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SUBCONTINUITY

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ABSTRACT. We give interesting characterizations using subcontinuity. Let X, Y be topological spaces. We study subcontinuity of multifunctions from X to Y and its relations to local compactness, local total boundedness and upper semicontinuity. If Y is regular, then F is subcontinuous iff $\overline{F}$ is USCO. A uniform space Y is complete iff for every topological space X and for every net $\{F_a\}$, $F_a \subset X \times Y$, of multifunctions subcontinuous at $x \in X$, uniformly convergent to F, F is subcontinuous at x. A Tychonoff space Y is Čech-complete (resp. $G_{\mathfrak{m}}$-space) iff for every topological space X and every multifunction $F \subset X \times Y$ the set of points of subcontinuity of F is a G_δ-subset (resp. $G_{\mathfrak{m}}$-subset) of X.

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1. Introduction

The notion of subcontinuity was introduced by Fuller [8] in 1968. Subcontinuous functions are generalizations of functions with compact codomain and it turned out that this notion complements the closed graph property in an interesting way:

$f\colon X \to Y$ with Hausdorff Y is continuous, iff it is subcontinuous and has closed graph.

There is a very natural extension of subcontinuity to multifunctions, introduced by Hrycay [22][1]. There is a large number of publications dealing with subcontinuity of either functions or multifunctions. Subcontinuity is often treated as a supplementary property, like before mentioned "upgrade" of closed graph property

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[1]This is the first reference we were able to find. Probably Smithson [34] did it independently in 1975, his definition is cited more frequently.

into continuity. Hrycay [22] used it for investigation of connected multifunctions (multifunctions taking connected sets onto connected sets). Subcontinuity has found many applications; in [10] it is used to study closed maps, in [3] to study topological groups, in [26] to study extensions of continuous functions and in [15] to characterize minimal USCO maps. Some generalizations of subcontinuity can be found in [12], or [1] in fuzzy topological spaces.

In the first part of this paper some basic facts about subcontinuity of multifunctions are summarized and it is proved how subcontinuity is related to upper semicontinuity. (This is an analogy to the relation of subcontinuity of a function and its continuity, which was one of the motivations of definition of subcontinuity.) Basic references to this part are [25], [23], [34]. There are also summarized some relations among subcontinuity, local compactness and local total boundedness, which are generalizations of the work of Mimna and Wingler [29] and our previous work [31]. The notion of local compactness for functions was introduced in [29], but as it is stated there, this property was used before by Doboš [32] as local boundedness; the notion of local total boundedness (for functions with values in metric space) was introduced in [31].

In the next parts some new concepts of subcontinuity are introduced, in particular weak subcontinuity [9] and subcontinuity with respect to hyperspace topologies. Under certain conditions subcontinuity with respect to the upper Vietoris topology is equivalent to subcontinuity. This opens new questions about characterizations of subcontinuity with respect to other hypertopologies.

We also study convergences of subcontinuous functions and multifunctions. We have proved that the uniform convergence preserves subcontinuity of multifunctions with values in complete uniform spaces and the Hausdorff uniform convergence preserves subcontinuity of multifunctions defined on a locally compact uniform space with values in a complete uniform space.

In the last part we investigate the type of the set of points of subcontinuity of a multifunction.

2. Preliminaries

We will work on topological spaces $(X,\ Y)$ with no other properties, e.g. Hausdorff, etc.; unless explicitly stated otherwise. For basic references see [24] and [5]. Under a multifunction we understand $F \subset X \times Y$, although this is usually called a graph of a multifunction, and we suppose that for every $x \in X$, $F(x) = \big\{y \in Y :\ (x,y) \in F\big\}$ is non-empty. For $M \subset X$ put $F(M) = \bigcup\big\{F(x) : x \in M\big\}$.

Also for W a member of some uniformity (on topological space Y) we will use $W(M) = \{y : \ (x,y) \in W \ \text{ for some } \ x \in M\}$ and $W(y) = W(\{y\})$.

For a set A in topological space denote by $\overline{A}$, $\operatorname{int}(A)$ the closure and the interior of A, respectively. Nets in a topological space will be denoted by $\{x_a : a \in \mathcal{A}\}$, where $\mathcal{A}$ is some directed set, or more often, when it is not confusing, just by $\{x_a\}$. $x_a \to x$ means that $\{x_a\}$ converges to x. Frequently used directed sets will be: a (base of a) system of neighborhoods of x, $\mathcal{U}(x)$ with $U \geq V$, iff $U \subset V$; a (base of a) uniformity $\mathcal{W}$ with $W_1 \geq W_2$, iff $W_1 \subset W_2$ and a system of finite subsets $\mathcal{F}$ of a set $\mathcal{U}$ (denoted by Φ) with $\mathcal{F}_1 \geq \mathcal{F}_2$, iff $\mathcal{F}_1 \supset \mathcal{F}_2$. Direction on carthesian product is defined naturally.

A nonvoid family $\mathcal{G}$ is a filter whenever $\emptyset \notin \mathcal{G}$; if $A \in \mathcal{G}$ and $A \subset B$ then $B \in \mathcal{G}$; and if $A, B \in \mathcal{G}$ then $A \cap B \in \mathcal{G}$. A filter $\mathcal{G}$ is an ultrafilter iff it is maximal with respect to $\subset$. $\mathcal{N}(x)$ denotes the neighborhood filter, i.e. the filter of all neighborhoods of x in a topological space X. We say that $\mathcal{G}$ converges to x (resp. $x \in \lim \mathcal{G}$) iff $\mathcal{N}(x) \subset \mathcal{G}$. In a uniform space $(X,\mathcal{U})$, a filter $\mathcal{G}$ is called Cauchy iff for every $U \in \mathcal{U}$ there is $G \in \mathcal{G}$ with $G \times G \subset U$. For a multifunction F and a filter $\mathcal{G}$, the family $\{F(G) : \ G \in \mathcal{G}\}$ is a base of a filter, denoted by $F\mathcal{G}$. Following definitions are from [4]. We say that a filter $\mathcal{G}$ is compactoid iff for every ultrafilter $\mathcal{H} \supset \mathcal{G}$, we have $\lim \mathcal{H} \neq \emptyset$. We say that a filter $\mathcal{G}$ is totally bounded iff for every $U \in \mathcal{U}$ there is finite $M \subset X$ such that $U(M) \in \mathcal{G}$.

Fundamental facts about multifunctions and hypertopologies can be found in [2]. Recall some of them. We say that F is upper semicontinuous (USC) at x, iff for every open $U \supset F(x)$ there is a neighborhood V of x with $F(V) \subset U$, moreover if $F(x)$ is compact, we say that F is USCO at x. F is lower semicontinuous (LSC) at x, iff for every open U with $U \cap F(x) \neq \emptyset$ there is a neighborhood V of x such that for every $\hat{x} \in V$, $F(\hat{x}) \cap U \neq \emptyset$. F is mixed semicontinuous (MSC) at x, iff for every open $U \supset F(x)$ there is a neighborhood V of x such that for every $\hat{x} \in V$, $F(\hat{x}) \cap U \neq \emptyset$. This property is used in [6] and [30, p. 272].

We say that F has a closed graph, iff $F = \overline{F}$. According to [25], F is p-upper semi continuous (p-USC) at x, iff $F(x) = \bigcap_{V \in \mathcal{U}(x)} \overline{F(V)}$. It is easy to see that $\overline{F}(x) = \bigcap_{V \in \mathcal{U}(x)} \overline{F(V)}$ and therefore F is p-USC at x, iff $F(x) = \overline{F}(x)$, i.e. it is a local version of closed graph property.

3. Subcontinuity, local compactness and local total boundedness

Fuller [8] defined subcontinuity of a function, however he defined also an inversely subcontinuous function, which can be considered as subcontinuous inverse of a function, which is generally a multifunction. So there is a natural extension of subcontinuity to multifunctions. The following definition is the local version of subcontinuity introduced by Hrycay [22, Definition 3.4].

Definition 3.1. Let X and Y be topological spaces and $F \subset X \times Y$ be a multifunction. F is subcontinuous (SC) at $x \in X$, iff for every net $\{x_a\}$ converging to x and for every $y_a \in F(x_a)$, the net $\{y_a\}$ has a cluster point.

F is SC, iff it is SC at x for all $x \in X$.

Note that a function f is subcontinuous at x, iff it is SC at x as a multifunction. If $G \subset F$ is a multifunction (i.e. $\emptyset \neq G(x) \subset F(x)$ for all $x \in X$), then G is SC at x whenever F is SC at x.

Definition 3.2. Let X and Y be topological spaces and $F \subset X \times Y$ be a multifunction. F is locally compact (LC) at $x \in X$, iff there is a neighborhood V of x and compact $K \subset Y$ with $F(V) \subset K$.

F is LC, iff it is LC at x for all $x \in X$.

Theorem 3.3. *Let X and Y be topological spaces and $F \subset X \times Y$ be a multifunction. The following are equivalent.*

(1) *F is SC at $x \in X$;*

(2) *$F\mathcal{N}(x)$ is compactoid;*

(3) *every selection of F is SC at $x \in X$;*

(4) *for every open cover $\mathcal{U}$ of Y there is a finite $\mathcal{F} \subset \mathcal{U}$ and a neighborhood V of $x \in X$ such that $F(V) \subset \bigcup \mathcal{F}$.*

Proof.

$1 \iff 2$: [4, Proposition 5.4]

$2 \iff 4$: [4, Theorems 3.7, 3.8]

$2 \implies 3$: If f is selection of F then $f\mathcal{N}(x)$ is finer than $F\mathcal{N}(x)$ and so it is compactoid.

$3 \implies 1$: This is obvious. □

Corollary 3.4. *Let X and Y be topological spaces and $F \subset X \times Y$ be a multifunction. If F is LC at $x \in X$, then it is SC at x. Conversely if Y is locally compact and F is SC at $x \in X$, it is also LC at x.*

P r o o f. If F is LC at $x \in X$, then there is a neighborhood V of x and a compact $K \subset Y$ with $F(V) \subset K$. For every open cover $\mathcal{U}$ of Y, there is a finite $\mathcal{F} \subset \mathcal{U}$ such that $K \subset \bigcup \mathcal{F}$, hence F is SC at x.

Suppose now that Y is locally compact and F is SC at $x \in X$. For every $y \in Y$ there is a compact neighborhood K_y of y. Put $\mathcal{U} = \{\operatorname{int}(K_y) : y \in Y\}$ and from subcontinuity of F we have a neighborhood V of x with $F(V) \subset K_{y_1} \cup \cdots \cup K_{y_n}$. □

Definition 3.5. Let X be a topological space, $(Y, \mathcal{W})$ be a uniform space and $F \subset X \times Y$ be a multifunction. F is locally totally bounded (LTB) at $x \in X$, iff for every $W \in \mathcal{W}$ there is a neighborhood V of x and a finite $M \subset Y$ with $F(V) \subset W(M)$.

F is LTB, iff it is LTB at x for all $x \in X$.

Let $(Y, \mathcal{W})$ be as above. For every open $W \in \mathcal{W}$, $\{W(y) : y \in Y\}$ is an open cover of Y. Thus we have from Theorem 3.3 that every multifunction SC at $x \in X$ is also LTB at x. The following characterization of the local total boundedness shows even a closer relation to subcontinuity.

Theorem 3.6. *Let X be a topological space, $(Y, \mathcal{W})$ be a uniform space and $F \subset X \times Y$ be a multifunction. The following are equivalent.*

(1) *F is LTB at $x \in X$;*
(2) *$F\mathcal{N}(x)$ is totally bounded;*
(3) *every selection of F is LTB at $x \in X$;*
(4) *for every net $\{x_a\}$ converging to x, a net $\{y_a\}$ with $y_a \in F(x_a)$ has a Cauchy subnet.*

P r o o f.

$1 \iff 2$: This is easy as well as $1 \implies 3$:

$3 \implies 4$: This is actually a corollary to [27, Theorem 2.8], but we will provide another proof. Let f be a selection of F, such that $y_a = f(x_a)$. Let $\mathcal{M}$ be a maximal family of subsets of Y, such that the net $\{f(x_a)\}$ is frequently in each member of $\mathcal{M}$ and the intersection of two members of $\mathcal{M}$ belongs to $\mathcal{M}$. $\mathcal{M}$ is an ultrafilter, thus we have the following property:

($*$) For every $N \subset Y$ either $N \in \mathcal{M}$ or $Y \setminus N \in \mathcal{M}$.

For every $W \in \mathcal{W}$ there is a neighborhood V of x and $\hat{y}_1, \ldots, \hat{y}_n$, such that $f(V) \subset W(\hat{y}_1) \cup \cdots \cup W(\hat{y}_n)$. Since the net $\{x_a\}$ is eventually in V, the net $\{f(x_a)\}$ is eventually in $f(V) \subset W(\hat{y}_1) \cup \cdots \cup W(\hat{y}_n)$. Using ($*$) we have that $W(\hat{y}_1) \cup \cdots \cup W(\hat{y}_n) \in \mathcal{M}$ and therefore (again using ($*$)) $W(\hat{y}_i) \in \mathcal{M}$, for some i; i.e. $\mathcal{M}$ contains small sets. Since the net $\{f(x_a)\}$ is frequently in each member of $\mathcal{M}$, by [24, Lemma 2.5] it has a subnet which is eventually in each member of $\mathcal{M}$ and thus it is Cauchy because $\mathcal{M}$ contains small sets.

$4 \implies 1$: Suppose F is not LTB at x, then there is $W \in \mathcal{W}$ such that for every neighborhood V of x and finite $M \subset Y$ there is $x_{(V,M)} \in V$ and $y_{(V,M)} \in F(x_{V,M}) \setminus W(M)$. Since $x_{(V,M)} \to x$, there is a Cauchy subnet $\{y_a\}$ of the net $\{y_{(V,M)}\}$; i.e. the net $\{y_a\}$ is eventually in $W(\hat{y})$ for some $\hat{y} \in Y$, contrary to supposition. $\square$

We see immediately that for Y complete, the subcontinuity coincides with the local total boundedness. Moreover we can prove that their coincidence characterizes complete uniform spaces. This is closely related to the [27, Theorem 2.2].

Theorem 3.7. *Let $(Y, \mathcal{W})$ be a uniform space. The following are equivalent.*

(1) $(Y, \mathcal{W})$ *is complete;*

(2) *for every topological space X and every multifunction $F \subset X \times Y$ the following holds: F is SC at $x \in X$, iff F is LTB at x.*

Proof. As we have noticed $1 \implies 2$ follows from Theorem 3.6. To prove $2 \implies 1$ suppose $(Y, \mathcal{W})$ is not complete. Let $(Y^*, \mathcal{W}^*)$ be its completion. Let $\{y_a\}$ be a Cauchy net in Y which has no limit point in Y. Let y^* be the limit point of the net $\{y_a\}$ in Y^*. For fixed $y_0 \in Y$ consider $f\colon (Y^*, \mathcal{W}^*) \to (Y, \mathcal{W})$ defined as follows:

$$f(y) = \begin{cases} y & y \in Y, \\ y_0 & \text{otherwise.} \end{cases}$$

Of course f is not SC at y^*, since $y_a \to y^*$ and $\{f(y_a)\}$ has no cluster point in Y. For every net $\{y_b\}$ converging to y^* either $f(y_b) = y_0$ frequently or $f(y_b) = y_b$ eventually and thus it has a Cauchy subnet in Y. By Theorem 3.6 we have that f is LTB at y^*, a contradiction. $\square$

Following propositions show some relations between subcontinuity and semicontinuity.

Proposition 3.8. ([25, Theorem 2.5]) *Let X and Y be topological spaces and $F \subset X \times Y$ be a multifunction. If F is USCO at $x \in X$, then it is SC at x.*

Proposition 3.9. ([25, Theorem 2.4]) *Let X and Y be topological spaces and $F \subset X \times Y$ be a multifunction. If F is SC and p-USC at $x \in X$, then F is USCO at x.*

For a multifunction $F \subset X \times Y$ denote by F^+ the multifunction defined by $F^+(x) = \bigcap_{V \in \mathcal{U}(x)} F(V)$, so we have $F \subset F^+ \subset \overline{F}$. Note that for any open $V_0 \subset X$ we have $F^+(V_0) = \bigcup_{x \in V_0} \big(\bigcap_{V \in \mathcal{U}(x)} F(V) \big)$ and since $V_0 \in \mathcal{U}(x)$ for every $x \in V_0$, we have that $\overline{F^{+}(V_0)} \subset \overline{F(V_0)}$ and therefore $\overline{F^{+}(V_0)} = \overline{F(V_0)}$. Analogously $\overline{F}(V_0) \subset \overline{F(V_0)}$. Theorem 3.3 immediately yields the following proposition.

Proposition 3.10. *F is SC at $x \in X$ iff F^{+} is SC at x.*

Theorem 3.11. *Let X and Y be topological spaces, let Y be regular and let $F \subset X \times Y$ be a multifunction. F is SC at $x \in X$ iff $\overline{F}$ is SC at x.*

Proof. Since $F \subset \overline{F}$ the subcontinuity of $\overline{F}$ implies the subcontinuity of F. For a filter $\mathcal{G}$ denote by $\overline{\mathcal{G}}$ the filter generated by $\{\overline{G} : G \in \mathcal{G}\}$. Now suppose F is SC at x then $F\mathcal{N}(x)$ is compactoid. From [4, Theorems 4.4, 4.9] follows that $\overline{F\mathcal{N}(x)}$ is compactoid. For every open V holds $\overline{F}(V) \subset \overline{F(V)}$, therefore $\overline{F}\mathcal{N}(x)$ is finer than $\overline{F\mathcal{N}(x)}$ and hence compactoid. $\square$

Corollary 3.12. *Let X and Y be topological spaces, let Y be regular and let $F \subset X \times Y$ be a multifunction. F is SC at $x \in X$ iff $\overline{F}$ is USCO at x. Particularly, a function $f : X \to Y$ is SC at x iff it is a selection of some multifunction that is USCO at x.*

Proposition 3.13. ([4, Corollary 5.7]) *Let X and Y be topological spaces, let Y be regular and let $F \subset X \times Y$ be a SC multifunction. If K is a relatively compact subset of X (i.e. $\overline{K}$ is compact), then $F(K)$ is relatively compact.*

Corollary 3.14. *Let X be a locally compact space, let Y be a regular space and let $F \subset X \times Y$ be a SC multifunction, then F is LC.*

4. Weak subcontinuity

Although the subcontinuity is a rather weak notion, there are several ways how to weaken it. Theorem 3.3 gives us three possibilities how to define a weak subcontinuity of a multifunction.

Definition 4.1. Let X and Y be topological spaces and $F \subset X \times Y$ be a multifunction.

F is W_1SC at $x \in X$, iff there is a selection of F that is SC at x.

F is W_2SC at $x \in X$, iff for every net $x_a \to x$ there is $y_a \in F(x_a)$ with a cluster point.

F is W_3SC at $x \in X$, iff for every open cover $\mathcal{U}$ of Y there is a neighborhood V of x and a finite $\mathcal{F} \subset \mathcal{U}$ such that for every $\hat{x} \in V$, $F(\hat{x}) \cap \left(\bigcup \mathcal{F}\right) \neq \emptyset$.

Note that a function (single-valued multifunction) is W_iSC at x, iff it is SC at x. In [9], the property W_2SC is called weak subcontinuity. This is probably the most natural way to define weak subcontinuity. The other two ways are useful for some characterizations of weak subcontinuity similar to subcontinuity.

Proposition 4.2. *Let X and Y be topological spaces and $F \subset X \times Y$ be a multifunction. If F is $W_i SC$ at $x \in X$ for $i = 1$ or 2, it is also $W_{i+1} SC$ at x. If Y is locally compact, then all three notions are equivalent.*

Proof. Obviously $\mathrm{W_1SC} \implies \mathrm{W_2SC}$. Now suppose that F is $\mathrm{W_2SC}$, but not $\mathrm{W_3SC}$ at x. There is an open cover $\mathcal{U}$ of Y such that for every neighborhood V of x and every finite $\mathcal{F} \subset \mathcal{U}$ there is $x_{(V,\mathcal{F})}$ with $F(x_{(V,\mathcal{F})}) \cap \left(\bigcup \mathcal{F}\right) = \emptyset$. Since $x_{(V,\mathcal{F})} \to x$ there is a net $\{y_{(V,\mathcal{F})}\}$, $y_{(V,\mathcal{F})} \in F(x_{(V,\mathcal{F})})$, with a cluster point $y \in U \in \mathcal{U}$. Hence there is $(V, \mathcal{F}) \geq (X, \{U\})$ such that $y_{(V,\mathcal{F})} \in U$ i.e. $y_{(V,\mathcal{F})} \in F(x_{(V,\mathcal{F})}) \cap \left(\bigcup \mathcal{F}\right)$, contrary to supposition.

At last suppose that Y is locally compact and F is $\mathrm{W_3SC}$. Every $y \in Y$ has a compact neighborhood K_y. Put $\mathcal{U} = \left\{\mathrm{int}(K_y) : \ y \in Y\right\}$. Then $\mathcal{U}$ is an open cover of Y. There is a neighborhood V of x and compact $K = K_{y_1} \cup \cdots \cup K_{y_n}$ such that for every $\hat{x} \in V$, $F(\hat{x}) \cap K \neq \emptyset$. Put

$$G(\hat{x}) = \begin{cases} F(\hat{x}) \cap K & \hat{x} \in V \\ F(\hat{x}) & \hat{x} \notin V \end{cases}$$

$G \subset F$ and $G(V) \subset K$, i.e. G is LC at x, hence G is SC at x and every selection of G is SC at x. □

Proposition 4.3. *Let X and Y be topological spaces and $F \subset X \times Y$ be a multifunction. If F is LSC at $x \in X$, then F is W_3SC at x.*

Proof. For every open cover $\mathcal{U}$ of Y there is $U \in \mathcal{U}$ with $F(x) \cap U \neq \emptyset$. From the lower semicontinuity of F we have a neighborhood V of x such that for every $\hat{x} \in V$, $F(\hat{x}) \cap U \neq \emptyset$. □

Theorem 4.4. *Let X and Y be topological spaces and $F \subset X \times Y$ be a multifunction. If F is W_2SC and p-USC at $x \in X$ then F is MSC at x.*

Proof. Suppose F is not MSC at x. Then there is open U with $F(x) \subset U$ such that for every $V \in \mathcal{U}(x)$ there is $x_V \in V$ with $F(x_V) \cap U = \emptyset$. Since $x_V \to x$ and F is $\mathrm{W_2SC}$ we have a net $\{y_V\}$, $y_V \in F(x_V)$, with a cluster point $y \notin U$, a contradiction since $y \in \overline{F}(x) = F(x)$. □

Theorem 4.5. *Let X and Y be topological spaces and $F \subset X \times Y$ be a multifunction. If F is MSC at x and $F(x)$ is a subset of compact $K \subset Y$, then F is W_3SC at x.*

Proof. Let $\mathcal{U}$ be an open cover of Y. Then there is finite $\mathcal{F} \subset \mathcal{U}$ with $F(x) \subset \bigcup \mathcal{F}$ and since F is MSC at x we have a neighborhood U of x such that for every $\hat{x} \in U$, $F(\hat{x}) \cap \left(\bigcup \mathcal{F}\right) \neq \emptyset$. □

5. Subcontinuity with respect to hypertopologies

Now we will view a multifunction as a mapping $F\colon X \to \mathcal{B}(Y)$, where $\mathcal{B}(Y)$ is some subfamily of $\mathcal{P}(Y)$, the family of all subsets of Y. We will consider $CL(Y)$ — the family of closed non-empty subsets of Y, $\mathcal{K}(Y)$ — the family of compact non-empty subsets of Y, or $\mathcal{P}(Y)$ itself. $\mathcal{B}(Y)$ will be equipped with some hypertopology.

Definition 5.1. Let X and Y be topological spaces, τ be a hypertopology on $\mathcal{B}(Y)$ and $F\colon X \to \mathcal{B}(Y)$. F is τ-subcontinuous (τ-SC) at $x \in X$, iff for every net $x_a \to x$ the net $\{F(x_a)\}$ has a cluster point in $(\mathcal{B}(Y), \tau)$.

Lemma 5.2. *Let X and Y be topological spaces, τ_V^+ be the upper part of the Vietoris topology on $\mathcal{P}(Y)$ and $F \subset X \times Y$ be a multifunction SC at $x \in X$. If $x_a \to x$, then $F(x_a) \to \overline{F}(x)$ in $(\mathcal{P}(Y), \tau_V^+)$.*

Proof. Suppose $F(x_a) \not\to \overline{F}(x)$ in $(\mathcal{P}(Y), \tau_V^+)$. There is open U such that $\overline{F}(x) \subset U$ and for every a there is $b_a \geq a$ with $F(x_{b_a}) \setminus U \neq \emptyset$. Choose $y_{b_a} \in F(x_{b_a}) \setminus U$. Since $x_{b_a} \to x$, the net $\{y_{b_a}\}$ has a cluster point $y \in \overline{F}(x)$, a contradiction. $\square$

Corollary 5.3. *Let X and Y be topological spaces, τ_V^+ be the upper part of the Vietoris topology on $\mathcal{P}(Y)$ and $F\colon X \to \mathcal{P}(Y)$. If F is SC at $x \in X$, it is also τ_V^+-SC at x.*

Corollary 5.4. *Let X and Y be topological spaces, τ_V^+ be the upper part of the Vietoris topology on $CL(Y)$ and $F\colon X \to CL(Y)$. If F is SC at $x \in X$, it is also τ_V^+-SC at x.*

Corollary 5.5. *Let X and Y be topological spaces, Y be regular, τ_V^+ be the upper part of the Vietoris topology on $\mathcal{K}(Y)$ and $F\colon X \to \mathcal{K}(Y)$. If F is SC at $x \in X$, it is also τ_V^+-SC at x.*

Proof. By Corollary 3.12 $\overline{F}$ is USCO at x and therefore $\overline{F}(x)$ is compact. $\square$

Theorem 5.6. *Let X and Y be topological spaces, τ_V^+ be the upper part of the Vietoris topology on $\mathcal{K}(Y)$ and $F\colon X \to \mathcal{K}(Y)$. If F is τ_V^+-SC at $x \in X$, it is also SC at x.*

Proof. Let $\mathcal{U}$ be an open cover of Y. For $K \in K(Y)$ let $\mathcal{U}_K \subset \mathcal{U}$ be a finite cover of K. The family $\mathcal{G} = \left\{ \left(\bigcup \mathcal{U}_K\right)^+ :\ K \in K(Y) \right\}$ is an open cover in $(K(Y), \tau_V^+)$. Since F is τ_V^+-SC at $x \in X$, there is a neighborhood V of x and a finite subfamily $\mathcal{H} = \left\{ \left(\bigcup \mathcal{U}_{K_i}\right)^+ :\ i = 1, \ldots, n \right\}$ of $\mathcal{G}$ such that $\left\{ F(x) :\ x \in V \right\} \subset \bigcup \mathcal{H}$. (This is from Theorem 3.3, where F is viewed as a function.) Then $F(V) \subset \bigcup \mathcal{F}$, where $\mathcal{F} = \bigcup\limits_{i=1}^{n} \mathcal{U}_{K_i}$ is finite subset of $\mathcal{U}$. $\square$

Corollary 5.7. *Let X and Y be topological spaces, Y be regular and τ_V^+ be the upper part of the Vietoris topology on $\mathcal{K}(Y)$. $F\colon X \to \mathcal{K}(Y)$ is SC at $x \in X$, iff it is τ_V^+-SC at x.*

6. Convergence of subcontinuous multifunctions

We start with the definition of the uniform convergence and the uniform convergence on compacta for multifunctions.

Definition 6.1. (see [33]) Let X be a topological space, $(Y, \mathcal{W})$ be a uniform space. We say that a net of multifunctions $\{F_a\}$, $F_a \subset X \times Y$, converges uniformly to $F \subset X \times Y$ ($F_a \rightrightarrows F$), iff for every $W \in \mathcal{W}$ there is a_0 such that for every $a \geq a_0$, there holds $F_a(\hat{x}) \subset W(F(\hat{x}))$ and $F(\hat{x}) \subset W(F_a(\hat{x}))$ for every $\hat{x} \in X$.

We say that a net of multifunctions $\{F_a\}$, $F_a \subset X \times Y$, converges uniformly on compacta to $F \subset X \times Y$ ($F_a \overset{\mathcal{K}}{\to} F$), iff for every K, a compact subset of X, there holds $F_a|K \rightrightarrows F|K$, where $F|K \subset K \times Y$ is the restriction of F onto K.

There have recently been many papers devoted to the study of topologies and convergences on spaces of set-valued maps. There has been interest in studying extensions of natural topologies on the spaces of functions to the space of multifunctions [11], [14], [18], [21].

The following examples show that the uniform convergence need not preserve the subcontinuity.

Example 6.2. For every non-complete metric space (Y, d), there is a sequence $\{f_n\}$ of SC functions, $f_n\colon \mathbb{R} \to Y$, uniformly convergent to $f\colon \mathbb{R} \to Y$, which is nowhere SC.

P r o o f. Let $\{D_n :\ n \in \omega\}$ be a family of pairwise disjoint dense subsets of $\mathbb{R}$, with $\mathbb{R} = \bigcup_{n\in\omega} D_n$. Since Y is non-complete, it has a non-convergent Cauchy sequence $\{y_n\}$. For every $n \in \omega$ define $f_n\colon \mathbb{R} \to Y$ by

$$f_n(x) = \begin{cases} y_i & x \in D_i,\ \ i \leq n, \\ y_n & x \in D_i,\ \ i > n. \end{cases}$$

Since $f_n(\mathbb{R})$ is finite for every $n \in \omega$, every f_n is LC and therefore SC. One can easily see, that $f_n \rightrightarrows f$, where f is defined by $f(x) = y_i$ for $x \in D_i$. To see that f is nowhere SC, we can use Proposition 3.10. Suppose f is SC at some x, then f^+ is SC at x and since $f^+(x) = \{y_n :\ n \in \omega\}$, $\{y_n\}$ has to have a cluster point, contrary to supposition. $\square$

Example 6.3. For every non-complete uniform space $(Y, \mathcal{W})$, there is a topological space X and a net $\{f_a\}$ of SC functions, $f_a\colon X \to Y$, uniformly convergent to $f\colon X \to Y$, which is not SC.

Proof. Since $(Y, \mathcal{W})$ is non-complete, it has a non-convergent Cauchy net $\{y_a :\ a \in \mathcal{A}\}$. Put $X = \mathcal{A} \cup \{x_0\}$, where $x_0 \notin \mathcal{A}$. Let τ be a topology on X, such that every $a \in \mathcal{A}$ is an isolated point and the family $\{U_b :\ b \in \mathcal{A}\}$, where $U_b = \{a \in \mathcal{A} :\ a \geq b\} \cup \{x_0\}$, forms a base of neighborhoods of x_0. Define $f_a\colon X \to Y$, for every $a \in \mathcal{A}$, and $f\colon X \to Y$ by

$$f_a(x) = \begin{cases} y_0 & x = x_0, \\ y_a & x \in \mathcal{A},\ x \geq a, \\ y_x & \text{otherwise}, \end{cases} \qquad f(x) = \begin{cases} y_0 & x = x_0, \\ y_x & \text{otherwise}, \end{cases}$$

where y_0 is a fixed point of Y. It is easy to see that all f_a are SC. To verify that $f_a \rightrightarrows f$, let $W \in \mathcal{W}$. Since $\{y_a :\ a \in \mathcal{A}\}$ is Cauchy, there is a_0 with $(y_a, y_b) \in W$ for every $a, b \geq a_0$; and thus for every $a \geq a_0$ and $x \in X$, $(f_a(x), f(x)) \in W$. Since $\{y_a :\ a \in \mathcal{A}\}$ has no cluster point and $a \to x_0$, f is not SC at x_0. □

Similarly we have the following example for the pointwise convergence.

Example 6.4. For every non-compact topological space Y, there is a topological space X and a net $\{f_a\}$ of SC functions, $f_a\colon X \to Y$, pointwise convergent to $f\colon X \to Y$, which is nowhere SC.

Proof. Let us take a net $\{y_a :\ a \in \mathcal{A}\}$ in Y with no cluster point. We need $X = \bigcup\limits_{a \in \mathcal{A}} D_a$, such that D_a are dense and pairwise disjoint. We can take for example a Hilbert cube $X = [0,1]^{\mathfrak{n}}$, with sufficiently large cardinal $\mathfrak{n}$. Let $f\colon X \to Y$ be defined as follows: $f(x) = y_a$ for $x \in D_a$. Like in Example 6.2 f is nowhere SC. Let $\mathcal{M}$ be the family of finite subsets of $\mathcal{A}$ (directed by $\supset$). For every $M \in \mathcal{M}$ define $f_M\colon X \to Y$ by

$$f_M(x) = \begin{cases} y_a & x \in D_a,\ a \in M, \\ y_0 & \text{otherwise}, \end{cases}$$

where $y_0 \in Y$ is fixed. For every $M \in \mathcal{M}$, $f_M(X)$ is finite, and therefore every f_M is LC and hence SC. Since for fixed $x \in X$, $\{f_M(x) :\ M \in \mathcal{M}\}$ is eventually constant, we clearly have that $f_M(x) \to f(x)$. □

From previous examples we can see two things. The first one is, that the pointwise convergence is incompatible with subcontinuity, because we need compact Y, but then all (multi)functions are SC. The second one is, that for the case of the uniform convergence and a uniform Y, we need Y to be complete, and therefore it suffices to consider the local total boundedness instead of subcontinuity.

Theorem 6.5. *Let X be a topological space, $(Y, \mathcal{W})$ be a uniform space and $\{F_a\}$, $F_a \subset X \times Y$, be a net of multifunctions LTB at $x \in X$. If $F_a \rightrightarrows F \subset X \times Y$, then F is LTB at x.*

Proof. For every $W \in \mathcal{W}$ choose U, with $U \circ U \subset W$. Since $F_a \rightrightarrows F$, there is a_0 such that for every $\hat{x} \in X$ is $F(\hat{x}) \subset U(F_{a_0}(\hat{x}))$. Since F_{a_0} is LTB at x, there is a neighborhood V of x and finite $M \subset Y$ such that for every $\hat{x} \in V$ is $F_a(\hat{x}) \subset U(M)$ and thus $F(\hat{x}) \subset (U \circ U)(M) \subset W(M)$. □

Corollary 6.6. *Let X be a topological space, $(Y, \mathcal{W})$ be a complete uniform space and $\{F_a\}$, $F_a \subset X \times Y$ be a net of multifunctions SC at $x \in X$. If $F_a \rightrightarrows F \subset X \times Y$, then F is SC at x.*

Corollary 6.7. *Let X be a locally compact topological space, $(Y, \mathcal{W})$ be a complete uniform space and $\{F_a\}$ be a net of multifunctions SC at $x \in X$. If $F_a \xrightarrow{\mathcal{K}} F$, then F is SC at x.*

In Corollary 6.7 it is sufficient that just x has a compact neighborhood.

From Corollary 6.6 and Example 6.3, we have the following characterization of complete uniform spaces.

Theorem 6.8. *Let $(Y, \mathcal{W})$ be a uniform space. The following are equivalent.*

(1) $(Y, \mathcal{W})$ *is complete;*

(2) *for every topological space X, for every $x \in X$ and for every net $\{F_a\}$, $F_a \subset X \times Y$ of multifunctions SC at $x \in X$, uniformly convergent to $F \subset X \times Y$, F is SC at x.*

Now we will consider convergences with respect to hypertopologies. Let τ be a hypertopology on $\mathcal{P}(X \times Y)$. We say that a net of multifunctions $\{F_a\}$, $F_a \subset X \times Y$ converges to a multifunction $F \subset X \times Y$, with respect to τ, iff it converges in $(\mathcal{P}(X \times Y), \tau)$. (Remember that we consider (multi)functions as subsets of $X \times Y$.)

Hyperspace topologies on multifunctions were studied by many authors [13], [17], [20], [28], [19], [16].

Example 6.9. For every non-compact topological space Y and non-empty topological space X, there is a net $\{F_a\}$, $F_a \subset X \times Y$ of SC multifunctions convergent to $F \subset X \times Y$ with respect to the Vietoris topology, which is nowhere SC.

Proof. Let us take a net $\{y_a : a \in \mathcal{A}\} \subset Y$ with no cluster point. Let $\mathcal{M}$ be the family of finite subsets of $\mathcal{A}$ (directed by $\supset$). For every $M \in \mathcal{M}$ let us define $F_M \subset X \times Y$ by $F_M(x) = \{y_a : a \in M\}$ for all $x \in X$, and define $F \subset X \times Y$ by $F(x) = \{y_a : a \in \mathcal{A}\}$ for all $x \in X$. Every F_M is obviously USCO and hence SC. Since $F_M \subset F$, we clearly have that if $F \in V^+$ then $F_M \in V^+$, for all $M \in \mathcal{M}$. If $F \in V^-$, then there is $(x, y_a) \in V$ for some $a \in \mathcal{A}$. For every $M \in \mathcal{M}$, $M \supset \{a\}$, we have $F_M \in V^-$. Thus $\{F_M : \ M \in \mathcal{M}\}$ converges to F with respect to the Vietoris topology. It is easy to see, that for every $x \in X$, F is not SC at x. ☐

If we want subcontinuity to be preserved under the convergence with respect to the Vietoris topology, we need compact Y, but this will force all multifunctions to be SC. Note that the previous example works well also for the Fell topology because it is weaker than the Vietoris topology. In the case of metric X and Y it works also for the Wijsman, proximal and ball proximal topologies (for definitions see [2]). At last we will discuss the convergence of graphs in the Hausdorff uniformity.

Recall that for $(X, \mathcal{W})$ a uniform space, sets

$$W_H = \big\{(A, B) : \ A, B \subset X, \ \ A \subset W(B), \ \ B \subset W(A)\big\},$$

where W runs through $\mathcal{W}$, form a base of the Hausdorff uniformity on $\mathcal{P}(X)$. This is a generalization of the Hausdorff metric: In particular if (X, d) is a metric space, then the above uniformity is pseudometrizable by the Hausdorff pseudometric

$$d_H(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup\big\{d(a, B) : \ a \in A\big\}$.

Definition 6.10. Let $(X, \mathcal{V})$ and $(Y, \mathcal{W})$ be uniform spaces. A net $\{F_a\}$, $F_a \subset X \times Y$ of multifunctions converges to a multifunction $F \subset X \times Y$ with respect to the Hausdorff uniformity ($F_a \xrightarrow{\mathcal{H}} F$), iff for every $V \in \mathcal{V}$ and every $W \in \mathcal{W}$ there is a_0 such that for every $a \geq a_0$, holds $F \subset (V \otimes W)(F_a)(= W \circ F_a \circ V^{-1})$ and $F_a \subset (V \otimes W)(F)(= W \circ F \circ V^{-1})$.

Theorem 6.11. *Let $(X, \mathcal{V})$ be a locally compact uniform space, $(Y, \mathcal{W})$ be a uniform space and $\{F_a\}$, $F_a \subset X \times Y$ be a net of multifunctions SC at $x \in X$. If $F_a \xrightarrow{\mathcal{H}} F \subset X \times Y$, then F is LTB at x.*

P r o o f. Since X is locally compact, there is $V \in \mathcal{V}$ such that $\overline{(V \circ V)(x)}$ is compact. According to Proposition 3.13 $\overline{F_a\big((V \circ V)(x)\big)}$ is compact for every a. For every $W \in \mathcal{W}$ there is $W_0 \in \mathcal{W}$ with $W_0 \circ W_0 \subset W$. Let $V_0 \in \mathcal{V}$ be such that $V_0^{-1} \subset V$. There is a such that $F \subset W_0 \circ F_a \circ V_0^{-1}$, i.e. $F(V(x)) \subset (W_0 \circ F_a \circ V_0^{-1} \circ V)(x) \subset W_0(K)$, where K is compact subset of Y. There is finite $M \subset Y$ with $K \subset W_0(M)$, hence $F(V(x)) \subset W(M)$ and therefore F is LTB at x. ☐

COROLLARY 6.12. *Let $(X,\mathcal{V})$ be a locally compact uniform space, $(Y,\mathcal{W})$ be a complete uniform space and $\{F_a\}$, $F_a \subset X\times Y$ be a net of multifunctions SC at $x \in X$. If $F_a \overset{\mathcal{H}}{\to} F \subset X\times Y$, then F is SC at x.*

7. Set of points of subcontinuity of a multifunction

Let us denote the set of points of subcontinuity of a multifunction F by $SC(F)$ and the set of points of upper semi-continuity of F by $USC(F)$. System (of sets) with cardinality $\mathfrak{m}$ is called $\mathfrak{m}$-system. An extension of the topological space X is a topological space $X^* \supset X$ in which X is dense.

DEFINITION 7.1. ([7, Definition 2.1]) A subset G of a topological space X is called $G_{\mathfrak{m}}$-subset, if it is an intersection of some open $\mathfrak{m}$-system. A Hausdorff space is said to be a $G_{\mathfrak{m}}$-space, if it is a $G_{\mathfrak{m}}$-subset of each of its Hausdorff extensions.

Instead of $G_{\aleph_0}$, we will use G_δ. A G_δ space is called Čech complete if it is a Tychonoff space.

DEFINITION 7.2. ([7, Definition 2.2]) A system $\{\mathcal{U}_i:\ i\in I\}$ of open covers of a topological space Y is said to be complete, if the following condition is satisfied: If $\mathcal{V}$ is a family of open subsets of Y with the finite intersection property such that $\mathcal{V}\cap\mathcal{U}_i \neq \emptyset$ for each $i \in I$, then $\bigcap\{\overline{V}:\ V\in\mathcal{V}\}\neq\emptyset$.

LEMMA 7.3. ([7, Theorem 2.8]) *Let X be a Tychonoff space. The following are equivalent:*

(1) *X is a $G_{\mathfrak{m}}$-space;*

(2) *X possesses a complete $\mathfrak{m}$-system of open covers;*

(3) *X is a $G_{\mathfrak{m}}$-subset of its Čech-Stone compactification βX.*

LEMMA 7.4. ([7, Proposition 2.13]) *Let $\{\mathcal{U}_i:\ i\in I\}$ be a complete system of open covers of a regular topological space Y. Suppose that $\mathcal{B}$ is a system of subsets of Y with finite intersection property such that for every $i\in I$ there is $B_i\in\mathcal{B}$ and finite $\mathcal{F}_i\subset\mathcal{U}_i$ with $B_i\subset\bigcup\mathcal{F}_i$; then $\bigcap\{\overline{B}:\ B\in\mathcal{B}\}\neq\emptyset$.*

Frolík also assumed that Y is Hausdorff, but it is not needed in his proof.

THEOREM 7.5. *Let X and Y be topological spaces, Y be regular, $F\subset X\times Y$ be a multifunction and $\{\mathcal{U}_i:\ i\in I\}$ be a complete family of open covers of Y. The following are equivalent.*

(1) *F is SC at $x\in X$;*

(2) *for every $i\in I$ there is a finite $\mathcal{F}\subset\mathcal{U}_i$ and a neighborhood V of $x\in X$ such that $F(V)\subset\bigcup\mathcal{F}$.*

P r o o f. From Theorem 3.3 we have (1) $\implies$ (2). For (2) $\implies$ (1) suppose there is an open cover $\mathcal{U}$ of Y such that for every finite $\mathcal{F} \subset \mathcal{U}$ and every neighborhood V of $x \in X$, $B_{\mathcal{F},V} = F(V) \setminus \left(\bigcup \mathcal{F}\right) \neq \emptyset$. The family $\mathcal{B} = \{B_{\mathcal{F},V}\}$ is a family of sets with the finite intersection property, and from Lemma 7.4, $\bigcap \{\overline{B} : B \in \mathcal{B}\} \neq \emptyset$; and hence $\mathcal{U}$ fails to be a cover, contrary to supposition. □

Theorem 7.6. *Let X and Y be topological spaces, Y be regular with complete $\mathfrak{m}$-system of open covers, $F \subset X \times Y$ be a multifunction, then $SC(F)$ is $G_{\mathfrak{m}}$-subset of X.*

P r o o f. Let $\{\mathcal{U}_i : i \in I\}$ be a complete $\mathfrak{m}$-system of open covers of Y. From Theorem 7.5 we have

$$SC(F) = \bigcap_{i \in I} \Big\{ x : (\exists \mathcal{F})(\exists V_x)\big(\mathcal{F} \subset \mathcal{U}_i \ \ \& \ \ \overline{F(V_x)} \subset \bigcup \mathcal{F}\big) \Big\},$$

where $\mathcal{F}$ is always finite and V_x is a neighborhood of x. □

Since a Čech-complete space has a countable complete family of open covers, we have:

Corollary 7.7. ([26, Lemma 1.13]) *Let X be a topological space, Y be a Čech-complete space and $F \subset X \times Y$ be a multifunction. Then $SC(F)$ is a G_δ-set.*

If $(Y, \mathcal{W})$ is a complete uniform space and $\mathcal{W}_0$ is a base of open elements of $\mathcal{W}$, the family $\big\{\{W(y) : y \in Y\} : W \in \mathcal{W}_0\big\}$ is a complete system of open covers of Y. Thus we have:

Corollary 7.8. *Let X be a topological space, $(Y, \mathcal{W})$ be a complete uniform space with the weight of the uniformity $w(Y, \mathcal{W}) = \mathfrak{m}$ and $F \subset X \times Y$ be a multifunction, then $SC(F)$ is a $G_{\mathfrak{m}}$-subset of X.*

Note that if F is compact-valued and has a closed graph then $USC(F) = SC(F)$. We have the following characterization of $G_{\mathfrak{m}}$-spaces.

Theorem 7.9. *Let Y be a Tychonoff space. The following are equivalent:*

1. *Y is $G_{\mathfrak{m}}$-space;*
2. *for every topological space X and every multifunction $F \subset X \times Y$, $SC(F)$ is a $G_{\mathfrak{m}}$-subset of X;*
3. *for every topological space X and every multifunction $F \subset X \times Y$ with the closed graph and compact values, $USC(F)$ is a $G_{\mathfrak{m}}$-subset of X.*

P r o o f.

(1) $\Longrightarrow$ (2) by Theorem 7.6 and Lemma 7.3. It remains to prove (3) $\Longrightarrow$ (1). Let $X = \beta Y$ be the Čech-Stone compactification of Y. For fixed $y_0 \in Y$ define a multifunction $F \subset X \times Y$ by $F(y) = \{y, y_0\}$ for $y \in Y$ and $F(x) = \{y_0\}$ for $x \in X \setminus Y$. F has a closed graph and compact values, therefore $Y = USC(F)$ has to be a $G_{\mathfrak{m}}$-subset of X and thus Y is $G_{\mathfrak{m}}$-space. □

The following corollary characterizes Čech-complete spaces as noted in [26].

Corollary 7.10. *Let Y be a Tychonoff space. The following are equivalent:*

(1) *Y is Čech-complete space;*
(2) *for every topological space X and every multifunction $F \subset X \times Y$, $SC(F)$ is a G_δ-set;*
(3) *for every topological space X and every multifunction $F \subset X \times Y$ with the closed graph and compact values, $USC(F)$ is a G_δ-set.*

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