VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-012-0019-8
Math. Slovaca **62** (2012), No. 3, 425–438

ON α-COMPLETE RETRACTS OF SPECKER LATTICE ORDERED GROUPS

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(*Communicated by Jiří Rachůnek*)

ABSTRACT. Let α be a cardinal. The notion of α-complete retract of a Boolean algebra has been studied by Dwinger. Specker lattice ordered groups were investigated by Conrad and Darnel. Assume that G is a Specker lattice ordered group generated by a Boolean algebra $B(G)$. The notion of α-complete retract of G can be defined analogously as in the case of Boolean algebras. In the present paper we deal with the relations between α-complete retracts of G and α-complete retracts of $B(G)$.

1. Introduction

Dwinger [4] introduced and studied α-complete retracts of Boolean algebras, where α is a cardinal. In particular, he proved that the classical Skolem's theorem is a consequence of his result on $\aleph_0$-complete retracts.

Specker lattice ordered groups were investigated by Conrad and Darnel [3]; cf. also the author [8], [11]. For each Specker lattice ordered group G there exists a generalized Boolean algebra $B(G)$ such that G is generated by $B(G)$. In the present paper we restrict ourselves to take case when $B(G)$ is a Boolean algebra.

The notion of α-complete retract of Specker lattice ordered groups can be defined analogously as in the case of Boolean algebras.

2010 Mathematics Subject Classification: Primary 06F15, 06E05.
Keywords: Specker lattice ordered group, Boolean algebra, retract, α-complete retract.
Supported by VEGA Agency grant 1/0539/08.
This work has been supported by the Center of Excellence SAS – Quantum Technologies.

Let G and $B(G)$ be as above. We show that there exists a one-to-one correspondence between retract mappings of G and retract mappings of $B(G)$. Further, we prove that if $B(G)$ is α-complete, then there exists a bijection between the system of all α-complete retracts of G and the system of all α-complete retracts of $B(G)$.

Retracts of lattice ordered groups have been investigated in [5], [6], [7], [9].

2. Preliminaries

For lattice ordered groups we apply the notation as in Birkhoff [1].

The symbol 0 denotes the zero real, the neutral element of a lattice ordered group or the least element of a Boolean algebra; the meaning of this symbol will be clear from the context.

We start by recalling the notion of Specker lattice ordered group corresponding to a Boolean algebra B (cf. [2], [3], [8], [11]). To avoid the trivial case, we always assume that B has more than one element.

We denote by Λ the set of all maximal proper filters of B. If $b \in B$, then b will be identified with the set $\Lambda(b)$ of all $\lambda \in \Lambda$ such that $b \in \lambda$. Let Z be the additive group of all integers with the natural linear order.

Consider the direct product

$$G_0 = \prod_{\lambda \in \Lambda} A_\lambda,$$

where $A_\lambda = Z$ for each $\lambda \in \Lambda$. For $a \in Z$ and $b \in B$ we denote by $a[b]$ the element of G_0 such that

$$a[b](\lambda) = \begin{cases} a, & \text{if } \lambda \in b, \\ 0, & \text{otherwise.} \end{cases}$$

Further, we denote by Z_B the set of all elements g of G_0 such that either $g = 0$ or g can be expressed in the form

$$g = a_1[c_1] + \dots + a_n[c_n], \tag{1}$$

where $a_1, \dots, a_n$ are nonzero elements of Z and $c_1, \dots, c_n$ are nonzero elements of B such that $c_{i(1)} \wedge c_{i(2)} = 0$ whenever $i(1)$ and $i(2)$ are distinct elements of the set $\{1, 2, \dots, n\}$. It is clear that Z_B is an ℓ-subgroup of G_0.

Without loss of generality we can assume that the elements $a_1, \dots, a_n$ are mutually distinct. In such a case, the expression (1) is said to be a standard Specker representation of g. Each nonzero element of G has a unique standard Specker representation. Put $Z_B = G$.

If $a_1[c_1]$ is as in (1) and $a_1 = 1$ then we write $a_1[c_1] = \overline{c}_1$. Thus we have

$$g = a_1\overline{c}_1 + \cdots + a_n\overline{c}_n.$$

Put $B(G) = \{\overline{b} : \ b \in B\}$. Hence $B(G) \subseteq G$; the partial order on G induces a partial order on $B(G)$. It is obvious that $B(G)$ is isomorphic to B; hence $B(G)$ is a Boolean algebra. It is well-known that $B(G)$ is a convex sublattice of the underlying lattice of G.

We say that G is the Specker lattice ordered group generated by the Boolean algebra B.

A non-zero ℓ-subgroup G_1 of G is said to be a Specker ℓ-subgroup of G if, whenever $\{\overline{b}_1, \dots, \overline{b}_n\}$ is an orthogonal system of elements of $B(G)$ and $a_1, \dots, a_n$ are mutually distinct nonzero integers such that $a_1\overline{b}_1 + \cdots + a_n\overline{b}_n \in G_1$, then for each $i \in \{1, 2, \dots, n\}$ we have $\overline{b}_i \in G_1$ and $\overline{b}_i' \in G_1$ ($\overline{b}_i'$ is the complement of $\overline{b}_i$ in $B(G)$).

Now, let us put $B_1 = \{b \in B : \ \overline{b} \in G_1\}$. It is easy to verify that G_1 is isomorphic to Z_{B_1}; hence G_1 is a Specker lattice ordered group.

We denote by $B(G_1)$ the set of all $\overline{b} \in B(G)$ such that $\overline{b} \in G_1$. Hence $B(G_1)$ is a Boolean algebra isomorphic to B_1. The relation between G_1 and $B(G_1)$ is analogous to that between G and $B(G)$.

Now, we recall the notion of α-complete retract of a Boolean algebra defined in [4].

Let α be a cardinal. Assume that A and X are α-complete Boolean algebras and that X is a subalgebra of A. We denote by X^* the α-complete subalgebra of A generated by X. (For a detailed definition of this notion and of the corresponding notion concerning Specker lattice ordered groups cf. Section 5 below.) The Boolean algebra X is called *α-complete retract* of A if there exists an α-complete homomorphic mapping of X^* onto X which leaves X elementwise fixed.

The definition of α-complete retract of a Specker lattice ordered group is analogous; for the sake of completeness, we formulate it in detail.

Let A and X be α-complete Specker lattice ordered groups such that X is a Specker ℓ-subgroup of A. Let X^* be the α-complete ℓ-subgroup of A which is generated by X. Suppose that there exists and α-complete homomorphism of X^* onto X which leaves X elementwise fixed. Then X is an *α-complete retract* in A.

We recall that a lattice ordered group A is *α-complete* if each upper-bounded subset A_1 of A with $0 \neq \operatorname{card} A_1 \leqq \alpha$ has the supremum in A. In such a case, the corresponding dual condition is also valid. An ℓ-subgroup X of A is an α-complete ℓ-subgroup of A if, whenever X_1 is a subset of X with $\operatorname{card} X_1 \leqq \alpha$ and the relation $\sup X_1 = x_1$ is valid in A, then $x_1 \in X$. A homomorphic mapping φ of a lattice ordered group A into a lattice ordered group B is called α-complete if it satisfies the following condition $(*)$ and the corresponding dual condition:

$(*)$ Whenever $\{x_i\}_{i\in I} \subseteq A$, $\operatorname{card} I \leqq \alpha$ and the relation $x = \bigvee_{i\in I} x_i$ is valid in A, then the relation $\varphi(x) = \bigvee_{i\in I} \varphi(x_i)$ is valid in B.

An analogous terminology is applied for Boolean algebras.

3. Retracts and retract mappings

Let B be a Boolean algebra. A homomorphism $\varphi\colon B \to B$ is a *retract mapping* if $\varphi(\varphi(x)) = \varphi(x)$ for each $x \in B$. Then the set $\varphi(B)$ is said to be a *retract* in B.

Let G_1 be a Specker ℓ-subgroup of a Specker lattice ordered group G and let ψ be a homomorphism of G onto G_1 such that $\psi(\psi(x)) = \psi(x)$ for each $x \in G$; then ψ is a *retract mapping* and G_1 is a *retract* in G.

In the present section we prove that there exists a one-to-one correspondence between retract mappings of the Specker lattice ordered group $G = Z_B$ and retract mappings of the Boolean algebra B. Further, some results of this section will be applied by the investigation of α-complete retracts.

Lemma 3.1. *Let ψ be a retract mapping on a Specker lattice ordered group $G = Z_B$. Then $\psi(B(G)) \subseteq B(G)$.*

Proof. Let $\overline{b} \in B(G)$. It suffices to deal with the case $\psi(\overline{b}) \neq 0$. Put $\psi(\overline{b}) = x$ and let

$$x = a_1\overline{b}_1 + \cdots + a_n\overline{b}_n \tag{1}$$

be the standard Specker representation of x. Then either there exists $i \in \{1, 2, \ldots, n\}$ with $a_i \neq 1$ or $n = 1$, $a_1 = 1$. In the latter case, $\psi(\overline{b}) = x = \overline{b}_1 \in B(G)$. Consider the former case. Since $\overline{b} > 0$, we must have $x > 0$, and thus, $a_i > 1$. Without loss of generality, we can suppose that $i = 1$.

The set $\psi(G)$ is a retract in G and hence it is a Specker ℓ-subgroup of G. Since $x \in \psi(G)$, we conclude that $\overline{b_1}$ and $\overline{b_1}'$ belong to $\psi(G)$. Hence $\psi(\overline{b}_1') = \overline{b}_1'$.

We have $\overline{b} \leqq \overline{b}_1 \vee \overline{b}_1'$. Then

$$a_1\overline{b}_1 \leqq x \leq \psi(\overline{b}) \leq \psi(\overline{b}_1) \vee \psi(\overline{b}_1') = \overline{b}_1 \vee \overline{b}_1'.$$

Since $\overline{b}_1 \wedge \overline{b}_1' = 0$ we get $(a_1\overline{b}_1) \wedge \overline{b}_1' = 0$, thus $a_1\overline{b}_1 \leqq \overline{b}_1$. We arrived at a contradiction. $\square$

Let ψ be as above. For each $b \in B(G)$ we put $\psi_0(b) = \psi(b)$. From Lemma 3.1 we immediately obtain:

Lemma 3.2. *For each retract mapping ψ on G, ψ_0 is a retract mapping on $B(G)$.*

Now let φ be a retract mapping on the Boolean algebra $B(G)$. We define a mapping $\varphi_0\colon G \to G$ as follows. We set $\varphi_0(0) = 0$. If $0 \neq x \in G$ and if (1) is the standard Specker representation of x, then we put

$$\varphi_0(x) = a_1\varphi(\overline{b}_1) + \dots + a_n\varphi(\overline{b}_n).$$

We need an auxiliary result concerning the representation of elements of G. Let x and (1) be as above; further, let

$$x_1 = a_{11}\overline{b}_{11} + \dots = a_{1m}\overline{b}_{1m}$$

be a standard Specker representation of an element $x_1 \in G$.

If $a \in Z$, $\overline{b} \in B(G)$ and either $a = 0$ or $\overline{b} = 0$, then we put $a\overline{b} = 0$.

We set $a_{n+1} = 0$, $\overline{b}_{n+1} = (\overline{b}_1 \vee \dots \vee \overline{b}_n)'$, $I = \{1, 2, \dots, n+1\}$, and similarly $a_{1,m+1} = 0$, $\overline{b}_{1,m+1} = (\overline{b}_{1,1} \vee \dots \vee \overline{b}_{1,m})'$, $J = \{1, 2, \dots, m+1\}$. Then we have

$$x = \sum_{i\in I} a_i\overline{b}_i = \bigvee_{i\in I} a_i\overline{b}_i,$$
$$x_1 = \sum_{j\in J} a_{1j}\overline{b}_{1j} = \bigvee_{j\in J} a_{1j}\overline{b}_{1j}.$$

Let $i \in I$ and $j \in J$. We put

$$\overline{c}_{ij} = \overline{b}_i \wedge \overline{b}_j, \qquad a_{ij} = a_i, \qquad a_{1,ij} = a_{1j}.$$

By a simple calculation (using the fact that the underlying lattice of G is distributive) we obtain

$$x = \sum_{(ij)\in I\times J} a_{ij}\overline{b}_{ij} = \bigvee_{(i,j)\in I\times J} a_{ij}\overline{b}_{ij},$$
$$x_1 = \sum_{(i,j)\in I\times J} a_{1,ij}\overline{b}_{ij} = \bigvee_{(i,j)\in I\times J} a_{1,ij}\overline{b}_{ij}.$$

Therefore we have the following result (using a modified notation):

Lemma 3.3. *Let x and x_1 be elements of G. Then they can be expressed in the form*

$$x = a_1\overline{b}_1 + \dots + a_n\overline{b}_n,$$
$$x_1 = a_{11}\overline{b}_1 + \dots + a_{1n}\overline{b}_n,$$

such that $\overline{b}_1, \dots, \overline{b}_n$ are mutually orthogonal elements of $B(G)$ and $a_1, \dots, a_n$, $a_{11}, \dots, a_{1n}$ are elements of Z.

Lemma 3.4. *For each retract mapping φ on $B(G)$, φ_0 is a retract mapping on G.*

Proof.

a) Let us remark that to calculate $\varphi_0(x)$, we can use any representation of x, not necessarily a standard one. It is easy to verify that whenever x, x_1 are elements of G and if they are represented as in Lemma 3.3, then

$$\begin{aligned}\varphi_0(x) &= a_1\varphi(\overline{b}_1) + \cdots + a_n\varphi(\overline{b}_n),\\ \varphi_0(x_1) &= a_{11}\varphi(\overline{b}_1) + \cdots + a_{1,n}\varphi(b_n),\\ x + x_1 &= (a_1 + a_{11})\overline{b}_1 + \cdots + (a_n a + a_{1,n})b_n,\\ \varphi_0(x + x_1) &= \varphi_0(x) + \varphi_0(x_1).\end{aligned}$$

Thus φ_0 is an endomorphism with respect to the operation $+$.

b) Now, let us deal with an analogous question concerning the operation $\vee$. Let us apply the notation as above.

We have

$$x = \bigvee_{i=1}^{n} a_i\overline{b}_i, \qquad x_1 = \bigvee_{i=1}^{n} a_{1,i}\overline{b}_i.$$

We also obtain

$$\varphi_0(x) = \bigvee_{i=1}^{n}(a_i\varphi(\overline{b}_i), \quad \varphi_0(x_1) = \bigvee_{i=1}^{n}(a_{1,i}\varphi(\overline{b}_i)),$$

$$x \vee x_1 = \bigvee_{i=1}^{n}(\sup(a_i, a_{1,i})\overline{b}_i),$$

$$\varphi_0(x \vee x_1) = \varphi_0(x) \vee \varphi_0(x_1).$$

Therefore φ_0 is an endomorphism with respect to the operation $\vee$. Further, since the operation $\wedge$ can be calculated by applying the operations $\vee$ and $-$, we conclude that φ_0 is also an endomorphism with respect to the operation $\wedge$.

c) We obviously have $\varphi_0(\varphi_0(x)) = \varphi_0(x)$ for each $x \in G$.

Evidently, if $x = a_1\overline{b}_1 + \cdots + a_n\overline{b}_n \in \varphi_0(G)$ and $a_1, \ldots, a_n$ are mutually distinct, then for all $i \in I$, both $\overline{b}_i$ and $\overline{b}_i'$ belong to $\varphi_0(G)$. Summarizing, we obtain that φ_0 is a retract mapping on the Specker lattice ordered group G. $\square$

From the definitions of ψ_0 and φ_0 we immediately obtain that the relations

$$(\psi_0)_0 = \psi, \qquad (\varphi_0)_0 = \varphi$$

are valid.

Hence if we put $f(\psi) = \psi_0$ for each retract mapping ψ on G, then f is a bijection of the system of all retract mappings on G onto the system of all retract mappings on $B(G)$. Since $B \simeq B(G)$, we get

Theorem 3.5. *Let $G = Z_B$ be a Specker lattice ordered group. Then there eixsts a one-to-one correspondence between retract mappings of G and retract mappings of B.*

4. Specker ℓ-subgroups

As above, let $G = Z_B$ be a Specker lattice ordered group. We denote by $S(G)$ the system of all Specker ℓ-subgroups of G.

From the definition of Specker ℓ-subgroup we immediately obtain

Lemma 4.1. *Let $H \in S(G)$. Then $H \cap B(G)$ is a subalgebra of the Boolean algebra $B(G)$.*

Lemma 4.2. *Let B_1 be a subalgebra of the Boolean algebra $B(G)$. We denote by H_1 the set of all elements x of G such that either $x = 0$ or x can be expressed in the form $x = a_1\overline{b}_1 + \dots + a_n\overline{b}_n$, where $a_1, \dots, a_n$ are nonzero integers and $\{b_1, \dots, b_n\}$ is an orthogonal system of elements of B_1. Then H_1 is a Specker ℓ-subgroup of G and $B(H_1) = B_1$.*

Proof. In view of Lemma 3.3, H_1 is an ℓ-subgroup of G. From this and from the definition of H_1 we conclude that H_1 is a Specker ℓ-subgroup of G and that $B(H_1) = B_1$. □

For a Boolean algebra B let $S(B)$ be the system of all subalgebras of B.

For each $H \in S(G)$ we put $\chi(H) = H \cap B(G)$. Let the systems $S(G)$ and $S(B)$ be partially ordered by the set-theoretical inclusion.

Lemmas 4.1 and 4.2 yield

Lemma 4.3. *The mapping $\chi\colon S(G) \to S(B(G))$ is a bijection. For $H_1, H_2 \in S(G)$ we have*

$$H_1 \subseteq H_2 \iff \chi(H_1) \subseteq \chi(H_2).$$

The following two assertions are easy to verify.

Lemma 4.4. *Let α be a cardinal and let G be α-complete. Assume that $\{\overline{b}_i\}_{i\in I} \subseteq B(G)$, $I \neq \emptyset$, $\operatorname{card} I \leqq \alpha$, $x \in G$ and that the relation $x = \bigvee_{i\in I} \overline{b}_i$ is valid in G. Then $x \in B(G)$.*

Lemma 4.5. *Let $\{\overline{b}_i\}_{i\in I}$ and x be as in* Lemma 4.4 *with the distinction that now the relation $x = \bigwedge_{i\in I} \overline{b}_i$ is valid in G. Then $x \in B(G)$.*

From Lemmas 4.4 and 4.5 we conclude:

Lemma 4.6. *Let α be a cardinal and suppose that G is α-complete. Then*

(i) *$B(G)$ is an α-complete sublattice of G;*

(ii) *$B(G)$ is α-complete.*

Lemma 4.7. *Let $\{\overline{b}_i\}_{i\in B} \subseteq B(G)$, $\overline{b} \in B(G)$ and assume that the relation $\bigvee_{i\in I} \overline{b}_i = \overline{b}$ is valid in $B(G)$. Then the same relation holds in G.*

Proof. Let $x \in G$ be an upper bound of the set $\{\overline{b}_i\}_{i\in I}$. Then $x \wedge \overline{b}$ is an upper bound of $\{\overline{b}_i\}_{i\in I}$, too. Using the convexity of $B(G)$, we obtain $\overline{b} \leq x$. □

Lemma 4.8. *Assume that $B(G)$ is α-complete. Then G is α-complete as well.*

Proof. It suffices to verify that if $\{g_i\}_{i\in I}$ is an upper bounded subset of G^+ and $\operatorname{card} I \leqq \alpha$, then this set has a supremum in G.

Let $x \in G$ and suppose that x is an upper bound of such set $\{g_i\}_{i\in I}$. Consider the standard Specker representation

$$x = a_1\overline{b}_1 + \cdots + a_n\overline{b}_n$$

of x; similarly, consider the standard Specker representations

$$g_i = a_{i1}\overline{b}_{i1} + \cdots + a_{in(i)}\overline{b}_{in(i)}$$

of elements g_i. We have

$$x = a_1\overline{b}_1 \vee \cdots \vee a_n\overline{b}_n,$$

$$g_i = a_{i1}\overline{b}_{i1} \vee \cdots \vee a_{in(i)}\overline{b}_{in(i)}.$$

The greatest element of the Boolean algebra B will be denoted by b^0. Put $a^* = \max\{a_1, \ldots, a_n\}$. Then $x \leqq a^*\overline{b^0}$, hence

$$a_{ij}\overline{b}_{ij} \leqq a^*\overline{b^0}, \quad 0 < a_{ij} \leqq a^*$$

for each $i \in I$ and each $j \in \{1, 2, \ldots, n(i)\}$.

The set of all upper bounds of the system $\{g_i\}_{i\in I}$ coincides with the set of all upper bounds of the system of all elements $a_{ij}\overline{b}_{ij}$ (under the notation as above).

For each integer k with $0 < k \leqq a^*$ we denote by P_k the set of all elements $a_{ij}\overline{b}_{ij}$ such that $a_{ij} = k$. If $P_k = \emptyset$, then we put $p_k = 0 \in G$. In the case $P_k \neq \emptyset$ we set

$$p_k = \sup P_k;$$

this supremum in G exists, since

$$\sup P_k = \sup\{a_{ij}\overline{b}_{ij}\} = \sup\{k\overline{b}_{ij}\} = k\sup\{\overline{b}_{ij}\}$$

(where $a_{ij} = k$). Namely, for each $i \in I$ there exists at most one j with $a_{ij} = k$ and thus $\operatorname{card}\{\overline{b}_{ij}\} \leqq \alpha$, hence $\sup\{\overline{b}_{ij}\}$ exists in $B(G)$. In view of Lemma 4.6, $\sup\{\overline{b}_{ij}\}$ in $B(G)$ is, at the same time, the supremum of the set $\{\overline{b}_{ij}\}$ in G.

Each element $a_{ij}\overline{b}_{ij}$ belongs exactly to one set P_k. Hence we obtain that the supremum of the system of elements $a_{ij}\overline{b}_{ij}$ is equal to $p_1 \vee p_2 \vee \dots \vee p_{a^*}$. $\square$

If $G_1 \in S(G)$, then the relation between G_1 and $\chi(G_1)$ is the same as the relation between G and $B(G)$. Hence from the previous lemmas we infer

Lemma 4.9. *Let $G_1 \in S(G)$. Then G_1 is α-complete if and only if the Boolean algebra $\chi(G_1)$ is α-complete.*

5. α-complete retracts

Let α be a cardinal and let G, B be as above. In the present section we assume that G is α-complete. According to Section 4, this is equivalent with the assumption that B is α-complete. Recall that the notion of α-complete homomorphism was defined in Section 2. In the present section we will deal with α-complete endomorphisms on Specker lattice ordered groups and on Boolean algebras.

If $\varphi\colon B(G) \to B(G)$ is a retract mapping on $B(G)$, then we define φ_0 in the same way as in Section 3. In view of Lemma 3.4, φ_0 is a retract mapping on G.

Analogously we proceed when ψ is a retract mapping on G. Then according to Lemma 3.2, ψ_0 is a retract mapping on $B(G)$.

Lemma 5.1. *Assume that ψ is an α-complete endomorphism on G and that it is, at the same time, a retract mapping on G. Then ψ_0 is an α-complete endomorphism on $B(G)$.*

Proof. It suffices to apply Lemma 4.6. $\square$

Now, let us assume that φ is an α-complete endomorphism on $B(G)$ and that it is, at the same time, a retract mapping on $B(G)$.

Lemma 5.2. *Let $x \in G^+$ and $\emptyset \neq X = \{g_i\}_{i\in I} \subseteq G^+$. Let $\operatorname{card} I \leq \alpha$. Assume that x is an upper bound of X. Put $X' = \varphi_0(X)$. Then $\varphi_0(\sup X) = \sup X'$.*

P r o o f. We apply the notation as in the proof of Lemma 4.8 with a slight modification. Namely, let

$$x = a_1\overline{b}_1 + \dots + a_n\overline{b}_n$$

be as in the mentioned proof and let $i \in I$. Since $\overline{0} \le g_i \le x$, there exists a representation of g_i having the form

$$g_i = a_{i1}\overline{b}_{i1} + \dots + a_{in}\overline{b}_{in}$$

such that for each $j \in \{1, 2, \dots n\}$ the relations

$$0 \le a_{ij} \le a_j, \quad \overline{0} \le \overline{b}_{ij} \le \overline{b}_j$$

are valid. Then $\overline{b}_{i,j(1)} \wedge \overline{b}_{i,j(2)} = \overline{0}$ whenever $j(1)$ and $j(2)$ are distinct and belong to the set $\{1, 2, \dots, n\}$. Thus

$$g_i = a_{i1}\overline{b}_{i1} \vee \dots \vee a_{in}\overline{b}_{in}.$$

Therefore, we have

$$\bigvee_{i\in I} g_i = \bigvee (a_{ij}\overline{b}_{ij}), \qquad \text{where} \quad i \in I \ \text{ and } \ j \in \{1, 2, \dots, n\}. \tag{1}$$

Let a^* be as in the proof of Lemma 4.8. For each integer k with $1 \le k \le a^*$ we denote by Q_k the set of all elements $\overline{b}_{ij}$ such that $a_{ij} = k$. Each element $\overline{b}_{ij}$ belongs to some Q_k, and this Q_k is uniquely determined. Let P_k be as in the proof of Lemma 4.8. If $Q_k \ne \emptyset$, then the elements of P_k have the form $k\overline{b}_{ij}$ with $\overline{b}_{ij} \in Q_k$. In this case, we have

$$\sup P_k = \bigvee_{\overline{b}_{ij}\in Q_k} (k\overline{b}_{ij}) = k \bigvee_{\overline{b}_{ij}\in Q_k} \overline{b}_{ij}; \tag{2}$$

we set $\sup P_k = p_k$. If $Q_k = \emptyset$, then we set $p_k = \overline{0}$. Therefore, in view of (1), we obtain

$$\bigvee_{i\in I} g_i = p_1 \vee p_2 \vee \dots \vee p_{a^*},$$

$$\varphi_0\Bigl(\bigvee_{i\in I} g_i\Bigr) = \varphi_0(p_1) \vee \varphi_0(p_2) \vee \dots \vee \varphi_0(p_{a^*}). \tag{3}$$

If $k \in \{1, 2, \dots, a^*\}$ and $Q_k \ne \emptyset$, then in view of (2), we have

$$\varphi_0(p_k) = \varphi_0\Bigl(k \bigvee_{\overline{b}_{ij}\in Q_k} \overline{b}_{ij}\Bigr) = k\varphi\Bigl(\bigvee_{\overline{b}_{ij}\in Q_k} \overline{b}_{ij}\Bigr).$$

According to the assumption, φ is an α-complete endomorphism on $B(G)$ and therefore

$$\varphi\Bigl(\bigvee_{\overline{b}_{ij}\in Q_k} \overline{b}_{ij}\Bigr) = \bigvee_{\overline{b}_{ij}\in Q_k} \varphi(\overline{b}_{ij}). \tag{4}$$

Further, for each $i \in I$, we have

$$\varphi_0(g_i) = a_{i1}\varphi(\overline{b}_{i1}) + \dots + a_{in}\varphi(\overline{b}_{in}).$$

If $j(1)$ and $j(2)$ are distinct elements of the set $\{1, 2, \dots, n\}$, then from $\overline{b}_{i,j(1)} \wedge \overline{b}_{i,j(2)} = \overline{0}$ we get $\varphi(\overline{b}_{i,j(1)}) \wedge \varphi(\overline{b}_{i,j(2)}) = \overline{0}$ and hence

$$\varphi_0(g_i) = a_{i1}\varphi(\overline{b}_{i1}) \vee \dots \vee a_{in}\varphi(\overline{b}_{in}).$$

Therefore

$$\bigvee_{i \in I} \varphi_0(g_i) = \bigvee (a_{ij}\varphi(\overline{b}_{ij})), \qquad \text{where} \quad i \in I \ \text{ and } \ j \in \{1, 2, \dots, n\}. \tag{5}$$

From (3), (4) and (5) we conclude that the relation

$$\varphi_0\Bigl(\bigvee_{i \in I} g_i\Bigr) = \bigvee_{i \in I} \varphi_0(g_i)$$

is valid. □

The following assertion is easy to verify.

Lemma 5.3. *Let $\emptyset \neq X = \{g_i\}_{i \in I}$ be an upper bounded subset of G and $i(0) \in I$. Let* $\operatorname{card} I \leq \alpha$. *Put $\{g_i \vee g_{i(0)}\}_{i \in I} = X_0$. Then,* $\sup X = \sup X_0$.

Lemma 5.4. *Let X be as in* Lemma 5.3. *Put $X' = \varphi_0(X)$. Then* $\varphi_0(\sup X) = \sup X'$.

Proof. We choose any $i(0) \in I$. Let X_0 be as in Lemma 5.3. In view of Lemma 5.3, $\sup X = \sup X_0$.

Similarly, we set $\{\varphi_0(g_i) \vee \varphi_0(g_{i(0)})\}_{i \in I} = X'_0$. Again, by Lemma 5.3 we have $\sup X' = \sup X'_0$.

Further, we put

$$\bigl\{t - g_{i(0)} : \ t \in X_o\bigr\} = X_{01}, \quad \bigl\{t - \varphi_0(g_{i(0)}) : \ t \in X'_0\bigr\} = X'_{01}.$$

Then we have

$$\varphi_0(X_{01}) = X'_{01}. \tag{6}$$

Moreover, we get

$$\sup X_{01} = \sup X_0 - g_{i(0)}, \qquad \sup X'_{01} = \sup X'_0 - \varphi_0(g_{i(0)}).$$

Obviously, $X_{01} \subseteq G^+$ and $\operatorname{card} X_{01} \leq \alpha$. According to Lemma 5.2 and (6), the relation

$$\varphi_0(\sup X_{01}) = \sup X'_{01}$$

is valid. We obtain

$$\begin{aligned}\varphi_0(\sup X) &= \varphi_0(\sup X_0) = \varphi_0(\sup X_{01} + g_{i(0)}) = \varphi_0(\sup X_{01}) + \varphi_0(g_{i(0)}) \\ &= \sup X'_{01} + \varphi_0(g_{i(0)}) = \sup X'_0 - \varphi_0(g_{i(0)}) + \varphi_0(g_{i(0)}) = \sup X'.\end{aligned}$$ □

Lemma 5.5. *Assume that φ is an α-complete endomorphism on $B(G)$ and that it is, at the same time, a retract mapping on $B(G)$. Then φ_0 is an α-complete endomorphism on G.*

Proof.

a) First, let us consider the condition concerning the operation $\vee$. The validity of this condition follows from Lemma 5.4.

b) Secondly, considering the operation $\wedge$ it suffices to take into account that the operation $\wedge$ can be expressed by means of the operation $-$ and $\vee$, and to apply the result from a). $\square$

Analogously we verify (using 4.9)

Lemma 5.5.1. *Let $G_1 \in S(G)$. Assume that ψ is an α-complete homomorphism on G_1 and a retract mapping on G_1. Then ψ_0 is an α-complete homomorphism on $\chi(G_1)$.*

Lemma 5.5.2. *Let $B_1 \in S(B(G))$. Assume that φ is an α-complete homomorphism and, at the same time, a retract mapping on B_1. Then φ_0 is an α-complete homomorphism on $\chi^{-1}(B_1)$.*

Let X be a subalgebra of B. Further, let X^* be an α-complete subalgebra of B such that

(i) $X \subseteq X^*$;

(ii) if X_1 is a subalgebra of B with $X \subseteq X_1 \subset X^*$, then X_1 fails to be α-complete subalgebra of B.

Then X^* is said to be *α-complete subalgebra of B generated by X*.

For Specker lattice ordered groups we apply the following analogous definition.

Let Y be a Specker ℓ-subgroup of G; further, let Y be an α-complete Specker ℓ-subgroup of G such that

(i) $Y \subseteq Y^*$;

(ii) if Y_1 is a Specker ℓ-subgroup of G such that $Y \subseteq Y_1 \subset Y^*$, then Y_1 is not an α-complete Specker ℓ-subgroup of G.

Then Y^* is said to be the *α-complete Specker ℓ-subgroup of G generated by Y*.

It is obvious that X^* is uniquely determined by X; simlarly, Y^* is uniquely determined by Y.

Let the mapping $\chi\colon S(G) \to S(B(G))$ be as in Section 4.

Lemma 5.6. *Let $Y \in S(G)$ and $X = \chi(Y)$. Then $X^* = \chi(Y^*)$.*

Proof. Let M_1 be the set of all $P \in S(G)$ such that P is an α-complete Specker ℓ-subgroup of G with $Y \subseteq P$. Similarly, let M_2 be the set of all $Q \in S(B(G))$ such that Q is an α-complete subalgebra of $B(G)$ with $X \subseteq Q$. Then Y^* is the intersection of all P belonging to M_1, and analogously, X^* is the intersection of all Q belonging to M_2.

If $Y_1 \in S(G)$ and $X_1 = \chi(Y_1)$, then in view of Lemma 4.9, Y_1 is an α-complete Specker ℓ-subgroup of G if and only if X_1 is an α-complete subalgebra of $B(G)$. Hence $\psi(M_1) = M_2$. Therefore according to Lemma 4.3 we obtain $\chi(Y^*) = X^*$. □

Lemma 5.7. *Assume that Y is an α-complete retract of G. Put $X = \chi(Y)$. Then X is an α-complete retract of $B(G)$.*

Proof. Let Y^* and X^* be as above. Then there exists α-complete endomorphism ψ on Y^* which leaves Y elementwise fixed. Put $\varphi = \psi_0$. Then in view of Lemma 5.5.1 we infer that X is an α-complete retract of $B(G)$. □

Lemma 5.8. *Assume that X is an α-complete retract in $B(G)$. We set $Y = \chi^{-1}(X)$. Then Y is an α-complete retract in G.*

Proof. We proceed analogously as in the proof of Lemma 5.7 with the distinction that instead of Lemma 5.5.1 we now apply Lemma 5.5.2. □

In view of Lemmas 5.7, 5.8 and 4.3 we conclude

Theorem 5.9. *Let $G = Z_B$ be a Specker lattice ordered group and let α be a cardinal. Assume that the Boolean algebra B is α-complete. There exists a one-to-one correspondence between α-complete retracts of G and α-complete retracts of $B(G)$.*

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Received 11. 1. 2010
Accepted 14. 4. 2010

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