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MULTIPLICATION OPERATORS ON LORENTZ-KARAMATA-BOCHNER SPACES

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ABSTRACT. In this small note, we will characterize the boundedness, compactness and closedness of the range of the multiplication operators on Lorentz-Karamata-Bochner spaces $L_{p,q;b}(\Omega, X)$ for $p, q \in (0, \infty]$.

1. Introduction

Throughout the paper (Ω, A, μ) will stand for a σ-finite measure space. For any two non-negative expressions (i.e. functions or functionals), A and B, the symbol $A \precsim B$ means that $A \leq cB$, for some positive constant c independent of the variables in the expressions A and B. If $A \precsim B$ and $B \precsim A$, we write $A \approx B$ and say that A and B are equivalent. Certain well-known terms such as Banach function space, rearrangement invariant Banach function space, associate space, absolutely continuous norm, etc. will be used frequently in the sequel without their definitions. However, the reader may be found their definitions e.g., in [5, 7].

Definition 1. A positive and Lebesgue measurable function b is said to be *slowly varying* (s.v.) on $(0, \infty)$ in the sense of Karamata if, for each $\varepsilon > 0$, $t^{\varepsilon} b\,(t)$ is equivalent to a non-decreasing function and $t^{-\varepsilon} b\,(t)$ is equivalent to a non-increasing function on $(0, \infty)$.

The detailed study of Karamata Theory, properties and examples of slowly varying functions can be found in [5, 13, 14] and [17, Chap. V]. Given a s.v. function b on $(0, \infty)$, we denote by γ_b the positive function defined by

$$\gamma_b\,(t) = b\left(\max\left\{t, \tfrac{1}{t}\right\}\right) \qquad \text{for all} \quad t > 0.$$

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It is known that any slowly varying function b on $(0,\infty)$ is equivalent to a slowly varying continuous function $\widetilde{b}$ on $(0,\infty)$. Consequently, without loss of generality, we assume that all slowly varying functions in question are continuous functions in $(0,\infty)$ [14]. We shall need the following property of s.v. functions, for which we refer to [13, Lemma 3.1].

Lemma 1. *Let b be a slowly varying function on $(0,\infty)$.*

(i) *Let $r \in \mathbb{R}$. Then b^r is a slowly varying function on $(0,\infty)$ and $\gamma_b^r(t) = \gamma_{b^r}(t)$ for all $t > 0$.*

(ii) *Given positive numbers ε and κ, $\gamma_b(\kappa t) \approx \gamma_b(t)$, i.e., there are positive constants c_ε and C_ε such that*

$$c_\varepsilon \min\{\kappa^{-\varepsilon}, \kappa^{\varepsilon}\}\gamma_b(t) \le \gamma_b(\kappa t) \le C_\varepsilon \max\{\kappa^{-\varepsilon}, \kappa^{\varepsilon}\}\gamma_b(t) \tag{1.1}$$

for all $t > 0$.

(iii) *Let $\alpha > 0$. Then*

$$\int_0^t \tau^{\alpha-1}\gamma_b(\tau)\,\mathrm{d}\tau \approx t^{\alpha}\gamma_b(t) \quad \text{and} \quad \int_t^\infty \tau^{-\alpha-1}\gamma_b(\tau)\,\mathrm{d}\tau \approx t^{-\alpha}\gamma_b(t) \tag{1.2}$$

for all $t > 0$.

Let f be a complex-valued measurable function defined on Ω. Then the nonnegative rearrangement of f is given by

$$f^*(t) = \inf\{s > 0 : \ \mu(\{w \in \Omega : \ |f(w)| > s\}) \le t\} \tag{1.3}$$

where we assume that $\inf \phi = \infty$. Also the average(maximal) function of f on $(0,\infty)$ is given by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s. \tag{1.4}$$

Note that $f^*(\cdot)$ and $f^{**}(\cdot)$ are nonincreasing and right continuous functions.

Definition 2. Let $p, q \in (0,\infty]$ and let b be a slowly varying function on $(0,\infty)$. The Lorentz-Karamata (LK) space $L_{p,q;b}(\Omega, A, \mu)$ is defined to be the set of all measurable functions such that

$$\|f\|_{p,q;b} := \left\| t^{\frac{1}{p}-\frac{1}{q}}\gamma_b(t)\, f^{**}(t)\right\|_{q;(0,\infty)} \tag{1.5}$$

is finite. Here $\|\cdot\|_{q;(0,\infty)}$ stands for the usual L_q (quasi-) norm over the interval $(0,\infty)$.

The Lorentz, Lorentz-Zygmund and generalised Lorentz-Zygmund spaces are all special cases of these spaces, obtained by making particular choices of the slowly varying function b. It is known that the LK spaces endowed with a convenient norm (1.5), is a rearrangment-invariant Banach function spaces with upper and lower Boyd indices both equal to $\frac{1}{p}$ and have absolutely continuous norm when $p \in (1,\infty)$ and $q \in [1,\infty)$. Also, the dual $(L_{p,q;b}(\Omega))^*$ coincides with the associate space $L_{p',q';b^{-1}}(\Omega)$ where $1 < p, q < \infty$ and p', q' are the conjugate exponent of p, q respectively. It is clear that, for $0 < p < \infty$, the LK space $L_{p,q;b}(\Omega)$ contains the characteristic function of every measurable subset of Ω with finite measure and hence, by linearity, every μ-simple function. In this case, with a little thought, it is easy to see that the set of simple functions is dense in the LK space as the LK spaces have absolutely continuous norm for $p \in (1,\infty)$ and $q \in [1,\infty)$.

Let X be a Banach space and for a strongly measurable function $f\colon \Omega \to X$ define a function $\|f\|$ as

$$\|f\|(w) = \|f(w)\| \qquad \text{for all} \quad w \in \Omega.$$

Then the Lorentz-Karamata-Bochner space $L_{p,q;b}(\Omega, X)$, is a rearrangement invariant-Bochner space for $p, q \in (0,\infty]$ where the norm is defined by

$$\|f\|_{p,q;b} := \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) \|f\|^{**}(t) \right\|_{q;(0,\infty)}.$$

The Lorentz-Karamata-Bochner space $L_{p,q;b}(\Omega, X)$ is a Banach space and we still have the density of simple functions in it and its dual is

$$(L_{p,q;b}(\Omega, X))^* = L_{p',q';b^{-1}}(\Omega, X^*)$$

where X^* has the Radon-Nikodym property, $1 < p < \infty$ and $1 \leq q < \infty$.

For a strongly measurable function $u\colon \Omega \to B(X)$, the class of all bounded operators on Banach space X, the multiplication transformation $M_u\colon L_{p,q;b}(\Omega, X) \to L(\Omega, X)$ is defined as

$$(M_u f)(w) = u(w) f(w) \qquad \text{for all} \quad w \in \Omega$$

where $L(\Omega, X)$ is the space of all strongly measurable functions. These transformations are studied on various spaces by many researchers in ([2, 3, 4, 8, 10] and [16]), in particular on L_p space in ([8, 16]), on Orlicz space in ([10]), on Lorentz space in ([2] and [3]) and on Lorentz-Bochner space in ([4]). It is natural to extend the study to more general class. For the detail of these spaces one can refer to ([5, 7, 8] and [17]) and the references therein.

2. Characterizations

In this section, the multiplication operators with some of its properties like invertibility, range and compactness are discussed. Let u be a strongly measurable operator valued map on Ω. Then the boundedness, invertibility and compactness of the multiplication operator M_u induced by u is characterized in terms of u.

Theorem 1. *The multiplication transformation $M_u\colon L_{p,q;b}(\Omega,X) \to L_{p,q;b}(\Omega,X)$ is bounded if and only if $u \in L^{\infty}(\Omega, B(X))$.*

P r o o f. Assume that $u \in L^{\infty}(\Omega, B(X))$ with

$$\|u\|_{\infty} = \inf\big\{k > 0 : \ \mu\big(\{w \in \Omega : \ \|u(w)\| > k\}\big) = 0\big\} < \infty.$$

For the nonnegative rearrangement of $M_u f$, one can find the distribution function of $M_u f = u \cdot f$ as

$$\begin{aligned} \mu_{M_u f}(s) &= \mu\big\{w \in \Omega : \ \|(M_u f)(w)\| > s\big\} \\ &= \mu\big\{w \in \Omega : \ \|u(w)(f(w))\| > s\big\} \\ &\le \mu\big\{w \in \Omega : \ \|u\|_{\infty} \|f(w)\| > s\big\} \\ &\le \mu\big\{w \in \Omega : \ \|u\|_{\infty} \|f(w)\| > s\big\} = \mu_{\|u\|_{\infty} f}(s). \end{aligned} \tag{2.1}$$

Hence for each $t \ge 0$, by (2.1) we get

$$\big\{s > 0 : \ \mu_{\|u\|_{\infty} f}(s) \le t\big\} \subseteq \big\{s > 0 : \ \mu_{M_u f}(s) \le t\big\} \tag{2.2}$$

and

$$\begin{aligned} \|M_u f\|^{*}(t) &= \inf\big\{s > 0 : \ \mu_{M_u f}(s) \le t\big\} \\ &\le \inf\big\{s > 0 : \ \mu_{\|u\|_{\infty} f}(s) \le t\big\} \\ &= \inf\big\{s > 0 : \ \mu\{w \in \Omega : \ \|u\|_{\infty} \|f(w)\| > s\} \le t\big\} \\ &= \|u\|_{\infty} \|f\|^{*}(t). \end{aligned} \tag{2.3}$$

Also, we write that $\|M_u f\|^{**}(t) \le \|u\|_{\infty} \|f\|^{**}(t)$ by (2.3). Therefore,

$$\begin{aligned} \|M_u f\|_{p,q,b} &= \Big\|t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) \|M_u f\|^{**}(t)\Big\|_{q,(0,\infty)} \\ &\le \Big\|t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) \|u\|_{\infty} \|f\|^{**}(t)\Big\|_{q,(0,\infty)} \\ &= \|u\|_{\infty} \Big\|t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) \|f\|^{**}(t)\Big\|_{q,(0,\infty)} = \|u\|_{\infty} \|f\|_{p,q,b} \end{aligned} \tag{2.4}$$

can be written. Consequently, M_u is a bounded operator on $L_{p,q;b}(\Omega, X)$ with $p, q \in (0, \infty]$ and $\|M_u\| \le \|u\|_{\infty}$ by (2.4).

Conversely, let us assume that $u \notin L^{\infty}(\Omega, B(X))$ whereas M_u is bounded. Then, we can find a sequence (E_{n_i}) of disjoint measurable subsets with finite measure such that $n_1 = 1$, $n_i \to \infty$ as $i \to \infty$ and

$$\|u(w)\| > n_i \qquad \text{for all} \quad w \in E_{n_i}.$$

For each $w \in E_{n_i}$, let $x_{w_i} \in X$ be such that $\|x_{w_i}\| = 1$ and $\|u(w)\, x_{w_i}\| > n_i$. For each i, define f_i as

$$f_i(w) = \begin{cases} x_{w_i}, & \text{if } w \in E_{n_i} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|f_i\|_{p,q,b} \precsim \mu(E_{n_i})^{\frac{q}{p}} \gamma_{b^q}\big(\mu(E_{n_i})\big)$ and so $f_i \in L_{p,q;b}(\Omega, X)$ for each i and $\|M_u f_i\|^*(t) \succsim n_i \|f_i\|^*(t)$. Hence $\|M_u f_i\|_{p,q,b} \succsim n_i \|f\|_{p,q,b}$. This contradicts the boundedness of M_u. □

Theorem 2. *If M_u is a linear transformation from $L_{p,q;b}(\Omega, X)$ to itself, then M_u must be bounded.*

Proof. Proof can be derived from the proof of [10, Theorem 2.4]. □

Theorem 3. *Let (Ω, A, μ) be a finite measure space. Then $M_u \colon L_{p,q;b}(\Omega, X) \to L_{p,q;b}(\Omega, X)$ is invertible with inverse M_ν for some $\nu \in L^\infty(\Omega, B(X))$ if and only if*

1. *$u(w)$ is invertible for μ-almost all $w \in \Omega$ and*
2. *there exists $\varepsilon > 0$ such that $\|u(w)(x)\| \geq \varepsilon \|x\|$ for all x in X and μ-almost all $w \in \Omega$.*

Proof. Suppose M_u is invertible and $M_u^{-1} = M_\nu$ for some $\nu \in L^\infty(\Omega, B(X))$. For each $x \in X$, define $C_x \colon \Omega \to X$ as

$$C_x(w) = x \qquad \text{for all} \quad w \in \Omega.$$

Then $\|C_x\|^*(t) = \|x\|\, \chi_{[0,\mu(\Omega)]}(t)$ and

$$\|C_x\|^{**}(t) = \begin{cases} \|x\|, & \text{if } 0 \leq t < \mu(\Omega) \\ \frac{\|x\|}{t}\mu(\Omega), & \text{if } t \geq \mu(\Omega). \end{cases}$$

This says that

$$\begin{aligned} \|C_x\|_{p,q,b} &= \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) \|C_x\|^{**}(t) \right\|_{q,(0,\infty)} \\ &= \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) \|C_x\|^{**}(t) \right\|_{q,(0,\mu(\Omega))} + \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) \|C_x\|^{**}(t) \right\|_{q,(\mu(\Omega),\infty)} \\ &\precsim \|x\|\, \mu(\Omega)^{q/p} \gamma_{b^q}(\mu(\Omega)) + \|x\|\, \mu(\Omega)^{q-q/p} \gamma_{b^q}(\mu(\Omega)) \\ &= \|x\|\, \mu(\Omega)^q \gamma_{b^q}(\mu(\Omega)) \end{aligned}$$

by Lemma 1. Hence $C_x \in L_{p,q;b}(\Omega, X)$. Since $M_u^{-1} = M_\nu$, we have $M_\nu M_u = M_u M_\nu$ and

$$u(w)\, \nu(w)\, x = \nu(w)\, u(w)\, x = x$$

for all $x \in X$ and for all $w \in \Omega$. This implies $u(w)$ is invertible for all $w \in \Omega$. Also for each $x \in X$ and $w \in \Omega$, we get

$$\|x\| = \|\nu(w)\, u(w)\, x\| \leq \|\nu(w)\|\, \|u(w)\, x\| \leq \|\nu\|_\infty \|u(w)\, x\|.$$

If we choose $\varepsilon = \frac{1}{\|\nu\|_\infty}$, then $\|u(w)x\| \geq \varepsilon \|x\|$ for all $x \in X$ and for all $w \in \Omega$.

Conversely, assume that the conditions (1) and (2) are true. If $\nu(w)$ is inverse of $u(w)$, then we can write

$$u(w)\,\nu(w)\,f(w) = \nu(w)\,u(w)\,f(w)$$

for all $w \in \Omega$ and $f \in L_{p,q;b}(\Omega, X)$. Therefore, $M_\nu M_u f = M_u M_\nu f = f$ for all $f \in L_{p,q;b}(\Omega, X)$. By condition (2),

$$\varepsilon \|\nu(w)x\| \leq \|\nu(w)\,u(w)\,x\| = \|x\|$$
$$\|\nu(w)x\| \leq \varepsilon^{-1}\|x\|$$

are found for all $x \in X$ and for all $w \in \Omega$. This implies that $\nu \in L^\infty(\Omega, B(X))$ and M_u is invertible. $\square$

Theorem 4. *Let $M_u \in B(L_{p,q;b}(\Omega, X))$. M_u has closed range if and only if there exists $\varepsilon > 0$ such that*

$$\varepsilon \|x\| \leq \|u(w)x\|$$

for all $x \in X$, for almost all $w \in S$ where S is the support of u.

Proof. Suppose that there exists some $\varepsilon > 0$ such that $\|u(w)x\| \geq \varepsilon\|x\|$ for all $x \in X$ and for almost all $w \in S$. Let $L_{p,q;b}(S)$ denote the subspace of all those f in $L_{p,q;b}(\Omega, X)$ which vanish outside S, then $L_{p,q;b}(S)$ is a closed invariant subspace of M_u. Also for each f in $L_{p,q;b}(S)$, we have

$$\|M_u f\| \geq \varepsilon \|f\|.$$

This gives $M_u|_{L_{p,q;b}(S)}$ has closed range and so of M_u.

Conversely, assume that M_u has closed range. Then there exists an $\varepsilon > 0$ such that

$$\|M_u f\|_{p,q,b} \geq \varepsilon \|f\|_{p,q,b} \qquad \text{for all} \quad f \in L_{p,q;b}(\Omega, X).$$

Let $E = \left\{w \in S : \ \|u(w)x\| < \frac{\varepsilon}{2}\|x\| \ \text{ for some } \ x \in X\right\}$. If $\mu(E) > 0$, then we can find a measurable subset F of E such that $0 < \mu(F) \leq \mu(E) < \infty$. For each $w \in F$, let $x_w \in X$ such that

$$\|u(w)x_w\| < \frac{\varepsilon}{2}\|x_w\| \qquad \text{with} \quad \|x_w\| = 1.$$

Define $f_F : \Omega \to X$ as

$$f_F(w) = \begin{cases} x_w & \text{if } w \in F \\ 0 & \text{otherwise.} \end{cases}$$

Then, $f_F \in L_{p,q;b}(S)$ and $\|M_u f\|_{p,q,b} < \varepsilon \|f\|_{p,q,b}$. Therefore, $\mu(E) = 0$ and $\|u(w)x\| \geq \frac{\varepsilon}{2}\|x\|$ μ-almost all $w \in S$ and for all $x \in X$. $\square$

Theorem 5. *$M_u \in B\left(L_{p,q;b}\left(\Omega, X\right)\right)$ is compact if $L_{p,q;b}\left(u_\varepsilon\right)$ is finite dimensional for each $\varepsilon > 0$, where*

$$L_{p,q;b}\left(u_\varepsilon\right) = \left\{f \in L_{p,q;b}\left(\Omega, X\right) : \ f \ \text{ vanishes outside } \ u_\varepsilon\right\}.$$

P r o o f. For each $n \in \mathbb{N}$, if we define

$$u_n\left(w\right) = \begin{cases} u\left(w\right), & \text{if } w \in u_{\frac{1}{n}} \\ 0, & \text{otherwise,} \end{cases}$$

then it is easy to see that all operators M_{u_n}'s are compact. Also, for each $f \in L_{p,q;b}\left(\Omega, X\right)$ and for all $s \geq 0$, we have

$$\left\{w \in \Omega : \ \left\|\left(u_n - u\right)\left(w\right)\left(f\left(w\right)\right)\right\| \geq s\right\} \subseteq \left\{w \in \Omega : \ \left\|\left(f\left(w\right)\right)\right\| \geq ns\right\}$$

and so

$$\left\|\left(M_{u_n} - M_u\right) f\right\|^*\left(t\right) \leq \frac{1}{n}\left\|f\right\|^*\left(t\right).$$

Therefore $\left\|\left(M_{u_n} - M_u\right) f\right\|_{p,q,b} \precsim \frac{1}{n}\left\|f\right\|_{p,q,b}$ and M_u is compact. □

Theorem 6. *If u is a strongly measurable operator valued map such that for some $k > 0$, $\left\|u\left(w\right)x\right\| \geq k\left\|x\right\|$ for all $x \in X$ whenever $\left\|u\left(w\right)\right\| \geq k$, then $M_u \in B\left(L_{p,q;b}\left(\Omega, X\right)\right)$ is compact if and only if $L_{p,q;b}\left(u_\varepsilon\right)$ is finite dimensional for each $\varepsilon > 0$.*

P r o o f. Assume that $M_u \in B\left(L_{p,q;b}\left(\Omega, X\right)\right)$ is compact. Then $M_u|_{L_{p,q;b}\left(u_\varepsilon\right)}$ is also a compact operator and

$$\left\|M_u \chi_{u_\varepsilon} f\right\|_{p,q,b} \succsim \varepsilon\left\|\chi_{u_\varepsilon} f\right\|_{p,q,b}$$

for each $f \in L_{p,q;b}\left(\Omega, X\right)$. Since $M_u|_{L_{p,q;b}\left(u_\varepsilon\right)}$ is compact and invertable, we say that $L_{p,q;b}\left(u_\varepsilon\right)$ is finite dimensional for each $\varepsilon > 0$. The converse of the proof follows from the preceeding theorem. □

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