_{2\alpha+\beta}}, & v_{j,j} &= L_\alpha, \quad 3 \leq j \leq k\,, \\
v_{j,j+1} = v_{j+1,j} &= \sqrt{(-1)^\alpha}, \quad 2 \leq j < k,
\end{aligned}$$

with $\alpha \in \mathbb{Z}^+$ and $\beta \in \mathbb{N}$, has successive determinants $|\mathbb{V}_{\alpha,\beta}(k)| = L_{\alpha k+\beta}$.

Using the family $\mathbb{M}_{\alpha,\beta}(k)$ they also derived the following generalization of identity (3)

$$F_{2mn} = F_{2m} \prod_{k=1}^{n-1} \left(L_{2m} - 2\cos\frac{k\pi}{n}\right). \tag{4}$$

In this paper we derive identity (4) in another way. Further we will find a generalization of (4) and similar factorizations for the Lucas numbers.

Throughout the paper we adopt the conventions (see e.g. [4]) that the product over an empty set is 1 and Iverson's notation

$$[P(k)] = \begin{cases} 1, & \text{if statement } P(k) \text{ is true;} \\ 0, & \text{if statement } P(k) \text{ is false.} \end{cases}$$

2. The preliminary results

Lemma 3. *Let n be any positive integer. Then*

$$L_{2n} = \prod_{k=1}^{n} \left(3 + 2\cos\frac{(2k-1)\pi}{2n} \right). \tag{5}$$

Proof. Replacing n by $2n$ in identity (1) for L_n and multiplying the j-th and $(2n+1-j)$-th factor, for $j = 1, 2, \dots, n$, we get

$$L_{2n} = \prod_{k=1}^{n} \left(1 - 4\cos\frac{(2(2n-k)+1)\pi}{4n} \cos\frac{(2k-1)\pi}{4n} \right).$$

Thus, the assertion follows from the formula

$$-2\cos\alpha\cos(\pi-\alpha) = 1 + \cos(\alpha-(\pi-\alpha))\cos(\alpha+(\pi-\alpha)),$$

which is valid for any real α, if we set $\alpha = \frac{(2k-1)\pi}{4n}$. $\square$

Remark 4. In the same way it can be proved by induction on m that the identity

$$L_{n2^m} = \prod_{k=1}^{n} \left(L_{2^m} + 2\cos\frac{(2k-1)\pi}{2n} \right)$$

holds for any positive integers n, m.

Lemma 5. *Let a, b be any real numbers and let $U_n(x)$ be the Chebyshev polynomial of the second kind. Denote $\mathbb{A}$, $\mathbb{B}$ the $n \times n$ symmetric tridiagonal matrices*

$$\mathbb{A} = \begin{pmatrix} a & 1 & & & & \\ 1 & a & 1 & & & \\ & 1 & a & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & a & 1 \\ & & & & 1 & a \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} a+b & 1 & & & & \\ 1 & a & 1 & & & \\ & 1 & a & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & a & 1 \\ & & & & 1 & a \end{pmatrix}.$$

Then $|\mathbb{A}| = \mathbb{U}_n\left(\frac{a}{2}\right)$ *and* $|\mathbb{B}| = b\,\mathbb{U}_{n-1}\left(\frac{a}{2}\right) + \mathbb{U}_n\left(\frac{a}{2}\right)$.

P r o o f. The first identity is obvious and the second relation can be derived by the expansion of the determinant of $\mathbb{B}$. Thus,

$$\begin{aligned}
|\mathbb{B}| &= (a+b)\begin{vmatrix} a & 1 & & & & \\ 1 & a & 1 & & & \\ & 1 & a & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & a & 1 \\ & & & & 1 & a \end{vmatrix} - \begin{vmatrix} a & 1 & & & \\ 1 & a & 1 & & \\ & 1 & a & 1 & \\ & & \ddots & \ddots & 1 \\ & & & 1 & a \end{vmatrix} \\
&= (a+b)\mathbb{U}_{n-1}\left(\frac{a}{2}\right) - \mathbb{U}_{n-2}\left(\frac{a}{2}\right) \\
&= \left(a\mathbb{U}_{n-1}\left(\frac{a}{2}\right) - \mathbb{U}_{n-2}\left(\frac{a}{2}\right)\right) + b\mathbb{U}_{n-1}\left(\frac{a}{2}\right) = \mathbb{U}_n\left(\frac{a}{2}\right) + b\mathbb{U}_{n-1}\left(\frac{a}{2}\right).
\end{aligned}$$

□

Corollary 6. ([5, (1.8), Corollary of Lemma 3]) *Between the Chebyshev polynomials of the first and the second kind the following holds for any positive integer n*

$$T_n(x) = U_n(x) - x\,U_{n-1}(x).$$

Lemma 7. *Let m, p be any positive integers. Then*

$$\left\lceil \frac{L_{2m+p}}{L_{m+p}} \right\rceil = \left\lceil \frac{F_{2m+p}}{F_{m+p}} \right\rceil = L_m + [2 \nmid m]$$

and

$$\left\lceil \frac{F_{2m+p}}{F_{m+p}} \right\rceil + \left\lfloor \frac{F_{2m+p}}{F_{m+p}} \right\rfloor = \left\lceil \frac{L_{2m+p}}{L_{m+p}} \right\rceil + \left\lfloor \frac{L_{2m+p}}{L_{m+p}} \right\rfloor = 2L_m + (-1)^{m+1}.$$

P r o o f. Using identities ([7, p. 177]),

$$L_{k+n} = L_n\,L_k + (-1)^{n+1}L_{k-n} \tag{6}$$

and

$$F_{k+n} = L_n\,F_k + (-1)^{n+1}F_{k-n}, \tag{7}$$

where k, n are any integers, we get

$$\left\lceil \frac{L_{2m+p}}{L_{m+p}} \right\rceil = \left\lceil L_m + (-1)^{m+1}\frac{L_p}{L_{m+p}} \right\rceil = \begin{cases} L_m + 1\,, & \text{if } m \text{ is odd;} \\ L_m\,, & \text{if } m \text{ is even} \end{cases}$$

and

$$\left\lceil \frac{F_{2m+p}}{F_{m+p}} \right\rceil = \left\lceil L_m + (-1)^{m+1}\frac{F_p}{F_{m+p}} \right\rceil = \begin{cases} L_m + 1\,, & \text{if } m \text{ is odd;} \\ L_m\,, & \text{if } m \text{ is even.} \end{cases}$$

Thus, the first relation is proved. Similarly we obtain that

$$\left\lfloor \frac{F_{2m+p}}{F_{m+p}} \right\rfloor = \left\lfloor \frac{L_{2m+p}}{L_{m+p}} \right\rfloor = \begin{cases} L_m\,, & \text{if } m \text{ is odd;} \\ L_m - 1\,, & \text{if } m \text{ is even} \end{cases}$$

and the second identity follows. □

Lemma 8. *Let m be any positive integer and let p be any nonnegative integer. The symmetric tridiagonal matrices $\mathbb{L}_{m,p}(n)$, $n = 1, 2, 3, \dots$, whose elements are given by*

$$\begin{aligned} &l_{1,1} = L_{2m+p}\,, && l_{1,2} = l_{2,1} = \sqrt{L_p}, \\ &l_{k,k} = L_{2m}, \quad 2 \le k \le n\,, && l_{k,k+1} = l_{k+1,k} = 1, \quad 2 \le k < n, \end{aligned}$$

have successive determinants $|\mathbb{L}_{m,p}(n)| = L_{2mn+p}$.

Proof. The assertion is an obvious consequence of Theorem 2, Lemma 7 and identity (6). □

Lemma 9. *Let m be any positive integer and let p be any nonnegative integer. The symmetric tridiagonal matrices $\mathbb{F}_{m,p}(n)$, $n = 1, 2, 3, \dots$, whose elements are given by*

$$\begin{aligned} &f_{1,1} = F_{2m+p}\,, && f_{1,2} = f_{2,1} = \sqrt{F_p}, \\ &f_{k,k} = L_{2m}, \quad 2 \le k \le n\,, && f_{k,k+1} = f_{k+1,k} = 1, \quad 2 \le k < n, \end{aligned}$$

have successive determinants $|\mathbb{F}_{m,p}(n)| = F_{2mn+p}$.

Proof. The assertion is a clear consequence of Theorem 1, Lemma 7 and identity (7). □

Lemma 10. *Let m, n be any positive integers. Then*

$$F_{2mn} = F_{2m} \prod_{k=1}^{n-1} \left(L_{2m} - 2\cos\frac{k\pi}{n} \right).$$

Proof. We will consider the matrix

$$\mathbb{F}_m(n) := \mathbb{F}_{m,0}(n) = \begin{pmatrix} F_{2m} & 0 & & & \\ 0 & L_{2m} & 1 & & \\ & 1 & L_{2m} & 1 & \\ & & \ddots & \ddots & 1 \\ & & & 1 & L_{2m} \end{pmatrix}$$

with $|\mathbb{F}_m(n)| = F_{2mn}$ according to Lemma 9. Let

$$\mathbb{L}_m(n-1) := \mathbb{L}_{m,0}(n-1) = \begin{pmatrix} L_{2m} & 1 & & & \\ 1 & L_{2m} & 1 & & \\ & 1 & L_{2m} & 1 & \\ & & \ddots & \ddots & 1 \\ & & & 1 & L_{2m} \end{pmatrix}$$

be the $(n-1)\times(n-1)$ symmetric tridiagonal matrix. Therefore we have $|\mathbb{F}_m(n)| = F_{2m}|\mathbb{L}_m(n-1)|$. Let λ_k and γ_k, $k = 1, 2, \ldots, n-1$, be the eigenvalues of $\mathbb{L}_m(n-1)$ and $\mathbb{G}_m(n-1) = \mathbb{L}_m(n-1) - L_{2m}\,\mathbb{I}$, respectively. Thus, $\lambda_k = L_{2m} + \gamma_k$. Eigenvalues γ_k of $\mathbb{G}_m(n-1)$ are the roots of the characteristic equation

$$|\mathbb{G}_m(n-1) - \gamma\,\mathbb{I}| = \begin{vmatrix} -\gamma & 1 & & & \\ 1 & -\gamma & 1 & & \\ & 1 & -\gamma & 1 & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -\gamma \end{vmatrix} = 0,$$

which, setting $-\frac{\gamma}{2} = \cos\theta$ and using properties of the Chebyshev polynomials of the second kind, can be rewritten to the equation $\frac{\sin(n\theta)}{\sin\theta} = 0$, with the roots

$$\theta_k = \frac{k\,\pi}{n}, \qquad k = 1, 2, \ldots, n-1.$$

Therefore $\gamma_k = -2\cos\frac{k\pi}{n}$ and

$$|\mathbb{L}_m(n-1)| = \prod_{k=1}^{n-1} \lambda_k = \prod_{k=1}^{n-1} \left(L_{2m} - 2\cos\frac{k\pi}{n} \right).$$

□

Corollary 11. *Let m, n be any positive integers. Then*

$$|\mathbb{L}_m(n)| = \frac{F_{2m(n+1)}}{F_{2m}}.$$

3. The main results

Theorem 12. *Let m, n be any positive integers and let p be any nonnegative integer. Then*

$$L_{2mn+p} = \frac{L_{2m+p}}{L_{2m}} \prod_{k=1}^{n} \left(L_{2m} - 2\cos\theta_k \right),$$

where θ_k, $k = 1, 2, \ldots, n$, are the roots of the equation

$$L_{2m}L_p \cos(n\theta)\sin\theta + 5\,F_{2m}F_p \sin(n\theta)\cos\theta = 0.$$

P r o o f. Consider the matrix

$$\mathbb{L}_{m,p}(n) = \begin{pmatrix} L_{2m+p} & \sqrt{L_p} & & & \\ \sqrt{L_p} & L_{2m} & 1 & & \\ & 1 & L_{2m} & 1 & \\ & & \ddots & \ddots & 1 \\ & & & 1 & L_{2m} \end{pmatrix}$$

with $|\mathbb{L}_{m,p}(n)| = L_{2mn+p}$ according to Lemma 8. Let

$$\mathbb{C}_{m,p}(n) = \left(\mathbb{I} + \left(\frac{L_{2m}}{L_{2m+p}} - 1\right) e_1 e_1^T\right) \mathbb{L}_{m,p}(n),$$

where e_1 is the first column of the identity matrix $\mathbb{I}$ of the order n. Thus, it is easy to see that $|\mathbb{C}_{m,p}(n)| = \frac{L_{2m}}{L_{2m+p}} |\mathbb{L}_{m,p}(n)|$. Let λ_k and γ_k, $k = 1, 2, \ldots, n-1$, be the eigenvalues of $\mathbb{C}_{m,p}(n)$ and $\mathbb{G}_{m,p}(n) = \mathbb{C}_{m,p}(n) - L_{2m}\,\mathbb{I}$, respectively. Then obviously $\lambda_k = L_{2m} + \gamma_k$. The eigenvalues γ_k of $\mathbb{G}_{m,p}(n)$ are the roots of the characteristic equation $|\mathbb{G}_{m,p}(n) - \gamma\,\mathbb{I}| = 0$. As

$$|\mathbb{G}_{m,p}(n) - \gamma\,\mathbb{I}| = \frac{L_{2m} L_p}{L_{2m+p}} \left| \left(\mathbb{I} + \left(\frac{L_{2m+p}}{L_{2m}\sqrt{L_p}} - 1\right) e_1 e_1^T\right) \cdot \right.$$
$$\left. \cdot\, (\mathbb{G}_{m,p}(n) - \gamma\,\mathbb{I}) \left(\mathbb{I} + \left(\frac{1}{\sqrt{L_p}} - 1\right) e_1 e_1^T\right) \right|,$$

γ_k are also roots of the equation

$$\begin{vmatrix} -\gamma\frac{L_{2m+p}}{L_{2m}L_p} & 1 & & & \\ 1 & -\gamma & 1 & & \\ & 1 & -\gamma & 1 & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -\gamma \end{vmatrix} = 0.$$

Using identity (6) we obtain the equation

$$\begin{vmatrix} -\gamma + \frac{L_{p-2m}}{L_{2m}L_p}\gamma & 1 & & & \\ 1 & -\gamma & 1 & & \\ & 1 & -\gamma & 1 & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -\gamma \end{vmatrix} = 0,$$

which can be rewritten with respect to Lemma 5 as

$$\gamma\,\frac{L_{p-2m}}{L_{2m}L_p}\,U_{n-1}\left(-\frac{\gamma}{2}\right) + U_n\left(-\frac{\gamma}{2}\right) = 0.$$

Setting $-\frac{\gamma}{2} = \cos\theta$ we have

$$-2\cos\theta\,\frac{L_{p-2m}}{L_{2m}L_p}\,U_{n-1}(\cos\theta) + U_n(\cos\theta) = 0. \tag{8}$$

Using relation (2) we can write equation (8) in the form

$$-2\cos\theta\,\frac{L_{p-2m}}{L_{2m}L_p}\,\frac{\sin(n\theta)}{\sin\theta} + \frac{\sin((n+1)\theta)}{\sin\theta} = 0 \tag{9}$$

and therefore consequently

$$\begin{aligned}
\frac{2\,L_{p-2m}}{L_{2m}L_p}\sin(n\theta)\cos\theta &= \sin(n\theta)\cos\theta + \cos(n\theta)\sin\theta,\\
\frac{2\,L_{p-2m} - L_{2m}L_p}{L_{2m}L_p}\sin(n\theta)\cos\theta &= \cos(n\theta)\sin\theta,\\
\frac{L_{p-2m} - L_{p+2m}}{L_{2m}L_p}\sin(n\theta)\cos\theta &= \cos(n\theta)\sin\theta.
\end{aligned}$$

Now, using the well-known identity $L_{k+n} - (-1)^n L_{k-n} = 5\,F_n\,F_k$ (see [7, p. 177]) we have

$$-5\,F_{2m}F_p\sin(n\theta)\cos\theta = L_{2m}L_p\cos(n\theta)\sin\theta. \tag{10}$$

The proof is over. □

Remark 13. For $p = 0$ the roots of equation (10) are $\theta_k = (2k-1)\,\frac{\pi}{2n}$, where $k = 1, 2, \ldots, n$, therefore it is possible to express the factorization of L_{2mn} in the form

$$L_{2mn} = \prod_{k=1}^{n}\left(L_{2m} + 2\cos\frac{(2k-1)\pi}{2n}\right).$$

Theorem 14. *Let m, n be any positive integers and let p be any nonnegative integer. Then*

$$F_{2mn+p} = \begin{cases} F_{2m}\prod\limits_{k=1}^{n-1}\left(L_{2m} - 2\cos\frac{k\pi}{n}\right), & \text{for } p = 0,\\ \frac{F_{2m+p}}{L_{2m}}\prod\limits_{k=1}^{n}\left(L_{2m} - 2\cos\theta_k\right), & \text{for } p \neq 0,\end{cases}$$

where θ_k, $k = 1, 2, \ldots, n$, are roots of the equation

$$F_{2m}L_p\sin(n\theta)\cos\theta + L_{2m}F_p\cos(n\theta)\sin\theta = 0.$$

Proof. The assertion for $p = 0$ follows from Lemma 10. The assertion for $p \neq 0$ can be proved analogously as in Theorem 12. Therefore we only sketch the proof. We use the matrix

$$\mathbb{F}_{m,p}(n) = \begin{pmatrix} F_{2m+p} & \sqrt{F_p} & & & \\ \sqrt{F_p} & L_{2m} & 1 & & \\ & 1 & L_{2m} & 1 & \\ & & \ddots & \ddots & 1 \\ & & & 1 & L_{2m} \end{pmatrix}$$

with the determinant $|\mathbb{F}_{m,p}(n)| = F_{2mn+p}$ according to Lemma 9. Let us denote

$$\mathbb{C}_{m,p}(n) = \left(\mathbb{I} + \left(\frac{L_{2m}}{F_{2m+p}} - 1\right) e_1 e_1^T\right) \mathbb{F}_{m,p}(n)$$

and $\mathbb{G}_{m,p}(n) = \mathbb{C}_{m,p}(n) - L_{2m}\,\mathbb{I}$. It is obvious $|\mathbb{C}_{m,p}(n)| = \frac{L_{2m}}{F_{2m+p}} |\mathbb{F}_{m,p}(n)|$. As

$$|\mathbb{G}_{m,p}(n) - \gamma\,\mathbb{I}| = \frac{L_{2m}F_p}{F_{2m+p}} \left| \left(\mathbb{I} + \left(\frac{F_{2m+p}}{L_{2m}\sqrt{F_p}} - 1\right) e_1 e_1^T\right) \cdot \right.$$

$$\left. \cdot\, (\mathbb{G}_{m,p}(n) - \gamma\,\mathbb{I}) \left(\mathbb{I} + \left(\frac{1}{\sqrt{F_p}} - 1\right) e_1 e_1^T\right) \right|,$$

the characteristic equation for the eigenvalues γ of $\mathbb{G}_{m,p}(n)$ can be rewritten to the form

$$\sin(n\theta)\cos\theta = -\frac{L_{2m}F_p}{F_{2m}L_p}\cos(n\theta)\sin\theta. \tag{11}$$

This finishes the proof. □

Corollary 15. *Let m, n be any positive integers and let p be any nonnegative integer. Then*

$$F_{2m}\,F_{2mn+p} + F_p\,F_{2m(n-1)} = F_{2m+p}\,F_{2mn}$$

and

$$F_{2m}\,L_{2mn+p} + L_p\,F_{2m(n-1)} = L_{2m+p}\,F_{2mn}.$$

Proof. We only prove the first identity because the second one can be proved similarly. Let $\mathbb{L}_m(n)$ and $\mathbb{F}_{m,p}(n)$ be the matrices used in the proof of Lemma 10 and Theorem 14, respectively. Then

$$\begin{vmatrix} F_{2m+p} & \sqrt{F_p} & & & \\ \sqrt{F_p} & L_{2m} & 1 & & \\ & 1 & L_{2m} & 1 & \\ & & \ddots & \ddots & 1 \\ & & & 1 & L_{2m} \end{vmatrix} = F_{2m+p}\,|\mathbb{L}_m(n-1)| - F_p\,|\mathbb{L}_m(n-2)|$$

and with respect to Corollary 11

$$F_{2mn+p} = F_{2m+p}\,\frac{F_{2mn}}{F_{2m}} - F_p\,\frac{F_{2m(n-1)}}{F_{2m}}. \qquad \square$$

Corollary 16. *Let m be any positive integer and let p be any nonnegative integer. Then*

$$L_{4m+p} = \frac{L_{2m+p}}{L_{2m}} \left(L_{2m} - \sqrt{\frac{L_{2m}L_p}{L_{2m+p}}} \right) \left(L_{2m} + \sqrt{\frac{L_{2m}L_p}{L_{2m+p}}} \right),$$

$$L_{6m+p} = L_{2m+p} \left(L_{2m} - \sqrt{1 + \frac{L_{2m}L_p}{L_{2m+p}}} \right) \left(L_{2m} + \sqrt{1 + \frac{L_{2m}L_p}{L_{2m+p}}} \right),$$

$$\begin{aligned} L_{8m+p} = \frac{L_{2m+p}}{L_{2m}} & \left(L_{2m} - \sqrt{\frac{L_{2m}\,L_p + 2\,L_{2m+p} + \sqrt{L_{2m}^2\,L_p^2 + 4\,L_{2m+p}^2}}{2\,L_{2m+p}}} \right) \\ & \left(L_{2m} + \sqrt{\frac{L_{2m}\,L_p + 2\,L_{2m+p} + \sqrt{L_{2m}^2\,L_p^2 + 4\,L_{2m+p}^2}}{2\,L_{2m+p}}} \right) \\ & \left(L_{2m} - \sqrt{\frac{L_{2m}\,L_p + 2\,L_{2m+p} - \sqrt{L_{2m}^2\,L_p^2 + 4\,L_{2m+p}^2}}{2\,L_{2m+p}}} \right) \\ & \left(L_{2m} + \sqrt{\frac{L_{2m}\,L_p + 2\,L_{2m+p} - \sqrt{L_{2m}^2\,L_p^2 + 4\,L_{2m+p}^2}}{2\,L_{2m+p}}} \right), \end{aligned}$$

$$\begin{aligned} L_{10m+p} = L_{2m+p} & \left(L_{2m} - \sqrt{\frac{L_{2m}\,L_p + 3\,L_{2m+p} + \sqrt{L_{p-2m}^2 + 4\,L_{p+2m}^2}}{2\,L_{2m+p}}} \right) \\ & \left(L_{2m} + \sqrt{\frac{L_{2m}\,L_p + 3\,L_{2m+p} + \sqrt{L_{p-2m}^2 + 4\,L_{p+2m}^2}}{2\,L_{2m+p}}} \right) \\ & \left(L_{2m} - \sqrt{\frac{L_{2m}\,L_p + 3\,L_{2m+p} - \sqrt{L_{p-2m}^2 + 4\,L_{p+2m}^2}}{2\,L_{2m+p}}} \right) \\ & \left(L_{2m} + \sqrt{\frac{L_{2m}\,L_p + 3\,L_{2m+p} - \sqrt{L_{p-2m}^2 + 4\,L_{p+2m}^2}}{2\,L_{2m+p}}} \right). \end{aligned}$$

P r o o f. We obtain these factorizations by setting consecutively $n = 2, 3, 4, 5$ into the assertion of Theorem 12. $\square$

4. Conclusion

Setting various values of p into the statements of Theorem 12 and Theorem 14 we obtain distinct factorizations of the given subsequence of the Fibonacci or Lucas numbers. For example, we have the following factorizations of L_{8m} setting $p = 4m$, $p = 2m$ and $p = 0$ into the first, the second and the third identity in Corollary 16:

$$L_{8m} = (L_{2m} - \sqrt{3})(L_{2m} + \sqrt{3}) \left(L_{2m} - \sqrt{1 + \frac{L_{2m}}{L_{6m}}}\right) \left(L_{2m} + \sqrt{1 + \frac{L_{2m}}{L_{6m}}}\right),$$

$$L_{8m} = (L_{2m} - \sqrt{2})(L_{2m} + \sqrt{2}) \left(L_{2m} - \sqrt{2 + \frac{2}{L_{4m}}}\right) \left(L_{2m} + \sqrt{2 + \frac{2}{L_{4m}}}\right),$$

$$L_{8m} = \left(L_{2m} - \sqrt{2+\sqrt{2}}\right) \left(L_{2m} + \sqrt{2+\sqrt{2}}\right) \left(L_{2m} - \sqrt{2-\sqrt{2}}\right) \left(L_{2m} + \sqrt{2-\sqrt{2}}\right).$$

REFERENCES

[1] CAHILL, N. D.—D'ERRICO, J. R.—NARAYAN, D. A.—NARAYAN, J. Y.: *Fibonacci determinants*, College Math. J. **3** (2002), 221–225.

[2] CAHILL, N. D.—D'ERRICO, J. R.—SPENCE, J. P.: *Complex factorizations of the Fibonacci and Lucas numbers*, Fibonacci Quart. **41** (2003), 13–19.

[3] CAHILL, N. D.—NARAYAN, D. A.: *Fibonacci and Lucas numbers as tridiagonal matrix determinants*, Fibonacci Quart. **42** (2004) 216–221.

[4] GRAHAM, R. L.—KNUTH, D. E.—PATASHNIK, O.: *Concrete Mathematics: a Foundation for Computer Science* (2nd ed.), Addison-Wesley Publishing Company, Amsterdam, 1994.

[5] MORGADO, J.: *Note on the Chebyshev polynomials and applications to the Fibonacci numbers*, Port. Math. (N.S.) **52** (1995), 363–378.

[6] STRANG, G.: *Introduction to Linear Algebra* (2nd ed.), Wellesley-Cambridge Press, Wellesley, MA, 1998.

[7] VAJDA, S.: *Fibonacci and Lucas Numbers and the Golden Section.* Ellis Horwood Books Math. Appl., Ellis Horwood Ltd./Halsted Press; Chichester/New York etc., 1989.

Received 30. 1. 2010
Accepted 8. 3. 2011

Department of Mathematics
Faculty of Scince
University of Hradec Králové
Rokitanského 62
CZ–PSC Hradec Králové
CZECH REPUBLIC

E-mail: jaroslav.seibert@uhk.cz
pavel.trojovsky@uhk.cz