VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-012-0009-x
Math. Slovaca **62** (2012), No. 2, 281–292

$\mathcal{A}$-SUMMATION PROCESS AND KOROVKIN-TYPE APPROXIMATION THEOREM FOR DOUBLE SEQUENCES OF POSITIVE LINEAR OPERATORS

Sevda Karakuş — Kamil Demirci

(*Communicated by Wladyslaw Wilczynski*)

ABSTRACT. The aim of this paper is to present a Korovkin-type approximation theorem on the space of all continuous real valued functions on any compact subset of the real two-dimensional space by using a $\mathcal{A}$-summation process. We also study the rates of convergence of positive linear operators with the help of the modulus of continuity.

©2012
Mathematical Institute
Slovak Academy of Sciences

1. Introduction

The study of the Korovkin type approximation theory is an area of active research, which deals with the problem of approximating a function by means of a sequence of positive linear operators. Most of the classical approximation operators tend to converge to the value of the function being approximated. However, at points of discontinuity, they often converge to the average of the left and right limits of the function. The main purpose of using summability theory has always been to make a nonconvergent sequence converge. Some results regarding matrix summability for positive linear operators may be found in the paper [1], [2], [14].

In this paper we give a Korovkin-type approximation theorem on the space of all continuous real valued functions on any compact subset of the real two-dimensional space by using a $\mathcal{A}$-summation process. Also we study the rates of convergence of positive linear operators with the help of the modulus of continuity.

2010 Mathematics Subject Classification: Primary 41A25, 41A36.
Keywords: matrix summability, positive linear operators, Korovkin theory, modulus of continuity, rates of convergence.

A double sequence $x = \{x_{m,n}\}_{m,n\in\mathbb{N}}$ is convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{m,n} - L| < \varepsilon$ whenever $m, n > N$. In this case L is called the Pringsheim limit of x and is denoted by $P\text{-}\lim x = L$ (see [12]).

If there exists a positive number M such that $|x_{m,n}| \le M$ for all $(m,n) \in \mathbb{N}^2 = \mathbb{N}\times\mathbb{N}$, then $x = \{x_{m,n}\}$ is said to be bounded. Note that in contrast to the case for single sequences, a convergent double sequence need not to be bounded.

Let

$$A = [a_{j,k,m,n}], \qquad j,k,m,n \in \mathbb{N},$$

be a four-dimensional infinite matrix. For a given double sequence $x = \{x_{m,n}\}$, the A-transform of x, denoted by $Ax := \{(Ax)_{j,k}\}$, is given by

$$(Ax)_{j,k} = \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} x_{m,n}, \qquad j,k \in \mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(j,k) \in \mathbb{N}^2$. We say that a sequence x is $A-$summable to l if the A -transform of x exists for all $j,k \in \mathbb{N}$ and convergent in the Pringsheim's sense i.e.,

$$\lim_{p,q} \sum_{m\in\mathbb{N}}^{p} \sum_{n\in\mathbb{N}}^{q} a_{j,k,m,n} x_{m,n} = y_{j,k} \qquad \text{and} \qquad \lim_{j,k} y_{j,k} = l.$$

In summability theory, a two-dimensional matrix transformation is said to be regular if it maps every convergent sequence in to a convergent sequence with the same limit. The well-known characterization of regularity for two dimensional matrix transformations is known as Silverman-Toeplitz conditions (see, for instance, [5]). In 1926, Robison [13] presented a four dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a double P-convergent sequence is not necessarily bounded. The definition and the characterization of regularity for four dimensional matrices is known as Robison-Hamilton conditions, or briefly, RH-regularity. (see, [6, 13])

Recall that a four dimensional matrix $A = [a_{j,k,m,n}]$ is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The Robison-Hamilton conditions state that a four dimensional matrix $A = [a_{j,k,m,n}]$ is RH-regular if and only if

(i) $P\text{-}\lim\limits_{j,k} a_{j,k,m,n} = 0$ for each $(m,n) \in \mathbb{N}^2$,

(ii) $P\text{-}\lim\limits_{j,k} \sum\limits_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} = 1$,

(iii) $P\text{-}\lim\limits_{j,k} \sum\limits_{m\in\mathbb{N}} |a_{j,k,m,n}| = 0$ for each $n \in \mathbb{N}$,

(iv) $P\text{-}\lim\limits_{j,k} \sum\limits_{n\in\mathbb{N}} |a_{j,k,m,n}| = 0$ for each $m \in \mathbb{N}$,

(v) $\sum_{(m,n)\in\mathbb{N}^2} |a_{j,k,m,n}|$ is P-convergent for each $(j,k)\in\mathbb{N}^2$,

(vi) there exist finite positive integers A and B such that $\sum_{m,n>B} |a_{j,k,m,n}| < A$ holds for every $(j,k)\in\mathbb{N}^2$.

Now let $\mathcal{A} := \left\{A^{(i,l)}\right\} = \left\{a^{(i,l)}_{j,k,m,n}\right\}$ be a sequence of four-dimensional infinite matrices with non-negative real entries. For a given double sequence of real numbers, $x = \{x_{m,n}\}$ is said to be $\mathcal{A}$-summable to l if

$$P\text{-}\lim_{j,k} \sum_{(m,n)\in\mathbb{N}^2} a^{(i,l)}_{j,k,m,n} x_{m,n} = l$$

uniformly in i and l. If $A^{(i,l)} = A$, four-dimensional infinite matrix, then $\mathcal{A}$-summability is the A-summability for four-dimensional infinite matrix.

Some results regarding matrix summability method for double sequences may be found in the papers [10], [11], [15].

Now let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix, and let $K \subset \mathbb{N}^2$. Then, a real double sequence $x = \{x_{m,n}\}$ is said to be A -statistically convergent to a number L if, for every $\varepsilon > 0$,

$$P\text{-}\lim_{j,k} \sum_{(m,n)\in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \left\{(m,n)\in\mathbb{N}^2 : \ |x_{m,n} - L| \geq \varepsilon\right\}.$$

In this case we write $\mathrm{st}^2_A\text{-}\lim_{m,n} x_{m,n} = L$. Observe that, a P-convergent double sequence is A-statistically convergent to the same value but the converse does not hold true.

We should note that if we take $A = C(1,1)$, which is the double Cesáro matrix, then $C(1,1)$-statistical convergence coincides with the notion of statistical convergence for double sequence, which was introduced in [7, 9]. Finally, if we replace the matrix A by the identity matrix for four-dimensional matrices, then A-statistical convergence reduces to the Pringsheim convergence.

2. A Korovkin type theorem

The space of all continuous real valued functions on a compact subset D of $\mathbb{R}^2$, the real two-dimensional space is denoted by $C(D)$. This space is equipped with the supremum norm

$$\|f\|_{C(D)} = \sup_{(x,y)\in D} |f(x,y)| \qquad (f \in C(D)).$$

A sequence $\{L_{m,n}\}$ of positive linear operators of $C(D)$ into itself is called an $\mathcal{A}$-summation process on $C(D)$ if $\{L_{m,n}(f)\}$ is $\mathcal{A}$-summable to f for every $f \in C(D)$, i.e.,

$$P\text{-}\lim_{j,k\to\infty}\left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}f - f \right\|_{C(D)} = 0, \quad \text{uniformly in } i \text{ and } l \tag{2.1}$$

where it is assumed that the series in (2.1) converges for each $i, l, j, k \in \mathbb{N}$ and f.

In [3] we have studied conditions so that

$$P\text{-}\lim_{j,k\to\infty} \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} \|L_{m,n}f - f\|_{C(D)} = 0, \quad \text{uniformly in } i \text{ and } l \tag{2.2}$$

for all $f \in C(D)$.

In this paper, we establish a theorem of the Korovkin type with respect to the convergence behavior (2.1) for a double sequence of positive linear operators of $C(D)$ into itself. So the results of type (2.1) are extensions of type (2.2).

Let $\{L_{m,n}\}$ be sequence of positive linear operators of $C(D)$ into itself such that

$$\sup_{i,l,j,k} \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} \|L_{m,n}(1)\|_{C(D)} < \infty. \tag{2.3}$$

Furthermore, for $i, l, j, k \in \mathbb{N}$ and $f \in C(D)$, let

$$B_{j,k}^{(i,l)} f = \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f$$

which is well defined by (2.3) and belongs to $C(D)$.

Then Volkov [17] obtained the following Korovkin theorem.

Theorem 2.1. *Let $\{L_{m,n}\}$ be a sequence of positive linear operators acting from $C(D)$ into itself. Then, for all $f \in C(D)$*

$$P\text{-}\lim_{m,n} \|L_{m,n}f - f\|_{C(D)} = 0$$

if and only if

$$P\text{-}\lim_{m,n} \|L_{m,n}f_r - f_r\|_{C(D)} = 0, \quad (r = 0, 1, 2, 3)$$

where $f_0(x,y) = 1,\ f_1(x,y) = x,\ f_2(x,y) = y,\ f_3(x,y) = x^2 + y^2$.

A-statistical analog of Theorem 2.1 [4] can be given as follows.

Theorem 2.2. *Let $\{L_{m,n}\}$ be a sequence of positive linear operators acting from $C(D)$ into itself and $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix. Then, for all $f \in C(D)$*

$$\mathrm{st}_A^2\text{-}\lim_{m,n} \|L_{m,n}f - f\|_{C(D)} = 0$$

if and only if

$$\text{st}_A^2\text{-}\lim_{m,n} \|L_{m,n} f_r - f_r\|_{C(D)} = 0 \qquad (r = 0, 1, 2, 3)$$

where $f_0(x, y) = 1,\ f_1(x, y) = x,\ f_2(x, y) = y,\ f_3(x, y) = x^2 + y^2.$

If we replace the matrix A in Theorem 2.2 by the Cesáro matrix $C(1, 1)$, we immediately get the statistical Korovkin result.

Now we give the following generalization by using a $\mathcal{A}$-summation process.

Theorem 2.3. *Let* $\mathcal{A} = \{A^{(i,l)}\}$ *be a sequence of four-dimensional infinite matrices with non-negative real entries so that*

$$\sup_{i,l,j,k} \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} < \infty. \tag{2.4}$$

Let $\{L_{m,n}\}$ *be a sequence of positive linear operators acting from* $C(D)$ *into itself. Assume that* (2.3) *holds. Then, for all* $f \in C(D)$

$$P\text{-}\lim_{j,k} \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f - f \right\|_{C(D)} = 0 \quad \textit{uniformly in } i \textit{ and } l \tag{2.5}$$

if and only if

$$P\text{-}\lim_{j,k} \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f_r - f_r \right\|_{C(D)} = 0 \quad \textit{uniformly in } i \textit{ and } l \qquad (r = 0, 1, 2, 3) \tag{2.6}$$

where $f_0(x, y) = 1,\ f_1(x, y) = x,\ f_2(x, y) = y,\ f_3(x, y) = x^2 + y^2.$

Proof. Since each $f_r \in C(D)$, $(r = 0, 1, 2, 3)$, the implication (2.5) $\Longrightarrow$(2.6) is obvious.

Suppose now that (2.6) holds. Then, since $f \in C(D)$, then for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(u, v) - f(x, y)| < \varepsilon$ holds for all $(u, v) \in D$ satisfying $|u - x| < \delta$ and $|v - y| < \delta$. Let

$$D_\delta := [x - \delta, x + \delta] \times [y - \delta, y + \delta] \cap D.$$

Thus we may write that

$$\begin{aligned} &|f(u, v) - f(x, y)| \\ &= |f(u, v) - f(x, y)|\, \chi_D(u, v) + |f(u, v) - f(x, y)|\, \chi_{D\setminus D_\delta}(u, v) \\ &< \varepsilon + 2M \chi_{D\setminus D_\delta}(u, v), \end{aligned}$$

where χ_D denotes the characteristic function of the set D and $M := \|f\|_{C(D)}$. Observe that

$$\chi_{D\setminus D_\delta}(u, v) \le \frac{(u - x)^2}{\delta^2} + \frac{(v - y)^2}{\delta^2}. \tag{2.7}$$

Combining (2.7) with (2.7) we have

$$|f(u,v)-f(x,y)| \leq \varepsilon + \frac{2M}{\delta^2}\left\{(u-x)^2+(v-y)^2\right\} \tag{2.8}$$

for all u, v, x, y. Now using the linearity and the positivity of the operators $L_{m,n}$ and considering inequality (2.8) and (2.4) , we obtain

$$\begin{aligned}
&\left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(f;x,y) - f(x,y)\right| \\
&\leq \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(|f(u,v)-f(x,y)|;x,y) \\
&\quad + |f(x,y)|\left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(1;x,y) - 1\right| \\
&\leq \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}\left(\varepsilon + \frac{2M}{\delta^2}\left\{(u-x)^2+(v-y)^2\right\};x,y\right) \\
&\quad + \|f\|_{C(D)}\left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(1;x,y) - 1\right| \\
&\leq \varepsilon + \varepsilon\left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(1;x,y) - 1\right| \\
&\quad + \frac{2M}{\delta^2} \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}\left((u-x)^2;x,y\right) \\
&\quad + \frac{2M}{\delta^2} \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}\left((v-y)^2;x,y\right) \\
&\quad + M\left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(1;x,y) - 1\right| \\
&\leq \varepsilon + (\varepsilon + M)\left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f_0 - f_0\right| \\
&\quad + \frac{2M}{\delta^2}\Bigg\{\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}\left(u^2;x,y\right) - 2x \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(u;x,y) \\
&\quad + x^2 \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(1;x,y) + \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}\left(v^2;x,y\right) \\
&\quad - 2y \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(v;x,y) + y^2 \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(1;x,y)\Bigg\}
\end{aligned}$$

$$\begin{aligned}
&\leq \varepsilon + (\varepsilon + M)\left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f_0 - f_0\right| \\
&\quad + \frac{2M}{\delta^2}\left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}\left(u^2+v^2;x,y\right) - \left(x^2+y^2\right)\right| \\
&\quad + \frac{4M}{\delta^2}|x|\left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}\left(u;x,y\right) - x\right| \\
&\quad + \frac{2M}{\delta^2}x^2\left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}\left(f_0;x,y\right) - f_0\right| \\
&\quad + \frac{4M}{\delta^2}|y|\left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}\left(v;x,y\right) - y\right| \\
&\quad + \frac{2M}{\delta^2}y^2\left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}\left(f_0;x,y\right) - f_0\right| \\
&\leq \varepsilon + K\left\{\left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f_0 - f_0\right| + \left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f_1 - f_1\right|\right. \\
&\qquad\left. + \left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f_2 - f_2\right| + \left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f_3 - f_3\right|\right\}
\end{aligned}$$

where

$$K := \max\left\{\varepsilon + M + \tfrac{2M}{\delta^2}\left(A^2+B^2\right), \tfrac{4M}{\delta^2}A, \tfrac{4M}{\delta^2}B, \tfrac{2M}{\delta^2}\right\},$$
$$A := \max\left\{|a|,|b|\right\}, \qquad B := \max\left\{|c|,|d|\right\}.$$

Then, taking supremum over $(x,y)\in D$ we have

$$\begin{aligned}
&\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}\left(f;x,y\right) - f\left(x,y\right)\right\|_{C(D)} \\
&\leq \varepsilon + K\left\{\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f_0 - f_0\right\|_{C(D)} + \left\|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f_1 - f_1\right\|_{C(D)}\right. \\
&\quad\left. + \left\|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f_2 - f_2\right\|_{C(D)} + \left\|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f_3 - f_3\right\|_{C(D)}\right\}.
\end{aligned}$$

Using (2.6) and by taking limit as $j,k\to\infty$, we obtain the desired result. $\square$

If we take $A^{(i,l)} = I$, I being the four-dimensional identity matrix, then we immediately get Theorem 2.1.

Example 2.1. Now we present an example such that our new approximation result works but its classical case does not work. Let $I = [0,1]$ and $D := I \times I$. Now we consider the double sequence $\{T_{m,n}\}$ of positive linear operators defined by

$$T_{m,n}(f;x,y) = (1+\alpha_{m,n})\, B_{m,n}(f;x,y)$$

where $\{B_{m,n}\}$ is the Bernstein polynomials of two variables defined on $C(D)$ by

$$B_{m,n}(f;x,y) = \sum_{j=0}^{m}\sum_{k=0}^{n} f\left(\frac{j}{m},\frac{k}{n}\right) c_m^j x^j (1-x)^{m-j}\, c_n^k y^k (1-y)^{n-k}$$

(see [16]) and $(\alpha_{m,n})$ is a double sequence defined by $\alpha_{m,n} = (-1)^{m+n}$.

A double sequence $x = \{x_{m,n}\}$ of real numbers is called almost convergent to a limit s if

$$P\text{-}\lim_{p,q\to\infty} \sup_{j,k\geq 0} \left| \frac{1}{pq} \sum_{m=j}^{j+p-1} \sum_{n=k}^{k+q-1} x_{m,n} - s \right| = 0,$$

that is, the average value of $\{x_{m,n}\}$ taken over any rectangle

$$\left\{(m,n):\ j \leq m \leq j+p-1,\ \ k \leq n \leq k+q-1\right\}$$

tends to s as both p and q tend to ∞, and this convergence is uniform in j and k. Now assume that $\mathcal{A} = \left\{A^{(i,l)}\right\} = \left\{a_{j,k,m,n}^{(i,l)}\right\}$ is a sequence of four-dimensional infinite matrices defined by $a_{j,k,m,n}^{(i,l)} = \frac{1}{jk}$ if $i \leq m \leq j+i-1$, $l \leq n \leq k+l-1$ and $a_{j,k,m,n}^{(i,l)} = 0$ otherwise. In this case $\mathcal{A}$-summability method reduces to almost convergence of double sequences introduced by Móricz [8]. Observe that $(\alpha_{m,n})$ is almost convergent to zero, but it is not convergent in Pringsheim's sense. Also $(\alpha_{m,n})$ is not statistically convergent.

We get

$$\begin{aligned}
B_{m,n}(f_0;x,y) &= f_0(x,y)\\
B_{m,n}(f_1;x,y) &= f_1(x,y)\\
B_{m,n}(f_2;x,y) &= f_2(x,y)\\
B_{m,n}(f_3;x,y) &= f_3(x,y) + \frac{x-x^2}{m} + \frac{y-y^2}{n}
\end{aligned}$$

where $f_0(x,y) = 1,\ f_1(x,y) = x,\ f_2(x,y) = y,\ f_3(x,y) = x^2+y^2$. Hence, for the double sequence $\{T_{m,n}\}$, since $(\alpha_{m,n})$ is almost convergent to zero, we conclude that $\{T_{m,n}\}$ satisfy the conditions of Theorem 2.3. Also, since $(\alpha_{m,n})$ is not convergent in Pringsheim's sense and statistical sense, $\{T_{m,n}\}$ is not satisfy Theorem 2.1 and Theorem 2.2.

3. Rates of convergence

In this section we study the rates of convergence of a sequence of positive linear operators mapping $C(D)$ into itself with the help of the modulus of continuity.

Let $f \in C(D)$. Then the modulus of continuity of f, defined to be

$$w(f;\delta) = \sup\Big\{ |f(u,v) - f(x,y)| : (u,v),(x,y) \in D \text{ and } \sqrt{(u-x)^2 + (v-y)^2} \le \delta \Big\}$$

for $\delta > 0$. It is easy to see that, for any $c > 0$ and all $f \in C(D)$

$$w(f;c\delta) \le (1 + [c])\, w(f;\delta)$$

where $[c]$ is defined to be the greatest integer less than or equal to c.

Then we have the following result.

Theorem 3.1. *Let $\mathcal{A} = \{A^{(i,l)}\}$ be a sequence of four-dimensional infinite matrices with non-negative real entries so that*

$$\sup_{i,l,j,k} \sum_{(m,n)\in\mathbb{N}^2} a^{(i,l)}_{j,k,m,n} < \infty.$$

Let $\{L_{m,n}\}$ be a sequence of positive linear operators acting from $C(D)$ into itself. Assume that (2.3) and the following conditions hold:

(i) $P\text{-}\lim_{j,k} \Big\| \sum_{(m,n)\in\mathbb{N}^2} a^{(i,l)}_{j,k,m,n} L_{m,n} f_0 - f_0 \Big\|_{C(D)} = 0$, *uniformly in i and l,*

(ii) $P\text{-}\lim_{j,k\to\infty} w(f;\delta) = 0$, *uniformly in i and l,*
where $\delta := \delta^{(i,l)}_{(j,k)} := \sqrt{\Big\| \sum_{(m,n)\in\mathbb{N}^2} a^{(i,l)}_{j,k,m,n} L_{m,n}(\varphi) \Big\|_{C(D)}}$
with $\varphi(u,v) = (u-x)^2 + (v-y)^2$,

(iii) $P\text{-}\lim_{j,k} w(f;\delta) \Big\| \sum_{(m,n)\in\mathbb{N}^2} a^{(i,l)}_{j,k,m,n} L_{m,n} f_0 - f_0 \Big\|_{C(D)} = 0$, *uniformly in i and l, where* $\delta := \delta^{(i,l)}_{(j,k)}$.

Then we have, for all $f \in C(D)$, $P\text{-}\lim_{j,k} \Big\| \sum_{(m,n)\in\mathbb{N}^2} a^{(i,l)}_{j,k,m,n} L_{m,n} f - f \Big\|_{C(D)} = 0$, *uniformly in i and l.*

Proof. Let $(x,y) \in D$ and $f \in C(D)$ be fixed. Using the linearity and the positivity of the operators $L_{m,n}$, for all $(m,n) \in \mathbb{N}^2$ and any $\delta > 0$, we have

$$\begin{aligned}
&\left| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f - f \right| \\
&\leq \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} \left(\left| f(u,v) - f(x,y) \right| ; x,y \right) \\
&\quad + \left| f(x,y) \right| \left| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} (1;x,y) - 1 \right| \\
&\leq \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} \left(w\left(f; \delta \frac{\sqrt{(u-x)^2 + (v-y)^2}}{\delta} \right); x,y \right) \\
&\quad + \|f\|_{C(D)} \left| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} (1;x,y) - 1 \right| \\
&\leq \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} \left(\left(1 + \left[\frac{\sqrt{(u-x)^2 + (v-y)^2}}{\delta} \right] \right) w(f;\delta); x,y \right) \\
&\quad + \|f\|_{C(D)} \left| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} (1;x,y) - 1 \right| \\
&\leq \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} w(f;\delta) L_{m,n} \left(\left(1 + \frac{(u-x)^2 + (v-y)^2}{\delta^2} \right); x,y \right) \\
&\quad + \|f\|_{C(D)} \left| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} (1;x,y) - 1 \right| \\
&\leq w(f;\delta) \left| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f_0 - f_0 \right| \\
&\quad + \|f\|_{C(D)} \left| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f_0 - f_0 \right| \\
&\quad + w(f;\delta) + \frac{w(f;\delta)}{\delta^2} \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} \left((u-x)^2 + (v-y)^2 ; x,y \right).
\end{aligned}$$

Then taking supremum over $(x,y)\in D$, we have

$$\begin{aligned}
&\left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f - f \right\|_{C(D)} \\
&\qquad \leq \|f\|_{C(D)} \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}\left(f_0;x,y\right) - f_0 \right\|_{C(D)} \\
&\qquad\quad + w\left(f;\delta\right) \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}\left(f_0;x,y\right) - f_0 \right\|_{C(D)} \\
&\qquad\quad + \frac{w\left(f;\delta\right)}{\delta^2} \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}\left(\varphi;x,y\right) \right\|_{C(D)} + w\left(f;\delta\right).
\end{aligned}$$

Now, if we take $\delta := \delta_{(j,k)}^{(i,l)} := \sqrt{\left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}\left(\varphi;x,y\right) \right\|_{C(D)}}$ then we may write that

$$\begin{aligned}
&\left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f - f \right\|_{C(D)} \\
&\qquad \leq \|f\|_{C(D)} \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}\left(f_0;x,y\right) - f_0 \right\|_{C(D)} \\
&\qquad\quad + w\left(f;\delta\right) \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}\left(f_0;x,y\right) - f_0 \right\|_{C(D)} + 2w\left(f;\delta\right)
\end{aligned}$$

and hence

$$\begin{aligned}
&\left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f - f \right\|_{C(D)} \\
&\qquad \leq K \left\{ \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f_0 - f_0 \right\|_{C(D)} \right. \\
&\qquad\quad \left. + w\left(f;\delta\right) \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} f_0 - f_0 \right\|_{C(D)} + w\left(f;\delta\right) \right\}
\end{aligned}$$

where $K = \max\left\{2, \|f\|\right\}$. Letting $j,k\to\infty$ and using (i), (ii) and (iii) the proof is completed. $\square$

REFERENCES

[1] ATLIHAN, Ö. G.—ORHAN, C.: *Matrix summability and positive linear operators*, Positivity **11** (2007), 387–389.

[2] ATLIHAN, Ö. G.—ORHAN, C.: *Summation process of positive linear operators*, Comput. Math. Appl. **56** (2008), 1188–1195.

[3] DEMİRCİ, K.—DİRİK, F.—KARAKUŞ, S.: *A-summability and Korovkin-type approximation theorem for double sequences of positive linear operators* (Submitted).

[4] DİRİK, F.—DEMİRCİ, K.: *Korovkin type approximation theorem for functions of two variables in statistical sense*, Turkish J. Math. **34** (2010), 73–83.

[5] HARDY, G. H.: *Divergent Series*, Oxford Univ. Press, London, 1949.

[6] HAMILTON, H. J.: *Transformations of multiple sequences*, Duke Math. J. **2** (1936), 29–60.

[7] MÓRICZ, F.: *Statistical convergence of multiple sequences*, Arch. Math. (Basel) **81** (2004), 82–89.

[8] MÓRICZ, F.—RHOADES, B. E.: *Almost convergence of double sequences and strong regularity of summability matrices*, Math. Proc. Cambridge Philos. Soc. **104** (1988), 283–294.

[9] MURSALEEN—EDELY, O. H. H.: *Statistical convergence of double sequences*, J. Math. Anal. Appl. **288** (2003), 223–231.

[10] MURSALEEN—SAVAŞ, E.: *Almost regular matrices for double sequences*, Studia Sci. Math. Hungar. **40** (2003), 205–212.

[11] PATTERSON, R. F.—SAVAŞ, E.: *Uniformly summable double sequences*, Studia Sci. Math. Hungar. **44** (2007), 147–158.

[12] PRINGSHEIM, A.: *Zur theorie der zweifach unendlichen zahlenfolgen*, Math. Ann. **53** (1900), 289–321.

[13] ROBISON, G. M.: *Divergent double sequences and series.* In: Amer. Math. Soc. Transl. Ser. 2, Vol. 28, Amer. Math. Soc., Providence, RI, 1926, pp. 50–73.

[14] RADU, C.: *A-summability and approximation of continuous periodic functions*, Studia Univ. Babeş-Bolyai Math. **52** (2007), 155–161.

[15] SAVAŞ, E.—RHOADES, B. E.: *Double summability factor theorems and applications*, Math. Inequal. Appl. **10** (2007), 125–149.

[16] STANCU, D. D.: *A method for obtaining polynomials of Bernstein type of two variables*, Amer. Math. Monthly **70** (1963), 260–264.

[17] VOLKOV, V. I.: *On the convergence of sequences of positive linear operators in the space of two variables*, Dokl. Akad. Nauk. SSSR (N.S.) **115** (1957), 17–19.

Received 23. 9. 2009
Accepted 10. 2. 2010

Department of Mathematics
Faculty of Arts and Science
Sinop University
Sinop, 57000
TURKEY

E-mail: skarakus@sinop.edu.tr
kamild@sinop.edu.tr