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ORDERED SEMIGROUPS WHOSE ELEMENTS ARE SEPARATED BY PRIME IDEALS

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ABSTRACT. In this paper we add one more characterization of intra-regular ordered semigroups in the existing bibliography by proving that the ordered semigroups whose elements are separated by prime ideals are actually the intra-regular ordered semigroups, a result which generalizes the corresponding result of semigroups (without order).

1. Introduction and prerequisites

A well known result by Stone [8] referring to the elements of a Boolean ring A separated by prime ideals of A is the following: If A is a Boolean ring containing elements a and b such that $ab \neq a$ or, equivalently, such that $a < b$ is false, then there exists a prime ideal P of A which contains b and not a; and, if A is a Boolean ring containing ideals A and B such that B is not a divisor of A, then there exists a prime ideal P in A which is a divisor of A but not of B (cf. [8, Theorem 64]). According to G. Szász, his paper in [9] has been inspired by the paper by Stone [8]. It is arguably the above mentioned theorem by Stone who let Szász to study semigroups whose elements are separated by prime ideals in [9] where he proved that these semigroups are actually the intra-regular semigroups. In the present paper we examine the result by Szász for ordered semigroups and we prove that the elements of an ordered semigroup S

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are separated by prime ideals if and only if (the ordered semigroup) S is intra-regular. Our result generalizes the corresponding result of semigroup (without order) given in [9].

A semigroup S is called *regular* if for each $a \in S$ there exists $x \in S$ such that $a = axa$, that is, if $a \in aSa$ for all $a \in S$. A semigroup S is called *intra-regular* if for every $a \in S$ there exist $x, y \in S$ such that $a = xa^2y$, which is equivalent to saying that $a \in Sa^2S$ for all $a \in S$ [1]. These conditions are also expressible as follows: S is regular (resp. intra-regular) if and only if $A \subseteq ASA$ (resp. $A \subseteq SA^2S$) for every $A \subseteq S$.

A subset T of an ordered semigroup (resp. semigroup) S is called *prime* [1, 3] if the complement $S \backslash T$ of T to S is either empty or it is a subsemigroup of S (that is either $S \backslash T = \emptyset$ or $S \backslash T \neq \emptyset$ and $a, b \notin T$ implies $ab \notin T$). Equivalent definitions are the following three definitions:

(1) If $a, b \in S$ such that $a, b \notin T$, then $ab \notin T$.
(2) If $a, b \in S$ such that $ab \in T$, then $a \in T$ or $b \in T$.
(3) If $A, B \subseteq S$ such that $AB \subseteq T$, then $A \subseteq T$ or $B \subseteq T$ [3].

A subset T of S is called *semiprime* [1, 4] if for every $a \in S$ such that $a^2 \in T$, we have $a \in T$; equivalently if $A \subseteq S$ such that $A^2 \subseteq T$, then $A \subseteq T$ [4]. An element a of an ordered semigroup (resp. semigroup) S is called *idempotent* if $a^2 = a$. An ordered semigroup (resp. semigroup) S is called *idempotent* [1, 2] if every element of S is so. An idempotent semigroup is also called a *band*. Bands were introduced by Klein-Barmen in 1940 who used the term "Schief".

If $(S, \cdot, \leq)$ is an ordered semigroup, we denote by $(H]$ the subset of S defined as follows:

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

A nonempty subset T of S is called an *ideal* of S [3] if

(1) $TS \subseteq T$, $ST \subseteq T$ and
(2) if $a \in T$ and $S \ni b \leq a$, then $b \in T$.

If T is an ideal of S, then $(T] = T$. For a nonempty subset A of S, we denote by $I(A)$ the ideal of S generated by A. Clearly, if A is an ideal of S, then $I(A) = A$. For $A = \{a\}$ we write $I(a)$ instead of $I(\{a\})$. We have $I(a) = (a \cup aS \cup Sa \cup SaS]$ [3]. S is called *intra-regular* [6, 7] if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$. Equivalent are the two definitions below:

(1) $a \in (Sa^2S]$ for every $a \in S$.
(2) $A \subseteq (SA^2S]$ for every $A \subseteq S$.

S is called *regular* [5] if for every $a \in S$ there exist $x \in S$ such that $a \leq axa$. This is equivalent to saying that $a \in (aSa]$ for all $a \in S$ or $A \subseteq (ASA]$ for all $A \subseteq S$.

The concepts of intra-regular, regular ordered semigroups generalize the corresponding concepts of intra-regular, regular semigroups as each intra-regular (resp. regular) semigroup endowed with the order $\leq \; := \{(x, y) \mid x = y\}$ (: the equality relation) is an intra-regular (resp. regular) ordered semigroup. Similarly, the concepts of ideals, prime ideals, semiprime ideals of ordered semigroups etc. generalize the corresponding concepts of semigroups. We denote by $\mathbb{N}$ the set $\{1, 2, \dots\}$ of natural numbers.

2. Main result

Definition 1. Let $(S, \cdot, \leq)$ be an ordered semigroup. We say that *the elements of S are separated by prime ideals* (of S) if the following assertion is satisfied: If $a, b \in S$ such that $b \notin I(a)$, then there exists a prime ideal P of S such that $a \in P$ and $b \notin P$.

Lemma 2. *An ordered semigroup S is intra-regular if and only if the ideals of S are semiprime.*

Proof.

$\Longrightarrow$: Let T be an ideal of S and $a \in S$ such that $a^2 \in T$. Since S is intra-regular, we have $a \in (Sa^2S] \subseteq (STS] \subseteq (T] = T$, thus T is semiprime.

$\Longleftarrow$: Let $a \in S$. Since $(Sa^2S]$ is an ideal of S, $(a^2)^2 = a^4 \in (Sa^2S]$ and $(Sa^2S]$ is semiprime, we have $a^2 \in (Sa^2S]$, and $a \in (Sa^2S]$, so S is intra-regular. $\square$

Notation 3. We denote by $\langle b \rangle$ the subsemigroup of S defined by

$$\langle b \rangle := \{b^n \mid n \in \mathbb{N}\}$$

(i.e. the cyclic subsemigroup of S generated by b).

Lemma 4. *Let S be an ordered semigroup, T a semiprime ideal of S and $a \in S$. If $a^n \in T$ for some $n \in \mathbb{N}$, then $a \in T$.*

Proof. For $n = 1, 2$ it is clear. Assume that $a^m \in T$ implies $a \in T$ for some $m \geq 3$, and let $a^{m+1} \in T$. Then $a \in T$. Indeed: Since $m \geq 3$, we have $a^{m-1} \in S$. Then we have $(a^m)^2 = a^{2m} = a^{m+1+m-1} = a^{m+1}a^{m-1} \in TS \subseteq T$. Since T is semiprime and $a^m \in S$ such that $(a^m)^2 \in T$, we have $a^m \in T$. Then, by our assumption, we have $a \in T$. $\square$

Lemma 5. *Let S be an ordered semigroup, T a semiprime ideal of S and $b \in S$ such that $b \notin T$. Then $T \cap \langle b \rangle = \emptyset$.*

Proof. Let $x \in T \cap \langle b \rangle$. Then $T \ni x = b^n$ for some $n \in \mathbb{N}$. Since T is a semiprime ideal of S, $b \in S$, $n \in \mathbb{N}$, and $b^n \in T$, by Lemma 4, we have $b \in T$ which is not possible. $\square$

Lemma 6. *If $(S, \cdot, \leq)$ is an ordered semigroup and $\mathcal{B}$ a nonempty family of ideals of S, then the union $\bigcup_{B \in \mathcal{B}} B$ is an ideal of S.*

Proof. First of all the set $\bigcup_{B \in \mathcal{B}} B$ is a nonempty subset of S since each ideal of S is nonempty. Moreover we have

$$\Big(\bigcup_{B \in \mathcal{B}} B\Big)S = \bigcup_{B \in \mathcal{B}} BS \subseteq \bigcup_{B \in \mathcal{B}} B, \qquad S\Big(\bigcup_{B \in \mathcal{B}} B\Big) = \bigcup_{B \in \mathcal{B}} SB \subseteq \bigcup_{B \in \mathcal{B}} B.$$

Let now $x \in \bigcup_{B \in \mathcal{B}} B$ and $S \ni y \leq x$. Suppose $x \in B$ for some $B \in \mathcal{B}$. Since B is an ideal of S, we have $y \in B \subseteq \bigcup_{B \in \mathcal{B}} B$. $\square$

As a result, the union of two ideals of S is again an ideal of S.

Lemma 7. *If S is an intra-regular ordered semigroup, then $I(a) = (SaS]$ for every $a \in S$.*

Proof. Let $a \in S$. Since S is intra-regular, we have $a \in (Sa^2S] \subseteq (SaS]$, and since $(SaS]$ is an ideal of S containing a, we have $I(a) \subseteq (SaS]$. On the other hand, $(SaS] \subseteq (a \cup aS \cup Sa \cup SaS] = I(a)$. Thus $I(a) = (SaS]$. $\square$

Lemma 8. *If S is an intra-regular ordered semigroup, then $I(ab) = I(ba)$ for every $a, b \in S$.*

Proof. Let $a, b \in S$. By Lemma 7, we have $(ab)^2 = a(ba)b \in (SbaS] = I(ba)$. Since S is intra-regular and $I(ab)$ is an ideal of S, by Lemma 2, $I(ab)$ is semiprime, so $ab \in I(ba)$, then $I(ab) \subseteq I(ba)$. By symmetry, we get $I(ba) \subseteq I(ab)$, thus we have $I(ab) = I(ba)$. $\square$

Lemma 9. *If S is an intra-regular ordered semigroup, then $I(a) \cap I(b) = I(ab)$ for each $a, b \in S$.*

Proof. First of all, in any ordered semigroup S, for any $a, b \in S$, we have $I(ab) \subseteq I(a) \cap I(b)$. In fact, let $a, b \in S$. Since $a \in I(a)$, we have $ab \in I(a)S \subseteq I(a)$, then $I(ab) \subseteq I(a)$. Since $b \in I(b)$, we have $ab \in SI(b) \subseteq I(b)$, then $I(ab) \subseteq I(b)$. Thus we have $I(ab) \subseteq I(a) \cap I(b)$. Let now S be an intra-regular ordered semigroup, $a, b \in S$ and $c \in I(a) \cap I(b)$. By Lemma 7, we have

$c \in I(a) = (SaS]$, thus $c \le uav$ for some $u, v \in S$. Since $c \in I(b) = (SbS]$, we have $c \le xby$ for some $x, y \in S$. Then by Lemmas 7 and 8, we get

$$c^2 \le x(byua)v \in S(byua)S \subseteq (S(byua)S] = I((byu)a) = I(abyu),$$

then $c^2 \in (I(abyu)] = I(abyu)$. Since S is intra-regular, by Lemma 2, the ideal $I(abyu)$ of S is semiprime, thus $c \in I(abyu)$. Then, applying first Lemma 8 and then Lemma 7, we obtain

$$c \in I(uaby) = (uaby \cup uabyS \cup Suaby \cup SuabyS] \subseteq (SabS] = I(ab).$$

□

Theorem 10. *The elements of an ordered semigroup S are separated by prime ideals (of S) if and only if S is intra-regular.*

Proof.

$\Longrightarrow$: Suppose S is not intra-regular. Then, by Lemma 2, there exists an ideal T of S and an element a of S such that $a^2 \in T$ and $a \notin T$. Since $a^2 \in T$ and T is an ideal of S, we have $I(a^2) \subseteq I(T) = T$. Since $a \notin T$, we have $a \notin I(a^2)$. Since $a \notin I(a^2)$, by hypothesis, there exists a prime ideal P of S such that $a^2 \in P$ and $a \notin P$. P as a prime ideal, is a semiprime ideal of S as well. Since $a^2 \in P$ and P is semiprime, we have $a \in P$. Impossible.

$\Longleftarrow$: Let $a, b \in S$ such that $b \notin I(a)$. Then there exists a prime ideal P of S such that $a \in P$ and $b \notin P$. In fact: Since S is intra-regular and $I(a)$ an ideal of S, by Lemma 2, $I(a)$ is semiprime. Since $I(a)$ is a semiprime ideal of S and $b \notin I(a)$, by Lemma 5, we have $I(a) \cap \langle b \rangle = \emptyset$. We consider the set

$$\mathcal{A} := \big\{T \mid T \text{ ideal of } S \text{ such that } I(a) \subseteq T \text{ and } T \cap \langle b \rangle = \emptyset\big\}.$$

Since $I(a) \in \mathcal{A}$, the set $\mathcal{A}$ is nonempty, so the set $\mathcal{A}$ endowed with the inclusion relation $\subseteq$ is an ordered set. The set $P := \bigcup\limits_{T \in \mathcal{A}} T$ is an upper bound of $\mathcal{A}$. Indeed: By Lemma 6, the set $\bigcup\limits_{T \in \mathcal{A}} T$ is an ideal of S. As $I(a) \subseteq T$ for each $T \in \mathcal{A}$, we have $I(a) \subseteq \bigcup\limits_{T \in \mathcal{A}} T$. Finally, $\Big(\bigcup\limits_{T \in \mathcal{A}} T\Big) \cap \langle b \rangle = \emptyset$. Indeed: If $x \in \Big(\bigcup\limits_{T \in \mathcal{A}} T\Big) \cap \langle b \rangle$, then $x \in T$ for some $T \in \mathcal{A}$ and $x \in \langle b \rangle$. Then $x \in T \cap \langle b \rangle$, where $T \in \mathcal{A}$ which is impossible. Since $(\mathcal{A}, \subseteq)$ is an ordered set and P is an upper bound of $\mathcal{A}$, P is a maximal element of $\mathcal{A}$. That is, P is an ideal of S, $I(a) \subseteq P$, $P \cap \langle b \rangle = \emptyset$, and there is no element T in $\mathcal{A}$ such that $T \supset P$. Since $a \in I(a) \subseteq P$, we have $a \in P$. Since $b \in \langle b \rangle$ and $P \cap \langle b \rangle = \emptyset$, we have $b \notin P$. It remains to prove that the ideal P of S is prime. For this purpose, let $u, v \in S$ such that $u, v \notin P$. Then $uv \notin P$. In fact: We consider the sets $P' := P \cup I(u)$ and $P'' := P \cup I(v)$. Since $u \in P'$ and $u \notin P$, we have $P \subset P'$. Since $v \in P''$ and $v \notin P$, we have

$P \subset P''$. Since P' is an ideal of S such that $I(a) \subseteq P \subset P'$, we have $P' \cap \langle b \rangle \neq \emptyset$ (otherwise $\mathcal{A} \ni P' \supset P$ which is impossible). Since P'' is an ideal of S such that $I(a) \subseteq P \subset P''$, we have $P'' \cap \langle b \rangle \neq \emptyset$ (otherwise $\mathcal{A} \ni P'' \supset P$ which is impossible). Let now $m, n \in \mathbb{N}$ such that $b^m \in P'$ and $b^n \in P''$. Since P', P'' are ideals of S and S is intra-regular, by Lemma 2, P' and P'' are semiprime. Then, by Lemma 4, we have $b \in P'$ and $b \in P''$. Since $b \in P'$, we have $b \in P$ or $b \in I(u)$. If $b \in P$, then $b \in P \cap \langle b \rangle = \emptyset$ which is impossible. Thus we have $b \in I(u)$. Similarly we get $b \in I(v)$. Since S is intra-regular, by Lemma 9, we have $b \in I(u) \cap I(v) = I(uv)$. If $uv \in P$, then $I(uv) \subseteq P$, and $b \in P$ which is impossible. Thus we have $uv \notin P$. $\square$

Remark 11.

(A) If the elements of an ordered semigroup $(S, \cdot, \leq)$ are idempotent, then S is intra-regular. Indeed: Let $a \in S$. Then $a = a^2 = aa = a^2a^2 = aa^2a$, where $a \in S$. Since $\leq$ is reflexive, we have $(a, aa^2a) \in \leq$ i.e. $a \leq aa^2a$, where $a \in S$. As a result, *the elements of an idempotent ordered semigroup are separated by prime ideals.*

(B) A commutative ordered semigroup $(S, \cdot, \leq)$ is intra-regular if and only if it is regular. In fact:

$\Longrightarrow$: Let $a \in S$. Since S is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y = a(xy)a$, where $xy \in S$, so S is regular.

$\Longleftarrow$: Since S is regular, there exists $x \in S$ such that $a \leq axa$. Again since S is regular, there exists $z \in S$ such that $x \leq xzx$. Thus we have $a \leq a(xzx)a = xa^2(zx)$, where $x, zx \in S$, so S is intra-regular. As a result, *the elements of a commutative ordered semigroup S are separated by prime ideals if and only if S is regular.*

In the following, given a semigroup $(S, \cdot)$ we denote by $\leq$ the order on S defined by $\leq\ := \{(x, y) \mid x = y\}$ (: the equality relation on S). It is easy to see that $(S, \cdot, \leq)$ is an ordered semigroup.

Remark 12. A semigroup $(S, \cdot)$ is intra-regular if and only if the ordered semigroup $(S, \cdot, \leq)$ is so. Indeed:

$\Longrightarrow$: Let $a \in (S, \cdot, \leq)$. Since $(S, \cdot)$ is intra-regular, there exist $x, y \in S$ such that $a = xa^2y$. Then $(a, xa^2y) \in \leq$, so $a \leq xa^2y$, and $(S, \cdot, \leq)$ is intra-regular.

$\Longleftarrow$: Let $a \in (S, \cdot)$. Since $(S, \cdot, \leq)$ is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y$. Since $(a, xa^2y) \in \leq$, we have $a = xa^2y$, so $(S, \cdot)$ is intra-regular.

As an application of Theorem 10, we have the following

Corollary 13. ([9]) *The elements of a semigroup $(S, \cdot)$ are separated by prime ideals if and only if S is intra-regular.*

P r o o f.

$\Longrightarrow$: As the elements of $(S,\cdot)$ are separated by prime ideals of $(S,\cdot)$, the elements of $(S,\cdot,\leq)$ are separated by prime ideals of $(S,\cdot,\leq)$ as well (this is because a prime ideal of $(S,\cdot)$ is a prime ideal of $(S,\cdot,\leq)$). By Theorem 10, $(S,\cdot,\leq)$ is intra-regular. By Remark 12, $(S,\cdot)$ is intra-regular.

$\Longleftarrow$: Let $a, b \in S$. Since $(S,\cdot)$ is intra-regular, by Remark 12, $(S,\cdot,\leq)$ is intra-regular. By Theorem 10, there exists a prime ideal P of $(S,\cdot,\leq)$ such that $a \in P$ and $b \notin P$. Then P a prime ideal of $(S,\cdot)$, $a \in P$ and $b \notin P$. $\square$

Remark 14. Remark 12 remains true if we replace the word "intra-regular" by "regular". In the same way as in Corollary 13, applying the results mentioned in Remark 11, we get the following results of [9, Section 4]:

> *The elements of a band are separated by prime ideals. The elements of a commutative semigroup S are separated by prime ideals if and only if S is regular (in the sense of J. von Neumann).*

Remark 15. In Theorem 10 above we used the following: If p is an upper bound of an ordered set $(P,\leq)$, then p is a maximal element of P (Clearly, if $t \in P$, $t \geq p$ then, since p is an upper bound of P, we have $p \geq t$, so $t = p$). Using the same in the proof of [9, Theorem], we avoid to apply Zorn's Lemma which is not necessary. This is because if S is a semigroup (or ordered semigroup), $I(a)$ the ideal of S generated by a $(a \in S)$, $\mathcal{A}$ the set of (all) ideals T of S such that $I(a) \subseteq T$ and $T \cap \langle b \rangle = \emptyset$, and $\mathcal{B}$ a linearly ordered subset of $\mathcal{A}$, then to prove that $\bigcup\limits_{B \in \mathcal{B}} B$ is an upper bound of $\mathcal{B}$ in $\mathcal{A}$ (and apply Zorn's Lemma) we do not use the fact that $\mathcal{B}$ is linearly ordered (that is, for each $B_1, B_2 \in \mathcal{B}$, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$).

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