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DYNAMIC EFFECT ALGEBRAS

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ABSTRACT. We introduce the so-called tense operators in lattice effect algebras. Tense operators express the quantifiers "it is always going to be the case that" and "it has always been the case that" and hence enable us to express the dimension of time in the logic of quantum mechanics. We present an axiomatization of these tense operators and prove that every lattice effect algebra whose underlying lattice is complete can be equipped with tense operators. Such an effect algebra is called dynamic since it reflects changes of quantum events from past to future.

Effect algebras were introduced by D. J. Foulis and M. K. Bennett [4] as an abstraction of the Hilbert space effects which play an important role in the logic of quantum mechanics. Detailed source of the properties of effect algebras together with description of their alter ego called D-posets is summarized in [3]. However, this description does not corporate the dimension of time. This means that effect algebras can serve to describe the states of effects in a given time but they cannot reveal what these effects expressed in the past or what they will reveal in the next time. A similar problem was already solved for the classical propositional calculus by introducing the so-called tense operators in Boolean algebras, see [1]. For MV-algebras and for Łukasiewicz-Moisil algebras, the tense operators were introduced by D. Diaconescu and G. Georgescu in [2]. It is our aim to introduce such tense operators for lattice effect algebras and hence set up an axiomatic tool which enable us to consider quantum structures dynamically, i.e. to capture also time dimension in our investigations.

At first, let us recall the concept of effect algebra (see [4] or [3]). By an *effect algebra* is meant a system $\mathcal{E} = (E; +, 0, 1)$ where 0 and 1 are distinguished elements of E, $0 \neq 1$, and $+$ is a partial binary operation on E satisfying the following axioms for $p, q, r \in E$:

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(E1) if $p + q$ is defined then $q + p$ is defined and $p + q = q + p$

(E2) if $q + r$ is defined and $p + (q + r)$ is defined then $p + q$ and $(p + q) + r$ are defined and $p + (q + r) = (p + q) + r$

(E3) for each $p \in E$ there exists a unique $p' \in E$ such that $p + p' = 1$; p' is called a *supplement* of p

(E4) if $p + 1$ is defined then $p = 0$.

Having an effect algebra $\mathcal{E} = (E; +, 0, 1)$, we can introduce the *induced order* $\leq$ on E as follows

$$a \leq b \iff (\exists c \in E)(c + a = b)$$

(see e.g. [3]). An effect algebra $\mathcal{E}$ is called a *lattice effect algebra* if $(E; \leq)$ is a lattice (with respect to the induced order). Evidently, $0 \leq a \leq 1$ for each $a \in E$ and $a \leq b$ implies $b' \leq a'$ for the supplements. Of course, $1 = 0'$ and $x'' = x$. If the underlying lattice $(E; \leq)$ is complete, we will call $\mathcal{E}$ a *complete lattice effect algebra*. It is worth noticing that $a + b$ exists in an effect algebra $\mathcal{E}$ if and only if $a \leq b'$ (or equivalently, $b \leq a'$). This condition is usually expressed by the notation $a \perp b$ (we say that a, b are orthogonal).

The second concept which will be used are so-called tense operators. They are in certain sense quantifiers which quantify the time dimension of the logic under consideration. The semantical interpretation of these *tense operators* G and H is as follows. Consider a pair $(T, \leq)$ where T is a non-void set and $\leq$ is a partial order on T. Let $x \in T$ and $f(x)$ be a formula of a given logical calculus. We say that $G(f(t))$ *is valid* if for any $s \geq t$ the formula $f(s)$ is valid. Analogously, $H(f(t))$ is valid if $f(s)$ is valid for each $s \leq t$. Thus the unary operators G and H constitute an algebraic counterpart of the tense operations "it is always going to be the case that" and "it has always been the case that", respectively.

These tense operators were firstly introduced as operators on Boolean algebras (see [1]) by the axioms

(B1) $G(1) = 1, \quad H(1) = 1$

(B2) $G(x \wedge y) = G(x) \wedge G(y), \quad H(x \wedge y) = H(x) \wedge H(y)$

(B3) $x \leq GP(x), \quad x \leq HF(x)$,

where $F(x) = \neg G(\neg x)$ and $P(x) = \neg H(\neg x)$.

For MV-algebras, two more axioms were inserted in [2], namely

$$G(x \to y) \leq G(x) \to G(y) \quad \text{and} \quad H(x \to y) \leq H(x) \to H(y)$$

and

$$G(x) \oplus G(y) \leq G(x \oplus y) \quad \text{and} \quad H(x) \oplus H(y) \leq H(x \oplus y).$$

Since every lattice effect algebra is composed by means of blocks which are MV-algebras as it was shown by Z. Riečanová [5], this motivates us to apply a similar axiomatization of tense operators on effect algebras. However, the binary operation $+$ in effect algebras is only partial and hence we must consider its definability. Moreover, the operation of implication does not play the role in effect algebras but a unary operation of supplement is important. Hence, we can state in the following:

Definition. By a *dynamic effect algebra* is meant a triple $\mathcal{D} = (\mathcal{E}; G, H)$ such that $\mathcal{E} = (E; +, 0, 1)$ is a lattice effect algebra and G, H are mappings of E into itself satisfying

(T1) $G(1) = 1$ and $H(1) = 1$

(T2) $G(x \wedge y) = G(x) \wedge G(y), \quad H(x \wedge y) = H(x) \wedge H(y)$

(T3) if $x + y$ exists then $G(x) + G(y)$ exists and $H(x) + H(y)$ exists and $G(x) + G(y) \leq G(x + y), \quad H(x) + H(y) \leq H(x + y)$

(T4) $G(x') \leq G(x)', \quad H(x') \leq H(x)'$

(T5) $x \leq GP(x), \ x \leq HF(x)$, where $P(x) = H(x')'$ and $F(x) = G(x')'$.

Just defined G and H will be called *tense operators* of a dynamic effect algebra $\mathcal{D}$.

We can derive some basic properties.

Theorem 1. *Let $\mathcal{D} = (\mathcal{E}; G, H)$ be a dynamic effect algebra. Then*

(a) $G(0) = 0 = H(0)$

(b) $x \leq y$ *implies* $G(x) \leq G(y)$ *and* $H(x) \leq H(y)$

(c) $PG(x) \leq x$ *and* $FH(x) \leq x$

(d) $x \leq y$ *implies* $P(x) \leq P(y)$ *and* $F(x) \leq F(y)$.

Proof.

(a) By (T1) and (T4) we infer $G(0) = G(1') \leq G(1)' = 1' = 0$. Since $0 \leq x$ for each $x \in E$, we conclude $G(0) = 0$. Analogously can be shown $H(0) = 0$.

(b) Assume $x \leq y$. Then there exists $z \in E$ with $y = x + z$ and hence $G(y) = G(x + z) \geq G(x) + G(z) \geq G(x)$. Analogously for the operator H.

(c) By (T5) we have $x' \leq HF(x') = H(G(x)')$ which yields immediately $PG(x) = H(G(x)')' \leq x'' = x$. Similarly can be checked $FH(x) \leq x$.

(d) If $x \leq y$ then $y' \leq x'$ and, by (b), $H(y') \leq H(x')$. Thus $P(x) = H(x')' \leq H(y')' = P(y)$ and, similarly, also $F(x) \leq F(y)$. □

Example 1. Let $\mathcal{E} = (\{0, a, b, c, a', b', c', 1\}; +, 0, 1)$ be an effect algebra where the operation $+$ is determined by Table 1.

Then $\mathcal{E}$ is a complete lattice effect algebra whose underlying lattice is depicted in Figure 1. It contains just two blocks, namely $\{0, a, a', b, b', 1\}$ and $\{0, c, c', b, b', 1\}$ which are MV-algebras.

TABLE 1. Operation +

$+$	0	a	b	c	a'	b'	c'	1
0	0	a	b	c	a'	b'	c'	1
a	a	b'	a'	$-$	1	$-$	$-$	$-$
b	b	a'	$-$	c'	$-$	1	$-$	$-$
c	c	$-$	c'	b'	$-$	$-$	1	$-$
a'	a'	1	$-$	$-$	$-$	$-$	$-$	$-$
b'	b'	$-$	1	$-$	$-$	$-$	$-$	$-$
c'	c'	$-$	$-$	1	$-$	$-$	$-$	$-$
1	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$

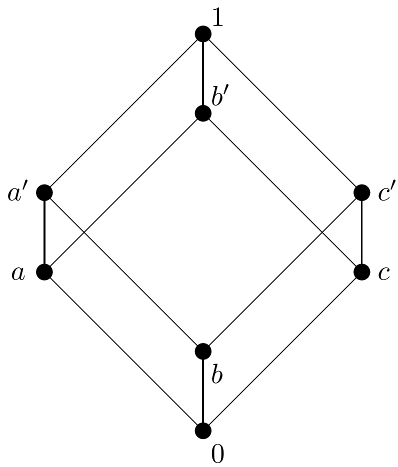

FIGURE 1. Underlying lattice of complete lattice effect algebra of Example 1

TABLE 2. operators G and H

x	0	a	b	c	a'	b'	c'	1
$G(x)$	0	0	0	0	0	b'	0	1
$H(x)$	0	0	b	0	b	0	b	1

Further, let us define on $\mathcal{E}$ the operators G and H by Table 2.

One can check that $(\mathcal{E}; G, H)$ is a dynamic effect algebra with two different non-trivial unary operators G and H.

In what follows, we are going to show how to introduce the tense operators on every complete lattice effect algebra in a "natural way". For this we firstly prove the following statement.

Theorem 2. *Let $(\mathcal{E}; G, H)$ be a dynamic effect algebra. If $\bigwedge\{x_i :\ i \in I\}$ exists then also*

$$\bigwedge\bigl\{G(x_i) :\ i \in I\bigr\} \qquad \textit{and} \qquad \bigwedge\bigl\{H(x_i) :\ i \in I\bigr\}$$

exist and

$$G\Bigl(\bigwedge\{x_i :\ i \in I\}\Bigr) = \bigwedge\bigl\{G(x_i) :\ i \in I\bigr\}$$

and

$$H\Bigl(\bigwedge\{x_i :\ i \in I\}\Bigr) = \bigwedge\bigl\{H(x_i) :\ i \in I\bigr\}.$$

Proof. Assume that $x_i \in E$ for $i \in I$ and $\bigwedge\{x_i :\ i \in I\}$ exists. Since $\bigwedge\{x_i :\ i \in I\} \leq x_i$, we have by (b) of Theorem 1 that $G\bigl(\bigwedge\{x_i :\ i \in I\}\bigr) \leq G(x_i)$ for each $i \in I$. Thus $G\bigl(\bigwedge\{x_i :\ i \in I\}\bigr)$ is a lower bound of the set $\bigl\{G(x_i) :\ i \in I\bigr\}$. Assume now that y is a lower bound of the set $\{G(x_i) :\ i \in I\}$. By (d) and (e) of Theorem 1 we have

$$P(y) \leq PG(x_i) \leq x_i$$

thus $P(y) \leq \bigwedge\{x_i :\ i \in I\}$. Hence,

$$y \leq GP(y) \leq G\Bigl(\bigwedge\{x_i :\ i \in I\}\Bigr).$$

This proves that $\bigwedge\bigl\{G(x_i) :\ i \in I\bigr\}$ exists and is equal to $G\bigl(\bigwedge\{x_i :\ i \in I\}\bigr)$.

Analogously it can be proved for the operator H. □

Lemma 1. *Let $\mathcal{E} = (E; +, 0, 1)$ be a complete lattice effect algebra. Let $a_i, b_i, c_i \in E$ for $i \in I$ and assume $a_i \perp b_i$ for all $i \in I$. Then*

(1) $\bigwedge\{a_i :\ i \in I\} + \bigwedge\{b_i :\ i \in I\}$ *exists and*

$$\bigwedge\{a_i :\ i \in I\} + \bigwedge\{b_i :\ i \in I\} \leq \bigwedge\{a_i + b_i :\ i \in I\}$$

(2) $\bigwedge\{c_i' :\ i \in I\} \leq \bigl(\bigwedge\{c_i :\ i \in I\}\bigr)'$.

Proof.

(1) Let $a_i \perp b_i$ for each $i \in I$, i.e. $a_i + b_i$ exists. Then $a_i \leq b_i'$ and hence $\bigwedge\{a_i :\ i \in I\} \leq a_i \leq b_i'$ thus $\bigwedge\{a_i :\ i \in I\} + b_i$ exists for each $i \in I$. Hence $b_i \leq \bigl(\bigwedge\{a_i :\ i \in I\}\bigr)'$, thus also

$$\bigwedge\{b_i :\ i \in I\} \leq b_i \leq \Bigl(\bigwedge\{a_i :\ i \in I\}\Bigr)',$$

i.e. $\bigwedge\{a_i :\ i \in I\} + \bigwedge\{b_i :\ i \in I\}$ exists. Since

$$\bigwedge\{a_i :\ i \in I\} + \bigwedge\{b_i :\ i \in I\} \leq a_i + b_i$$

for each $i \in I$, we conclude

$$\bigwedge\{a_i :\ i \in I\} + \bigwedge\{b_i :\ i \in I\} \leq \bigwedge\{a_i + b_i :\ i \in I\}.$$

(2) Evidently, $\bigwedge\{c_i : i \in I\} \le c_i$ thus $\big(\bigwedge\{c_i : i \in I\}\big)' \ge c_i'$ for each $i \in I$, whence $\big(\bigwedge\{c_i : i \in I\}\big)' \ge \bigwedge\{c_i' : i \in I\}$. □

By a *frame* (see e.g. [2]) is meant a couple (T, R) where T is a non-void set and R is a binary relation on T. For our sake, we will assume that R is reflexive. The set T is considered to be a time scale and a relation R expresses a relationship "to be before" or "to be after". Having an effect algebra $\mathcal{E} = (E; +, 0, 1)$ and a non-void set T, we can produce the direct power $\mathcal{E}^T = (E^T; +, o, j)$ where the operation $+$ is defined and evaluated on $p, q \in E^T$ componentwise, i.e. $p + q$ is defined if $p(t) + q(t)$ is defined for each $t \in T$ and then $(p + q)(t) = p(t) + q(t)$. Moreover, o, j are such the elements of E^T that $o(t) = 0$ and $j(t) = 1$ for all $t \in T$.

We can state our main result.

Theorem 3. *Let $\mathcal{E} = (E; +, 0, 1)$ be a complete lattice effect algebra and let (T, R) be a frame. Define mappings G, H of E^T into itself as follows*

$$G(p)(x) = \bigwedge\{p(y) : xR\,y\} \qquad \textit{and} \qquad H(p)(x) = \bigwedge\{p(y) : yR\,x\}.$$

Then G, H are tense operators on $\mathcal{E}^T$, i.e. $\mathcal{D} = (\mathcal{E}^T; G, H)$ is a dynamic effect algebra.

Proof. Trivially we can verify $G(j) = j$ and $H(j) = j$ due to the fact that $j(t) = 1$ for each $t \in T$ thus (T1) holds. Prove (T3). Assume that $p, q \in E^T$ and $p + q$ exists. Hence, $p(y) + q(y)$ exists for each $y \in T$. By Lemma 1 also $\bigwedge\{p(y) : xR\,y\} + \bigwedge\{q(y) : xR\,y\}$ exists and $G(p)(x) + G(q)(x) = \bigwedge\{p(y) : xR\,y\} + \bigwedge\{q(y) : xR\,y\} \le \bigwedge\{p(y) + q(y) : xR\,y\} = G(p + q)(x)$ for each $x \in T$. Thus $G(p) + G(q) \le G(p + q)$. Analogously we can show $H(p) + H(q) \le H(p + q)$.

By Lemma 1, we can see (T4) immediately and (T2) follows by associativity of infima in complete lattices. It remains to prove (T5). Since

$$(H(p')')(x) = \Big(\bigwedge\{p'(y) : yR\,x\}\Big)' = \bigvee\{p(y) : yR\,x\}$$

due to De Morgan laws, we obtain

$$GP(p)(x) = G\big(H(p')'\big)(x) = \bigwedge_t\Big\{\bigvee_y\{p(y) : yR\,t\} : xR\,t\Big\}.$$

Since every member of the infima is greater or equal to $p(x)$, we conclude $GP(p)(x) \ge p(x)$ for each $x \in T$, i.e. $p \le GP(p)$. Analogously can be shown $p \le HF(p)$. □

Remark. If the relation R on a non-void set T is a partial order, i.e. our frame is $(T, \le)$, then $x \le y$ expresses the fact that y "follows" x and $y \le x$ means y "is before" x. Then the operator H as defined in Theorem 3 can be interpreted as "a history" of an element $p \in E^T$ and $G(p)$ is "a future" of p. More precisely,

$H(p)$ says that p was true in past with at least the same degree as p in present and $G(p)$ says that p will be true in future with at least the same degree as it is now.

Having a lattice effect algebra $\mathcal{E} = (E; +, 0, 1)$ and a frame (T, R), let G_t, H_t be tense operators on $\mathcal{E}$ for each $t \in T$. It is evident that the functions G, H defined on E^T componentwise, i.e. $G(p)(t) = G_t(p(t))$ and $H(p)(t) = H_t(p(t))$ are tense operators on $\mathcal{E}^T$. The aim of the following example is to show that this is not the only possibility how to define tense operators on $\mathcal{E}^T$.

Example 2. Let $\mathcal{E} = (E; +, 0, 1)$ be an effect algebra where $E = \{0, a, b, 1\}$ and the operation $+$ is determined by Table 3.

TABLE 3. operation $+$

$+$	0	a	b	1
0	0	a	b	1
a	a	1	$-$	$-$
b	b	$-$	1	$-$
1	1	$-$	$-$	$-$

Then $\mathcal{E}$ is a complete lattice effect algebra whose underlying lattice is depicted in Figure 2.

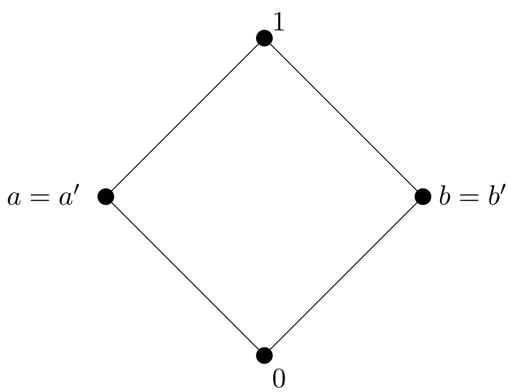

FIGURE 2. underlying lattice of complete lattice effect algebra of Example 2

One can easily check that there exists seven couples of tense operators on this effect algebra $\mathcal{E}$ which are indicated in Table 4 (since always $G(0) = 0 = H(0)$ and $G(1) = 1 = H(1)$, we list only outcomes on a and b).

However, if the frame is $(\{1, 2, 3\}, \leq)$ where $\leq$ is a natural order on integers, then E^T has $4^3 = 64$ elements and we can compute the tense operators G and H as defined in Theorem 3 (see Table 5).

TABLE 4. tense operators

x	a	b	a	b	a	b	a	b	a	b	a	b	a	b
$G(x)$	0	0	0	a	0	b	a	0	a	b	b	0	b	a
$H(x)$	0	0	b	0	0	b	a	0	a	b	0	a	b	a

One can easily check that e.g.

$$G((b,1,a)) = (0,a,a) \quad \text{and} \quad H((b,1,a)) = (b,b,0)$$

but, when working with G_t, H_t on $\mathcal{E}$ componentwise, it have to be $G((b,1,a))(2) = 1$ and $H((b,1,a))(2) = 1$ since $(b,1,a)(2) = 1$ and $G_t(1) = H_t(1)$.

Among other things, it means that the tense operators G, H (as defined in Theorem 3) cannot be decomposed into direct factors.

Table 5. operators G and H

x	$G(x)$	$H(x)$		x	$G(x)$	$H(x)$
(000)	(000)	(000)		$(b00)$	(000)	$(b00)$
$(00a)$	$(00a)$	(000)		$(b0a)$	$(00a)$	$(b00)$
$(00b)$	$(00b)$	(000)		$(b0b)$	$(00b)$	$(b00)$
(001)	(001)	(000)		$(b01)$	(001)	$(b00)$
$(0a0)$	(000)	(000)		$(ba0)$	(000)	$(b00)$
$(0aa)$	$(0aa)$	(000)		(baa)	$(0aa)$	$(b00)$
$(0ab)$	$(00b)$	(000)		(bab)	$(00b)$	$(b00)$
$(0a1)$	$(0a1)$	(000)		$(ba1)$	$(0a1)$	$(b00)$
$(0b0)$	(000)	(000)		$(bb0)$	(000)	$(bb0)$
$(0ba)$	$(00a)$	(000)		(bba)	$(00a)$	$(bb0)$
$(0bb)$	$(0bb)$	(000)		(bbb)	(bbb)	(bbb)
$(0b1)$	$(0b1)$	(000)		$(bb1)$	$(bb1)$	(bbb)
(010)	(000)	(000)		$(b10)$	(000)	$(bb0)$
$(01a)$	$(0aa)$	(000)		$(b1a)$	$(0aa)$	$(bb0)$
$(01b)$	$(0bb)$	(000)		$(b1b)$	(bbb)	(bbb)
(011)	(011)	(000)		$(b11)$	$(b11)$	(bbb)
$(a00)$	(000)	$(a00)$		(100)	(000)	(100)
$(a0a)$	$(00a)$	$(a00)$		$(10a)$	$(00a)$	(100)
$(a0b)$	$(00b)$	$(a00)$		$(10b)$	$(00b)$	(100)
$(a01)$	(001)	$(a00)$		(101)	(001)	(100)
$(aa0)$	(000)	$(aa0)$		$(1a0)$	(000)	$(1a0)$
(aaa)	(aaa)	(aaa)		$(1aa)$	(aaa)	$(1aa)$
(aab)	$(00b)$	$(aa0)$		$(1ab)$	$(00b)$	$(1a0)$
$(aa1)$	$(aa1)$	(aaa)		$(1a1)$	$(aa1)$	$(1aa)$
$(ab0)$	(000)	$(a00)$		$(1b0)$	(000)	$(1b0)$
(aba)	$(00a)$	$(a00)$		$(1ba)$	$(00a)$	$(1b0)$
(abb)	$(0bb)$	$(a00)$		$(1bb)$	(bbb)	$(1bb)$
$(ab1)$	$(0b1)$	$(a00)$		$(1b1)$	$(bb1)$	$(1bb)$
$(a10)$	(000)	$(aa0)$		(110)	(000)	(110)
$(a1a)$	(aaa)	(aaa)		$(11a)$	(aaa)	$(11a)$
$(a1b)$	$(0bb)$	$(aa0)$		$(11b)$	(bbb)	$(11b)$
$(a11)$	$(a11)$	(aaa)		(111)	(111)	(111)

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