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TORSION CLASSES OF GENERALIZED BOOLEAN ALGEBRAS

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ABSTRACT. Torsion classes and radical classes of lattice ordered groups have been investigated in several papers. The notions of torsion class and of radical class of generalized Boolean algebras are defined analogously. We denote by $\mathcal{T}_g$ and $\mathcal{R}_g$ the collections of all torsion classes or of all radical classes of generalized Boolean algebras, respectively. Both $\mathcal{T}_g$ and $\mathcal{R}_g$ are partially ordered by the class-theoretical inclusion. We deal with the relation between these partially ordered collection; as a consequence, we obtain that $\mathcal{T}_g$ is a Brouwerian lattice. W. C. Holland proved that each variety of lattice ordered groups is a torsion class. We show that an analogous result is valid for generalized Boolean algebras.

1. Introduction

Generalized Boolean algebras play an essential role in investigating certain types of lattice ordered groups and of vector lattices; in this connection, cf. the papers of Conrad and Darnel [3], [4], Conrad and Martínez [5], and the author [15], [16], [17].

The notion of torsion class of lattice ordered groups was introduced and studied by Martínez [20]. The author [10] defined and investigated the notion of radical class of lattice ordered groups.

A class C of lattice ordered groups is a torsion class if

(i) C is a radical class, and

(ii) C is closed with respect to homomorphisms.

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Torsion classes and radical classes of lattice ordered groups have been dealt with in a series of further papers. We mention here, e.g., the articles of Martínez [21], Conrad [1], the author [11] (for torsion classes) and Conrad [2], Darnel [6], Medvedev [22] (for radical classes).

For some types of lattice ordered algebraic structures, the notion of radical class can be defined analogously as in the case of lattice ordered groups; cf. [12] (generalized Boolean algebras) and [13] (MV-algebras).

Similarly, the notion of torsion class has been defined and studied in the theory of MV-algebras [14]. In the above mentioned papers [15] and [17], torsion classes of generalized Boolean algebras have been dealt with in connection with the theory of Carathéodory vector lattices or Specker lattice ordered groups, respectively.

We recall that generalized Boolean algebra is defined to be a distributive lattice B with a least element such that each principal ideal of B is a Boolean algebra.

We denote by $\mathcal{T}_g$ the collection of all torsion classes of generalized Boolean algebras; $\mathcal{T}_g$ is partially ordered by the class-theoretical inclusion. We show that $\mathcal{T}_g$ is a large collection (there exists an injective mapping of the class of all infinite cardinals into the collection $\mathcal{T}_g$). Nevertheless, we can apply the standard terminology for the mentioned partial order in $\mathcal{T}_g$; we recall that an analogous method has been applied by Martínez [20].

We prove that the partially ordered collection $\mathcal{T}_g$ is a Brouwerian lattice.

Let A be a nonempty class of generalized Boolean algebras which is closed with respect to isomorphisms. We denote by

$s_0(A)$ — the class of all generalized Boolean algebras B_1 such that B_1 is a convex subalgebra of some generalized Boolean algebras belonging to A;

$w(A)$ — the class of all generalized Boolean algebras which can be expressed as a weak direct product of some elements of A;

$\hom A$ — the class of all homomorphic images of elements of A.

Let $\mathcal{B}_0$ be the class of all generalized Boolean algebras and let $\emptyset \neq X \subseteq \mathcal{B}_0$. We prove the following results:

(i) The radical class generated by X is equal to $w(s_0(X))$.

(ii) The torsion class generated by X is equal to $w(s_0(\hom X))$.

(iii) If X is a radical class, then the torsion class generated by X is equal to $\hom X$.

Holland [8] proved the following theorem:

($*$) Each variety of lattice ordered groups is a torsion class.

Since each variety is closed with respect to homomorphisms, the essential part of the theorem $(*)$ consists in the assertion

$(*_1)$ Each variety of lattice ordered group is a radical class.

In the present paper we show that a result analogous to $(*_1)$ is valid for generalized Boolean algebras.

2. Preliminaries

For the sake of completeness, we recall some basic definitions.

Let $\mathcal{G}$ be the class of all lattice ordered groups and let $f\colon \mathcal{G} \to \mathcal{G}$ be a mapping such that for each $G \in \mathcal{G}$ the following conditions are satisfied:

(i) $f(G)$ is a convex ℓ-subgroup of G;

(ii) if G_1 is a convex ℓ-subgroup of G, then $f(G_1) = G_1 \cap f(G)$.

We say that f is a radical mapping on $\mathcal{G}$. We denote by F the class of all radical mappings on $\mathcal{G}$. A nonempty subclass C of $\mathcal{G}$ is defined to be a radical class if there exists a radical mapping f on $\mathcal{G}$ such that $C = \big\{G \in \mathcal{G} :\ f(G) = G\big\}$; we write $\varphi(f) = C$.

A radical class of lattice ordered groups is said to be a torsion class if it is closed with respect to homomorphisms.

Let $\mathbf{B} = (B; \wedge, \vee, 0)$ be a distributive lattice with the least element 0. Assume that for each $x \in B$, the lattice $\mathbf{B}_x = ([0,x]; \wedge, \vee, 0)$ is complemented, where $[0,x]$ is the corresponding interval in $\mathbf{B}$. Then $\mathbf{B}$ is a *generalized Boolean algebra.*

A concex sublattice of $\mathbf{B}$ containing the element 0 will be said to be a convex subalgebra of $\mathbf{B}$.

The notion of a homomorphism of generalized Boolean algebras is defined in a natural way (cf. Section 7).

Let $\mathcal{B}_0$ be the class of all generalized Boolean algebras.

We define the notions of radical mapping of $\mathcal{B}_0$ and radical class in $\mathcal{B}_0$ analogously as we did for lattice ordered groups; namely, we apply the above conditions (i) and (ii) with the distinction that instead of $\mathcal{G}, G$ and G_1 we now have $\mathcal{B}_0, B$ and B_1, where $B \in \mathcal{B}_0$ and B_1 is a convex subalgebra of B. A radical class which is closed with respect to homomorphisms is said to be a torsion class.

Let $B \in \mathcal{B}_0$. We denote by $c(B)$ the system of all convex subalgebras of B; this system is partially ordered by the set-theoretical inclusion. In view of [12], we have the following results.

Proposition 2.1. *The system $c(B)$ is a Brouwerian lattice.*

Proposition 2.2. *Let f be a radical mapping and $C_f = \{B_1 \in \mathcal{B} : f(B_1) = B_1\}$. Then $f(B)$ is the largest element of $c(B)$ belonging to the radical class C_f.*

Lemma 2.3. *Let $\emptyset \neq C_2 \subseteq \mathcal{B}_0$. Assume that the following conditions are satisfied.*

(i_1) *If $B \in C_2$ and $B_1 \in c(B)$, then $B_1 \in C_2$;*

(ii_1) *For each $B \in \mathcal{B}_0$, the set $C_2 \cap c(B)$ has a greatest element.*

Then C_2 is a radical class.

Let α be an infinite cardinal. We mention the following examples of radical classes:

- the class of all generalized Boolean algebras B such that each interval of B is α-complete;
- the class of all generalized Boolean algebras which are α-distributive.

In view of the above examples we conclude that the collection $\mathcal{R}_g$ of all radical classes of $\mathcal{B}_0$ is a proper class.

In the following section we show that the collection $\mathcal{T}_g$ of all torsion classes of generalized Boolean algebras is a proper class as well.

Both $\mathcal{R}_g$ and $\mathcal{T}_g$ are partially ordered by the class-theoretical inclusion.

3. Weak direct products

Assume that $(B_i)_{i\in I}$ is a nonempty indexed system of generalized Boolean algebras. Their direct product is defined in the usual way and it is denoted by $\prod_{i\in I} B_i$. The elements of the direct product are written in the form $x = (x_i)_{i\in I}$; we say that x_i is the component of x in B_i; instead of x_i we write also $x(B_i)$.

The set of all elements x such that the set $\{i \in I : x_i \neq 0\}$ is finite will be denoted by $(w)\prod_{i\in I} B_i$ and is said to be a weak direct product of the system $(B_i)_{i\in I}$. We have

$$(w)\prod_{i\in I} B_i \in c\Big(\prod_{i\in I} B_i\Big).$$

Lemma 3.1. *Let X be a radical class of generalized Boolean algebras. Then $s_0(X) = X = w(X)$.*

Proof.

a) The relation $s_0(X) = X$ is a consequence of definitions of a radical class and of a radical mapping.

b) Let us deal with the relation $X = w(X)$. We have $X \subseteq w(X)$. Assume that $B \in w(X)$. Hence B can be expressed in the form

$$B = (w) \prod_{i \in I} B_i,$$

where $B_i \in X$ for each $i \in I$.

Let i be a fixed element of I. We denote by $\overline{B}_i$ the set of all elements x of B such that $x_j = 0$ for each $j \in I$, $j \neq i$. Then $\overline{B}_i$ is a sublattice of B and $\overline{B}_i \simeq B_i$. For simplifying the notation, let us identify $\overline{B}_i$ and B_i.

There exists $f \in F$ with $\varphi(f) = X$. Put $f(B) = B_1$. If $i \in I$, then in view of the relation $B_i \in X$ we get $f(B_i) = B_i$. Applying the definition of a radical mapping again, we obtain $B_i \subseteq B_1$.

Let $x \in B$. Then the set $I_1 = \{i \in I : \ x_i \neq 0\}$ is finite. Put $I_1 = \{i(1), i(2), \dots, i(n)\}$. We have

$$x = x_{i(1)} \vee \dots \vee x_{i(n)},$$

whence $x \in B_1$. Thus $B_1 = B$ and $f(B) = B$. Therefore $B \in X$, which completes the proof. □

Lemma 3.2. *Let X be a nonempty class of generalized Boolean algebras such that $s_0(X) = X = w(X)$. Then X is a radical class.*

Proof.

a) In view of relation $s_0(X) = X$, the condition (i_1) of Lemma 2.3 is satisfied.

b) Now we will show that the condition (ii_1) of Lemma 2.3 is fulfilled as well. Let $B \in \mathcal{B}_0$. We denote by X_1 the set of all intervals of B which contain the element 0 and belong to the class X. If $[0, x_1]$ is such interval, then in view of a) we also have $[0, x_2] \in X_1$, where $x_2 < x_1$.

Let $[0, x]$ and $[0, y]$ belong to X_1. We put $x \wedge y = z$ and $x \vee y = t$. Further, let z_1 be the relative complement of z in the interval $[0, y]$. Then we have

$$z_1 \wedge x = (z_1 \wedge y) \wedge x = z_1 \wedge (y \wedge x) = z_1 \wedge z = 0,$$
$$z_1 \vee x = z_1 \vee (z \vee x) = (z_1 \vee z) \vee x = y \vee x = t.$$

In view of distributivity of B we get

$$[0, t] \simeq [0, z_1] \times [0, x]. \tag{1}$$

We denote by $\overline{X}_1$ the set of all elements $b \in B$ such that $[0, b]$ belongs to X_1. Under the notation as above, we have $x, y \in \overline{X}_1$. In view of a), the element

z also belongs to $\overline{X}_1$ and, moreover, both $[0, z_1]$ and $[0, x]$ are subsets of $\overline{X}_1$. The relation (1) and the condition $X = w(X)$ show that the interval $[0, t]$ is a subset of $\overline{X}_1$. Hence $\overline{X}_1$ is a convex subalgebra of B. It is obvious that $\overline{X}_1$ is the greatest element of the set $X \cap c(B)$. According to Lemma 2.3, we conclude that X is a radical class. □

The following assertion is easy to verify.

Lemma 3.3. *Let B_i $(i \in I)$ are elements of $\mathcal{B}_0$ and $B = (w) \prod_{i \in I} B_i$. If $B_1 \in c(B)$, then for each $i \in I$ we have $B_1 \cap B_i \in c(B_i)$ and $B_1 = (w) \prod_{i \in I} (B_1 \cap B_i)$.*

Proposition 3.4. *Let X be a nonempty class of generalized Boolean algebras. Put $Y = w(s_0(X))$.*

(i) *Y is a radical class.*

(ii) *If Y_1 is a radical class with $Y_1 \supseteq X$, then $Y_1 \supseteqq Y$.*

Proof. For each nonempty class Z of generalized Boolean algebras we have $s_0(s_0(Z)) = s_0(Z)$ and $w(w(Z)) = w(Z)$. Hence $w(Y) = Y$. From this and from Lemma 3.3 we obtain $s_0(Y) = s_0(w(s_0(X))) = Y$. Hence in view of Lemma 3.2 we conclude that (i) is valid.

Let Y_1 be a radical class with $Y_1 \supseteqq X$. Then

$$Y = w(s_0(X)) \subseteq w(s_0(Y_1)) = Y_1.$$

□

In view of Proposition 3.4 we say that $w(s_0(X))$ is the radical class generated by the class X.

For each infinite cardinal α we denote by X_α the collection of all generalized Boolean algebras B such that, whenever $x \in B$, then $\operatorname{card}[0, x] \leqq \alpha$.

Proposition 3.5. *Let α be an infinite cardinal. Then the collection X_α is a torsion class.*

Proof. It is obvious that the collection X_α is nonempty and that $s_0(X_\alpha) = X_\alpha$.

Let $(B_i)_{i \in I}$ be a nonempty system of generalized Boolean algebras such that $B_i \in X_\alpha$ for each $i \in I$; let B be the weak direct product of this system. Further, let $0 < x \in B$ and $I_1 = \{i \in I : x_i \neq 0\}$, where x_i is the component of x in B_i. Then the set I_1 is finite; we can write $I_1 = \{i(1), i(2), \ldots, i(n)\}$. We have $x = x_{i(1)} \vee \cdots \vee x_{i(n)}$; moreover, the interval $[0, x]$ is a direct product of the intervals $[0, x_{i(1)}], \ldots, [0, x_{i(n)}]$. Since $\operatorname{card}[0, x_{i(1)}] \leqq \alpha$, …, $\operatorname{card}[0, x_{i(n)}] \leqq \alpha$, we get

$$\operatorname{card}[0, x] \leqq \alpha^n = \alpha$$

and hence $B \in X_\alpha$.

In view of Lemma 3.2, we conclude that the collection X_α is a radical class. It remains to verify that X_α is closed with respect to homomorphisms.

Let $B \in X_\alpha$ and let $\varphi\colon B \to B_1$ be a homomorphism of B onto a generalized Boolean algebra B_1. Let $0 < x_1 \in B_1$. There exists $x \in B$ with $\varphi(x) = x_1$. Then we have $\varphi([0,x]) = [0,x_1]$. Since $\operatorname{card}[0,x] \leqq \alpha$ we obtain $\operatorname{card}[0,x_1] \leqq \alpha$ and hence $B_1 \in X_\alpha$. □

For each infinite cardinal α we put $f(\alpha) = X_\alpha$. If α_1 and α_2 are distinct infinite cardinals, then $f(\alpha_1) \neq f(\alpha_2)$. Thus f is an injective mapping of the class of all infinite cardinals into the collection $\mathcal{T}_g$. Hence the collection $\mathcal{T}_g$ is a proper class.

Assume that $\{B_i\}_{i\in I}$ is a system of ideals of a generalized Boolean algebra B such that, whenever $i(1)$ and $i(2)$ are distinct elements of I, then $B_{i(1)} \cap B_{i(2)} = \{0\}$.

Let $0 < z \in B$. Suppose that there exist sets of indices

$$I_1 = \{i(1), i(2), \ldots, i(n)\} \subseteq I,$$
$$I_2 = \{j(1), j(2), \ldots, j(m)\} \subseteq I$$

and elements

$$0 < x_1 \in B_{i(1)}, \ldots, 0 < x_n \in B_{i(n)},$$
$$0 < y_1 \in B_{j(1)}, \ldots, 0 < y_m \in B_{j(m)}$$

such that

$$z = x_1 \vee \cdots \vee x_n = y_1 \vee \cdots \vee y_m.$$

Lemma 3.6. *Under the notation as above, we have*

(i) $I_1 = I_2$;

(ii) *if* $k \in \{1,2,\ldots,n\}$, $\ell \in \{1,2,\ldots,m\}$ *and* $B_{i(k)} = B_{j(\ell)}$, *then* $x_k = y_\ell$.

Proof. Let $k \in \{1,2,\ldots,n\}$. Then

$$x_k = x_k \wedge z = x_k \wedge (y_1 \vee \cdots \vee y_m) = (x_k \wedge y_1) \vee \cdots \vee (x_k \wedge y_m).$$

Let $\ell \in \{1,2,\ldots,m\}$. If $B_{i(k)} \neq B_{j(\ell)}$, then $B_{i(k)} \cap B_{j(\ell)} = \{0\}$. Thus there exists $\ell \in \{1,2,\ldots,m\}$ with $B_{i(k)} = B_{j(\ell)}$. Moreover, if $\ell' \in \{1,2,\ldots,m\}$ and $\ell' \neq \ell$, then $B_{i(k)} \neq B_{j(\ell')}$ and so $x_k \wedge y_{\ell'} = 0$. We obtain $i(k) \in I_2$, hence $I_1 \subseteq I_2$. Analogously we can verify that the relation $I_2 \subseteq I_1$ is valid. Therefore $I_1 = I_2$.

Further, we have $x_k \leqq y_\ell$. By analogous argument we get $y_\ell \leqq x_k$ and hence $x_k = y_\ell$. □

Lemma 3.7. *Let B and $\{B_i\}_{i\in I}$ be as in* Lemma 3.6. *Assume that for each element $0 < z \in B$ there exists a set $I_1 = \{i(1), \dots, i(n)\} \subseteq I$ such that there are elements $0 < x_1 \in B_{i(1)}$, ..., $0 < x_n \in B_{i(n)}$ with $z = x_1 \vee \dots \vee x_n$. Then B is isomorphic to $(w) \prod_{i\in I} B_i$.*

Proof. Put $(w) \prod_{i\in I} B_i = B'$. We set $f(0) = 0$; for $z \in B$ with $z > 0$ let $f(z) = z'$, where $z' \in B'$ such that

$$z'(B_{i(1)}) = x_1, \dots, z'(B_{i(n)}) = x_n$$

and $z'(B_i) = 0$ whenever $i \in I \backslash I_1$.

In view of Lemma 3.6, f is a correctly defined mapping of B into B'. It is obvious that f is injective and that for any $z, t \in B$ we have

$$z \leqq t \iff f(z) \leqq f(t).$$

Let $0 < p \in B'$ and $I(p) = \big\{i \in I : \ p(B_i) \neq 0\big\}$. The set $I(p)$ is finite; we can write $I(p) = \{j(1), \dots, j(m)\}$. We denote $q = p(B_{j(1)}) \vee \dots \vee p(B_{j(m)})$. Then we obtain $f(q) = p$; thus the mapping f is surjective. Therefore f is an isomorphism of B onto B'. □

4. The partially ordered collection $\mathcal{T}_g$

It is obvious that the class $\mathcal{B}_0$ of all generalized Boolean algebras is the greatest element of the partially ordered collection $\mathcal{T}_g$. Let X_0 be the class of all one-element Boolean algebras; then X_0 is the least element of $\mathcal{T}_g$.

Assume that $\{X_i\}_{i\in I}$ is a nonempty subcollection of $\mathcal{T}_g$; put $X = \bigcap_{i\in I} X_i$. In view of the definition of $\mathcal{T}_g$, we have $X \in \mathcal{T}_g$. Then X is the greatest lower bound of the system $\{X_i\}_{i\in I}$ in $\mathcal{T}_g$ According to the fact that $\mathcal{T}_g$ has the greatest element and the least element, we conclude:

Proposition 4.1. *The partially ordered collection $\mathcal{T}_g$ is a complete lattice.*

Lemma 4.2. *Let $\emptyset \neq X \subseteq \mathcal{B}_0$. Assume that X is closed with respect to homomorphisms. Then the class $s_0(X)$ is closed with respect to homomorphisms.*

Proof. Put $s_0(X) = Y$ and let $B \in Y$. Assume that B_1 is a homomrphic image of B. Then without loss of generality we can suppose that there exists a congruence relation ρ on B such that $B_1 = B/\rho$. The congruence relation ρ is generated by an ideal A on B; in the usual way, we can write $B_1 = B/A$.

Since $B \in s_0(X)$ there exists $B_2 \in X$ such that B is an ideal of B_2. Then A is an ideal of B_2. We conclude that B/A is an ideal of B_2/A. Since X is closed with respect to homomorphisms, we get $B_2/A \in X$. Therefore $B_1 = B/A \in s_0(X)$. □

Lemma 4.3. *Let $\emptyset \neq X \subseteq \mathcal{B}_0$. Assume that X is closed with respect to homomorphisms. Then the class $w(X)$ is closed with respect to homomorphisms.*

Proof. Put $w(X) = Y$ and let $B \in Y$. Further, let B_1, ρ and A be as in the proof of Lemma 4.2.

Since $B \in w(X)$ there exists a system $\{B_i\}_{i\in I} \subseteq X$ such that $B = (w) \prod\limits_{i\in I} B_i$.

For each $z \in B$ we denote by $\overline{z}$ the congruence class taken with respect to ρ which contains the element z; for $Z \subseteq B$ we put $\{\overline{z} : z \in Z\} = \overline{Z}$.

Let $i \in I$. Let B_i^0 be the set of all elements t of B such that $t(B_j) = 0$ for each $j \in I$ with $j \neq i$. The set B_i^0 is partially ordered by the partial order induced from B. Then $B_i^0 \simeq B_i$, thus $B_i^0 \in X$.

In view of the congruence relation ρ, we conclude that $\overline{B_i^0}$ is an ideal of B_1 and $\overline{B_{i(1)}^0} \cap \overline{B_{i(2)}^0} = \{\overline{0}\}$ whenever $i(1)$ and $i(2)$ are distinct elements of I.

For each element $b_1 \in B_1$ there exists $z \in B$ with $b_1 = \overline{z}$. There are $i(1), \ldots, i(n) \in I$ and $x_1 \in B_{i(1)}, \ldots, x_n \in B_{i(n)}$ with $z = x_1 \vee \cdots \vee x_n$. Then $\overline{x_1} \in \overline{B_{i(1)}^0}, \ldots, \overline{x_n} \in \overline{B_{i(n)}^0}$ and $\overline{z} = \overline{x_1} \vee \cdots \vee \overline{x_n}$. Therefore, according to Lemma 3.7, we have $B_1 = (w) \prod\limits_{i\in I} \overline{B_i^0}$.

Obviously, $\overline{B_i^0}$ is a homomorphic image of B_i^0 and thus $\overline{B_i^0} \in X$. Hence $B_1 \in Y$. □

Lemma 4.4. *Let $\emptyset \neq X \subseteq \mathcal{B}_0$. Assume that the class X is closed with respect to homomorphisms. Then the class $w(s_0(X))$ is closed with respect to homomorphisms.*

Proof. This is a consequence of Lemma 4.2 and Lemma 4.3. □

For each $\emptyset \neq X \subseteq \mathcal{B}_0$ we denote by $\hom(X)$ the class of all homomorphic images of X.

Theorem 4.5. *Let $\emptyset \neq X \subseteq \mathcal{B}_0$. Put $w(s_0(\hom(X))) = Y$. Then*

(i) *Y is a torsion class and $Y \supseteq X$;*

(ii) *if Y_1 is a torsion class and $Y_1 \supseteq X$, then $Y_1 \supseteq Y$.*

Proof. The relation $Y \supseteq X$ is obvious. According to Proposition 3.4, Y is a radical class. Then in view of Lemma 4.4, we obtain that Y is a torsion class.

Let Y_1 be a torsion class with $Y_1 \supseteq X$. Then we have

$$Y_1 = w(s_0(\mathrm{hom}(Y_1))) \supseteq w(s_0(\mathrm{hom}(X))) = Y.$$

□

In view of Theorem 4.5, we say that Y is a torsion class generated by the class X.

According to Proposition 4.1, the collection $\mathcal{T}_g$ is a complete lattice. The lattice operations in $\mathcal{T}_g$ will be denoted by the symbols $\wedge$ and $\vee$. Similarly, the collection $\mathcal{R}_g$ is a complete lattice;this has been investigated in [12]. We denote now the lattice operations in $\mathcal{R}_g$ by $\wedge^r$ and $\vee^r$.

It is obvious that $\wedge$ coincides with the class-theoretical operation $\cap$ on the class $\mathcal{T}_g$. Analogously, $\wedge^r$ coincides with $\cap$ on $\mathcal{R}_g$.

Lemma 4.6. *Let $\emptyset \neq \{X_i\}_{i\in I} \subseteq \mathcal{T}_g$. Then*

$$\bigvee_{i\in I} X_i = w\Big(\bigcup_{i\in I} X_i\Big).$$

Proof. We put $\bigcup_{i\in I} X_i = Y$. Since $\mathrm{hom}(X_i) = X_i$ and $s_0(X_i) = X_i$ for each $i \in I$ we obtain $\mathrm{hom}(Y) = Y$ and $s_0(Y) = Y$. Thus $w(Y) = w(s_0(\mathrm{hom}\, Y))$. Then according to Theorem 4.5, we have $w(Y) \in \mathcal{T}_g$.

Let $i \in I$. Then $X_i \subseteq Y$ and hence $X_i \leqq w(Y)$ in $\mathcal{T}_g$. Further, let $Z \in \mathcal{T}_g$ and assume that $X_i \leqq Z$ for each $i \in I$. Thus $Y \subseteq Z$ and so $w(Y) \subseteq w(Z) = Z$. We have verified that $w(Y)$ is the least upper bound of $\{X_i\}_{i\in I}$ in $\mathcal{T}_g$. □

Applying an analogous argument (without dealing with $\mathrm{hom}(X_i)$), we obtain

Lemma 4.7. *Let $\emptyset \neq \{X_i\}_{i\in I} \subseteq \mathcal{R}_g$. Then*

$$\overset{r}{\bigvee_{i\in I}} X_i = w\Big(\bigcup_{i\in I} X_i\Big).$$

In view of the definitions of of $\mathcal{R}_g$ and $\mathcal{T}_g$, the relation $\mathcal{T}_g \subseteq \mathcal{R}_g$ is valid. From Lemma 4.6 and Lemma 4.7 we get

Lemma 4.8. *Let $\emptyset \neq \{X_i\}_{i\in I} \subseteq \mathcal{T}_g$. Then*

$$\bigwedge_{i\in I} X_i = \overset{r}{\bigwedge_{i\in I}} X_i, \qquad \bigvee_{i\in I} X_i = \overset{r}{\bigvee_{i\in I}} X_i.$$

Proposition 4.9. (Cf. [12].) *Let $\{X_i\}_{i\in I} \subseteq \mathcal{R}_g$ and $Y \in \mathcal{R}_g$. Then*

$$Y \wedge \bigvee_{i\in I} X_i = \bigvee_{i\in I} (Y \wedge X_i). \tag{1}$$

Proposition 4.10. *Let $\{X_i\}_{i\in I} \subseteq \mathcal{T}_g$ and $Y \in \mathcal{T}_g$. Then the relation* (1) *is valid.*

In view of Proposition 4.1 and Proposition 4.10, we have:

Theorem 4.11. *The partially ordered collection $\mathcal{T}_g$ is a Brouwerian lattice.*

5. Torsion classes generated by radical classes

In the present section we deal with the construction of a torsion class generated by a radical class. As application, we prove a result related to Theorem 4.5.

Lemma 5.1. *Let X be a radical class of generalized Boolean algebras. Put $\hom X = Y$. Then $s_0(Y) = Y$.*

Proof. Let $B \in s_0(Y)$. Thus there exists $B_1 \in Y$ such that B is an ideal of B_1. Further, there exists $B_2 \in X$ such that B_1 is a homomorphic image of B_2. Without loss of generality we can assume that there exists an ideal A of B_2 with $B_1 = B_2/A$.

We denote by by $\overline{x}$ the coset of B_1 containing the element $x \in B_2$, and we put $B_3 = \{x \in B_2 : \overline{x} \in B\}$. Then B_3 is an ideal of B_2, thus $B_3 \in X$. Also, A is an ideal of B_3 and B_3/A is isomorphic to B. Therefore $B \in \hom X$. We have verified that $s_0(Y) \subseteq Y$. Since $Y \subseteq s_0(Y)$, we get the relation $s_0(Y) = Y$. □

Let $\{B'_i\}_{i\in I}$ be a system of generalized Boolean algebras and for each $i \in I$, let A_i be an ideal of B'_i; put $B'_i/A_i = B_i$. We set

$$(w)\prod_{i\in I} B_i = B, \tag{1}$$

$$(w)\prod_{i\in I} B'_i = B'. \tag{2}$$

Then $A = (w)\prod\limits_{i\in I} A_i$ is an ideal of B'.

Let $b' \in B'$, $b' > 0$. There exist distinct indices $i(1), i(2), \dots, i(n) \in I$ and uniquely determined elements $0 < x'_1 \in B'_{i(1)}, \dots, 0 < x'_n \in B'_{i(n)}$ such that

$$b' = x'_1 \vee x'_2 \vee \dots \vee x'_n. \tag{3}$$

For $i \in \{1, 2, \dots, n\}$ let x_i be the congruence class in B'_i/A_i containing the element x'_i; we put

$$\varphi(b') = x_1 \vee x_2 \vee \dots \vee x_n.$$

From the relations (1), (2) and (3) we obtain that the following assertions is valid.

Lemma 5.2. *Under the notation as abve, φ is a homomorphism of the generalized Boolean algebra B' onto B and the kernel of this homomorphism is equal to A.*

Lemma 5.3. *Let X and Y be as in* Lemma 5.1. *Then $w(Y) = Y$.*

Proof. Let $\{B_i\}_{i\in I} \subseteq Y$ and $B = (w)\prod_{i\in I} B_i$. Let $i \in I$. We have $B_i \in \operatorname{hom}(X)$, hence there exists $B'_i \in X$ such that B_i is a homomorphic image of B'_i. Without loss of generality, we can assume that there is an ideal A_i of B'_i and that $B_i = B'_i/A_i$. We put $(w)\prod_{i\in I} B'_i = B'$ and $A = (w)\prod_{i\in I} A_i$. Since X is a radical class, by Lemma 3.1 we get $B' \in X$. According to Lemma 5.2, B is a homomorphic image of B', thus $B \in \operatorname{hom}(X) = Y$.

We have verified that $w(Y) \subseteq Y$. Since $Y \subseteq w(Y)$, we have $Y = w(Y)$. ☐

Theorem 5.4. *Let X be a radical class of generalized Boolean algebras. Then* $\operatorname{hom} X$ *is the torsion class generated by X.*

Proof. According to Lemma 5.1 and Lemma 5.3, we get

$$w(s_0(\operatorname{hom}(X))) = w(\operatorname{hom}(X)) = \operatorname{hom}(X).$$

Then Theorem 4.5 finishes the proof. ☐

According to Theorem 4.5, Theorem 5.4 and in view of the fact that $w(s_0(Z))$ is a radical class for each $\emptyset \neq Z \subseteq \mathcal{B}_0$, we obtain

Corollary 5.5. *Let $\emptyset \neq Z \subseteq \mathcal{B}_0$. Then*

$$\operatorname{hom}(w(s_0(Z))) = w(s_0(\operatorname{hom}(Z))).$$

6. The class $\mathcal{B}_0^*$

Above we applied the usual convention that an algebraic structure and its underlying set were denoted by the same symbol. This convention cannot be used if two different algebraic structures have the same underlying set; now, we will deal with such situation. Therefore we need a slight modification of the previous notation.

Considering a generalized Boolean algebra, we will write it in the form $\mathbf{B} = (B; \wedge, \vee, 0)$, where B is the underlying set of $\mathbf{B}$.

Definition 6.1. Let $\mathbf{A} = (A; \wedge, \vee, 0, g)$ be an algebra of type $(2, 2, 0, 2)$ such that

(i) $(A; \wedge, \vee, 0)$ is a distributive lattice with the least element 0;

(ii) for each $x, y \in A$ we have

$$x \wedge g(x, y) = 0, \qquad x \vee g(x, y) = x \vee y. \tag{1}$$

Then $\mathbf{A}$ is said to be a b^*-algebra.

Lemma 6.2. *Let* $\mathbf{A}$ *be a* b^**-algebra. Put* $\mathbf{B} = (A; \wedge, \vee, 0)$*. Then* $\mathbf{B}$ *is a generalized Boolean algebra.*

Proof. Let $x, z \in A, x \leqq z$. Then

$$x \wedge g(x, z) = 0, \quad x \vee g(x, z) = z.$$

This yields that the interval $[0, z]$ of $\mathbf{B}$ is a complemented distributive lattice. Therefore $\mathbf{B}$ is a generalized Boolean algebra. □

Lemma 6.3. *Let* $\mathbf{B} = (B; \wedge, \vee, 0)$ *be a generalized Boolean algebra. For* $x, y \in B$ *we denote by* $g(x, y)$ *the relative complement of* x *in the interval* $[0, x \vee y]$ *of* $\mathbf{B}$*. Put* $\mathbf{A} = (B; \wedge, \vee, 0, g)$*. Then* $\mathbf{A}$ *is a* b^**-algebra.*

Proof. The condition (i) of Definition 6.1 obviously holds; further, in view of the definition of g, the relations (1) are valid. □

If $\mathbf{A}$ and $\mathbf{B}$ are as in 6.3, then we put $\mathbf{A} = \mathbf{B}^*$. The structures $\mathbf{B}$ and $\mathbf{B}^*$ have the same underlying set.

Let $\mathcal{B}_0^*$ be the class of all b^*-algebras. The mapping $\psi\colon \mathcal{B}_0 \to \mathcal{B}_0^*$ defined by $\psi(\mathbf{B}) = \mathbf{B}^*$ is a bijection (cf. Lemma 6.2 and Lemma 6.3). In fact, the algebraic structures $\mathbf{B}$ and $\mathbf{B}^*$ are only "formally" different.

Let us remark that for abstract algebras, similar situations have been studied in detail by Marczewski and his successors; here, the terms 'weak isomorphism' and 'weak automorphism' were applied (cf. Marczewski [18], [19], Goetz [7] and Traczyk [23]). Cf. also the author [9] for the particular case of distributive lattices.

Let $\mathbf{B}^* = (B; \wedge, \vee, 0, g)$ be a b^*-algebra. The notion of subalgebra of $\mathbf{B}^*$ is defined in the usual way.

Further, let $\mathbf{B} = (B; \wedge, \vee, 0)$ be a generalized Boolean algebra and let B_1 be a nonempty subset of B containing the element 0 such that B_1 is closed with respect to the operations $\wedge$ and $\vee$. Assume that the lattice $\mathbf{B}_1 = (B; \wedge, \vee, 0)$ is a generalized Boolean algebra. Then we say that $\mathbf{B}_1$ is a subalgebra of the generalized Boolean algebra $\mathbf{B}$. If $\mathbf{B}' \in c(\mathbf{B})$, then, obviously, $\mathbf{B}'$ is a subalgebra of $\mathbf{B}$.

Due to Lemma 6.2 and Lemma 6.3 we obtain the following results.

Lemma 6.4.1. *Let* $\mathbf{B} = (B; \wedge, \vee, 0)$ *be a generalized Boolean algebra and let* B_1 *be a nonempty subset of* B*. Then the following conditions are equivalent:*

(i) $\mathbf{B}_1 = (B_1, \wedge, \vee, 0)$ *is a subalgebra of the generalized Boolean algebra* $\mathbf{B}$*;*

(ii) $\mathbf{B}_1^* = (B_1, \wedge\vee, 0, g)$ *is a subalgebra of the* b^**-algebra* $B^* = (B, \wedge, \vee, 0, g)$*.*

Lemma 6.4.2. *Let* $\mathbf{B}$ *and* B_1 *be as in* Lemma 6.4.1*. Further, assume that* B_1 *is a convex subset of* B*. Then the following conditions are equivalent:*

(i) $\mathbf{B}_1 = (B_1, \wedge, \vee, 0)$ *is a convex subalgebra of the generalized Boolean algebra* $\mathbf{B}$*;*

(ii) $\mathbf{B}_1^* = (B_1, \wedge, \vee, 0, g)$ *is a convex subalgebra of the* b^**-algebra* $B^* = (B, \wedge, \vee, 0, g)$*.*

Lemma 6.4.3. *Let* $\mathbf{B} \in \mathcal{B}_0$ *and let* $(\mathbf{B}_i)_{i\in I}$ *be an indexed system of elements of* $\mathcal{B}_0$*. Then the following conditions are equivalent:*

(i) $\mathbf{B} \simeq \prod\limits_{i\in I} \mathbf{B}_i$*;*

(ii) $\mathbf{B}^* \simeq \prod\limits_{i\in I} \mathbf{B}_i^*$*.*

7. Varieties of b^*-algebras and of generalized Boolean algebras

Using identities involving polynomials constructed by the operations $\wedge$, $\vee$, 0 and g we can define the notion of variety of b^*-algebras in the usual way. In more detail, we proceed as follows. Let S be a nonempty system of pairs (f_1, f_2) where f_1 and f_2 are polynomials of mentioned type. Let $C(S)$ be the class of all b^*-algebras satisfying all identities $f_1 = f_2$ where $(f_1, f_2) \in S$. Then $C(S)$ is said to be a variety of b^*-algebras.

If C_1 is a subclass of $\mathcal{B}_0$, then we put

$$C_1^* = \{\mathbf{B}^* : \mathbf{B} \in C_1\}.$$

Definition 7.1. Let C_1 be a subclass of $\mathcal{B}_0$. We say that C_1 is a variety of generalized Boolean algebras if the class C_1^* is a variety of b^*-algebras.

Theorem 7.2. *Each variety of generalized Boolean algebras is a radical class.*

Proof. Let X be a variety of generalized Boolean algebras. In view of Proposition 3.4 we have to verify that $w(s_0(X)) = X$.

a) Let $B_1 \in s_0(X)$. Hence there exists $B \in X$ with $B_1 \in c(B)$. Thus $B^* \in X^*$; further, according to Lemma 6.4.1, we obtain that B_1^* is a subalgebra of B^*. The fact that each variety of algebras is closed with respect to subalgebras yields that B_1^* belongs to X^* and so B_1 belongs to X. Therefore $s_0(X) = X$.

b) Now, assume that B is an element of $w(X)$. Thus it can be expressed in the form

$$B = (w) \prod_{i \in I} B_i,$$

where all B_i belong to X. According to Lemma 6.4.3 we have

$$B^* = (w) \prod_{i \in I} B_i^*$$

and all B_i^* belong to X^*.

Since each variety of algebras is closed with respect to direct products, we get that $\prod_{i \in I} B_i^*$ belongs to X^*. Further, $(w) \prod_{i \in I} B_i^*$ is a subalgebra of $\prod_{i \in I} B_i^*$; thus B^* belongs to X^* as well. Hence $B \in X$ and so $w(X) = X$.

Summarizing, in view of a) we get $w(x_0(X)) = X$, as desired. □

Let us now apply homomorphisms of generalized Boolean algebras for describing varieties.

Assume that $\mathbf{B}_1 = (B_1; \wedge, \vee, 0_1)$ and $\mathbf{B}_2 = (B_2; \wedge, \vee, 0_2)$ are generalized Boolean algebras and that φ is a mapping of B_1 into B_2. It seems to be natural to define φ to be a homomorphism of $\mathbf{B}_1$ into $\mathbf{B}_2$ if the following conditions are satisfied:

(1) $\varphi_1(0_1) = 0_2$ and $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y), \varphi(x \vee y) = \varphi(x) \vee \varphi(y)$ for each $x, y \in B_1$;

(2) the set $\varphi(B_1)$ is an underlying set of a subalgebra of $\mathbf{B}_2$.

Lemma 7.3. *Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be as above and let φ be a mapping of the set B_1 into B_2 such that the condition* (1) *is satisfied. Suppose that $x, z \in B_1$ and that x_1 is a relative complement of x in the interval $[0_1, z]$ of $\mathbf{B}_1$. Then $\varphi(x_1)$ is a relative complement of $\varphi(x)$ in the interval $[0_2, \varphi(z)]$ of $\mathbf{B}_2$.*

Proof. We have

$$x \wedge x_1 = 0_1, \qquad x \vee x_1 = z.$$

Thus in view of (1) we obtain $\varphi(0_1) = 0_2$ and

$$\varphi(x) \wedge \varphi(x_1) = 0_2, \qquad \varphi(x) \vee \varphi(x_1) = \varphi(z).$$

□

Corollary 7.4. *Let* $\mathbf{B}_1, \mathbf{B}_2$ *and* φ *be as in* Lemma 7.3. *Then* φ *is a homomorphism of* $\mathbf{B}_1$ *into* $\mathbf{B}_2$ *if and only if the condition* (1) *is satisfied.*

Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be as above. Consider the b^*-algebras $\mathbf{B}_1^*$ and $\mathbf{B}_2^*$. The homomorphism of $\mathbf{B}_1^*$ into $\mathbf{B}_2^*$ is defined in the obvious way. The b^*-algebras $\mathbf{B}_1^*$ and $\mathbf{B}_2^*$ are written in the form

$$\mathbf{B}_1^* = (B_1; \wedge, \vee, 0_1, g), \quad \mathbf{B}_2^* = (B_2; \wedge, \vee, 0_2, g).$$

Lemma 7.5. *Let* φ *be a homomorphism of* $\mathbf{B}_1$ *into* $\mathbf{B}_2$*, where* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *are as above. Then*

$$\varphi(g(x, y) = g(\varphi(x), \varphi(y)) \qquad \text{for each} \quad x, y \in B_1. \tag{3}$$

Proof. This is a consequence of Lemma 7.3. □

Lemma 7.6. *Let* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *be as above. Assume that* φ *is a mapping of* B_1 *into* B_2*. Then the following conditions are equivalent:*

(i) φ *is a homomorphism of* $\mathbf{B}_1$ *into* $\mathbf{B}_2$*.*

(ii) φ *is a homomorphism of* $\mathbf{B}_1^*$ *into* $\mathbf{B}_2^*$*.*

Proof. The implication (ii) $\Longrightarrow$ (i) is obviously valid. Assume that the condition (ii) holds. In view of (1) and according to Lemma 7.5 we conclude that the condition (ii) is satisfied. □

Let C be a nonempty subclass of $\mathcal{B}_0$. As usual, we denote by HC, SC and PC the class of all homomorphic images, all subalgebras or all direct products of elements of C, respectively.

As above, let $C^* = \{\mathbf{B}^* : \mathbf{B} \in C\}$. The classes HC^*, SC^* and PC^* are defined analogously as in the case of the class C.

Lemma 7.7. *Let* $\emptyset \neq C \subseteq \mathcal{B}_0$*. Then*

$$HC^* = (HC)^*, \quad SC^* = (SC)^*, \quad PC^* = (PC)^*.$$

Proof. The relation $HC^* = (HC)^*$ is a consequence of Lemma 7.6. In view of Lemma 6.4.1, the relation $SC^* = (SC)^*$ is valid. According to Lemma 6.4.3, $PC^* = (PC)^*$. □

Since in the definition ob b^*-algebra there are used only operations and no relations, we get that C^* is a variety if and only if $C^* = HSPC^*$. Above we have defined C to be a variety of generalized Boolean algebras if the class C^* is a variety of b^*-algebras. Thus in view of Lemma 7.7 we obtain

Proposition 7.8. *Let* $\emptyset \neq C \subseteq \mathcal{B}_0$*. Then* C *is a variety of generalized Boolean algebras if and only if* $HSPC = C$*.*

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