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GENERALIZED FUZZY n-ARY SUBHYPERGROUPS OF A COMMUTATIVE n-ARY HYPERGROUP

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ABSTRACT. In this paper, by means of a new idea, the concept of (invertible) $(\in, \in\vee q)$-fuzzy n-ary subhypergroups of a commutative n-ary hypergroup is introduced and some related properties are investigated. A kind of n-ary quotient hypergroup by an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup is provided and the relationships among $(\in, \in\vee q)$-fuzzy n-ary subhypergroups, n-ary quotient hypergroups and homomorphism are investigated. Several isomorphism theories of n-ary hypergroups are established.

1. Introduction

The concept of hyperstructure was first introduced by Marty [30] at the eighth Congress of Scandinavian Mathematicians in 1934. Later on, people have observed that hyperstructures have many applications in both pure and applied sciences. A comprehensive review of the theory of hyperstructures can be found in [6, 36, 15]. In a recent book of Corsini and Leoreanu [9], the authors have collected numerous applications of algebraic hyperstructures, especially those from the last fifteen years to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities. Sets endowed with one n-ary operation having different properties were investigated

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by many researchers. Such systems have many applications in different branches such as automata, quantum groups, combinatorics, cryptology, errordetecting and error-correcting coding theory and so on(see [18, 31, 35] for details). The concept of n-ary groups was first introduced by Dönte [17]. In his paper Dörnte observed that any n-ary groupoid (G, f) of the form $f(x_1, \ldots, x_n)$ is an n-ary group, where $(G, \cdot)$ is a group but for every $n > 2$ there are n-ary groups which are not of this form. n-hypergroups have been introduced by Davvaz and Vougiouklis in [16] as a generalization of hypergroups in the sense of Marty and then studied by Leoreanu and Davvaz [26] and Leoreanu and Corsini [25].

After introducing the concept of fuzzy sets by Zadeh in 1965 [41], there are many papers devoted to fuzzify the classical mathematics into fuzzy mathematics. On the other hand, because of the importance of group theory in mathematics as well as its many areas of application, the notion of fuzzy subgroups was defined by Rosenfeld in [33]. Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and so on. This provides sufficient motivations for researchers to review various concepts and results from the realm of abstract algebra to a broader framework of fuzzy setting, see [29]. The relationships between the fuzzy sets and algebraic hyperstructures (structures) have been considered by Corsini, Davvaz, Kehagias, Leoreanu, Vougiouklis, Yin, Zhan and others. The reader is refereed to [7, 8, 10, 11, 13, 21, 22, 24, 27, 34, 36, 38, 39, 40, 42, 43]. Using the notion "belongingness $(\in)$" and "quasi-coincidence (q)" of a fuzzy point with a fuzzy set introduced by Pu and Liu [32], the concept of (α, β)-fuzzy subgroups where α, β are any two of $\{\in, q, \in\vee q, \in\wedge q\}$ with $\alpha \neq \in\wedge q$ was introduced by Bhakat and Das [1] in 1992, in which the $(\in, \in\vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. The detailed study with $(\in, \in\vee q)$-fuzzy subgroup has been considered in Bhakat and Das [2] and Bhakat [4, 3]. The concept of an $(\in, \in\vee q)$-fuzzy subring and ideal of a ring have been introduced in Bhakat and Das [5]. In [12], Davvaz and Corsini introduced the concept of $(\in, \in\vee q)$-fuzzy subhyperquasigroups of hyperquasigroups. And the concept of interval-valued $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of an n-ary hypergroup was introduced and some related properties are investigated by Davvaz et al. [14].

In this paper, using the notion "belongingness" and "quasi-coincidence" of a fuzzy point with a fuzzy set, we will introduce a new ordering relation, called fuzzy inclusion or quasicoincidence relation. By using this new idea, we will introduce and investigate (invertible) $(\in, \in\vee q)$-fuzzy n-ary subhypergroups of a commutative n-ary hypergroup. The rest of this paper is organized as follows. In Section 2, we summarize some basic concepts in n-ary hyperstructures which will be used throughout the paper. In Section 3, using a new ordering relation, we introduce and study (invertible) $(\in, \in\vee q)$-fuzzy n-ary subhypergroups of a commutative n-ary hypergroup and several characterizations of them are presented. A kind of n-ary quotient hypergroup by an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup is also provided and studied. In Section 4, the relationships among $(\in, \in\vee q)$-fuzzy n-ary subhypergroups, n-ary quotient hypergroups and homomorphism are investigated. Several isomorphism theories of n-ary hypergroups are established. Some conclusions are given in the last section.

2. Fuzzy sets in n-ary hypergroups

In this section, we summarize some basic concepts (see [13] or [28]) which will be used throughout the paper.

We will concern primarily with a basic non-empty set H. Denote by H^n the cartesian product $H \times \cdots \times H$, where H appears n times. An element of H^n will be denoted by $(x_1, \ldots, x_n)$, where $x_i \in H$ for all $1 \leq i \leq n$. In general, a mapping $f\colon H^n \to \mathcal{P}^*(H)$ is called an *n-ary hyperoperation* and n is called the *arity* of the hyperoperation f. Let f be an n-ary hyperoperation on H and $A_1, \ldots, A_n$ subsets in H. Define

$$f(A_1, \ldots, A_n) = \bigcup \big\{ f(x_1, \ldots, x_n) \mid x_i \in A_i,\ 1 \leq i \leq n \big\}.$$

In the sequel, we shall denote the sequence $x_i, x_{i+1}, \ldots, x_j$ by x_i^j. For $j < i$, $x_i^j = \emptyset$. Thus

$$f(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_n)$$

will be written as $f(x_1^i,\ y_{i+1}^j,\ z_{j+1}^n)$. Also $f(a_1^i, x*)$ means $f(a_1^i, \underbrace{x, \ldots, x}_{n-i})$ for a_1^i, $x \in H$ and $1 \leq i \leq n$.

H with an n-ary hyperoperation $f\colon H^n \to \mathcal{P}^*(H)$ is called an *n-ary hypergroupoid* and will be denoted by (H, f). An n-ary hypergroupoid (H, f) is said to be *commutative* if for all $x_1^n \in H$, and any permutation ρ of $\{1, \ldots, n\}$, we have $f(x_1^n) = f(x_{\rho_1}, x_{\rho_2}, \ldots, x_{\rho_n})$. An n-ary hypergroupoid (H, f) is said to be an *n-ary semihypergroup* if the following associative axiom holds:

$$f\big(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}\big) = f\big(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}\big)$$

for all $i, j \in \{1, 2, \ldots, n\}$ and $x_1^{2n-1} \in H$. An n-ary semihypergroup is said to be an *n-ary hypergroup* if for all $y, y_1^n \in H$ and fixed $i \in \{1, \ldots, n\}$, there exists $x \in H$ such that $y \in f(y_1^{i-1}, x, y_{i+1}^n)$. An element e of H is called an *identity* of H if for any $x \in H$ and $i \in \{1, \ldots, n\}$, we have $x \in f(\underbrace{e, \ldots, e}_{i-1}, x, \underbrace{e, \ldots, e}_{n-i})$.

Definition 2.1. Let (H, f) be an n-ary hypergroup and K a non-empty subset of H. We say that K is an *n-ary subhypergroup* of H if the following conditions hold:

(i) K is closed under the n-ary hyperoperation f;

(ii) For all $k, k_1^n \in K$ and fixed $i \in \{1, \ldots, n\}$, there exists $x \in K$ such that $k \in f(k_1^{i-1}, x, k_{i+1}^n)$.

In what follows, (H, f) denotes a commutative n-ary hypergroup with an identity e unless otherwise stated.

An n-ary subhypergroup K of H is said to be *invertible* if for any $x, y \in H$, $x \in f(y, \underbrace{K, \ldots, K}_{n-1})$ implies $y \in f(x, \underbrace{K, \ldots, K}_{n-1})$.

Next we recall some fuzzy logic concepts. Fuzzy sets were introduced by Zadeh [41] as a generalization of crisp sets. Let X be a non-empty set. A fuzzy subset μ of X is defined as a mapping from X to $[0, 1]$. The set of all fuzzy subsets of X is denoted by $\mathcal{F}(X)$. For $\mu, \nu \in \mathcal{F}(X)$, by $\mu \subseteq \nu$ we mean that $\mu(x) \leq \nu(x)$ for all $x \in X$ (The $\subseteq$ is called *pointwise order* in lattice theory). And the *union* and *intersection* of μ and ν, denoted by $\mu \cup \nu$ and $\mu \cap \nu$, are defined as the fuzzy subsets of X by $(\mu \cup \nu)(x) = \mu(x) \vee \nu(x)$ and $(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x)$, respectively, for all $x \in X$.

For any $A \subseteq X$ and $r \in (0, 1]$, the fuzzy subset r_A of X is defined by

$$r_A(x) = \begin{cases} r & \text{if } x \in A, \\ 0, & \text{otherwise,} \end{cases}$$

for all $x \in X$. In particular, when $r = 1$, r_A is said to be the *characteristic function* of A, denoted by χ_A; when $A = \{x\}$, the r_A is said to be a *fuzzy point* with support x and value r and is denoted by r_x. A fuzzy point r_x is said to *belong to* (resp., be *quasi-coincident with*) a fuzzy subset μ, written as $r_x \in \mu$ (resp., $r_x \, q \, \mu$), if $\mu(x) \geq r$ (resp., $\mu(x) + r > 1$). If $\mu(x) \geq r$ or $\mu(x) + r > 1$, then we write $r_x \in\!\vee q \, \mu$.

For $\mu \in \mathcal{F}(X)$ and $r \in (0, 1]$. The sets $\mu_r = \big\{x \in X \mid \mu(x) \geq r\big\}$ and $[\mu]_r = \{x \in X \mid r_x \in\!\vee q \, \mu\}$ are called a *level subset* of μ and an *$\in\!\vee q$-level subset* of μ, respectively. And μ is said to have sup-property if for any $A \in \mathcal{P}^*(H)$, there exists $x_0 \in A$ such that $\mu(x_0) = \bigvee_{x \in A} \mu(x)$. We shall use the following abbreviated notation: the sequence $\mu(x_i), \mu(x_{i+1}), \ldots, \mu(x_j)$ will be denoted by $\mu_{x_i}^{x_j}$. For $j < i$, $\mu_{x_i}^{x_j} = 0$.

Now let us define a new ordering relation $\subseteq\!\vee q$ on $\mathcal{F}(X)$, which is called a *fuzzy inclusion or quasi-coincidence relation*, as follows.

For any $\mu, \nu \in \mathcal{F}(X)$, by $\mu \subseteq\vee q\ \nu$ we mean that $r_x \in \mu$ implies $r_x \in\vee q\ \nu$ for all $x \in X$ and $r \in (0, 1]$. Moreover, μ and ν are said to be $(0, 0.5)$-*fuzzy equal*, denoted by $\mu \approx \nu$, if $\mu \subseteq\vee q\ \nu$ and $\nu \subseteq\vee q\ \mu$.

In the sequel, unless otherwise stated, $M(r_1, r_2, \ldots, r_n)$, where n is a positive integer, will denote $r_1 \wedge r_2 \wedge \cdots \wedge r_n$ for all $r_1, \ldots, r_n \in [0, 1]$, $\overline{\in\vee q}$ means $\in\vee q$ does not hold and $\overline{\subseteq\vee q}$ implies $\subseteq\vee q$ is not true.

Lemma 2.2. *Let $\mu, \nu \in \mathcal{F}(X)$. Then $\mu \subseteq\vee q\ \nu$ if and only if $\nu(x) \geq M(\mu(x), 0.5)$ for all $x \in X$.*

P r o o f. Assume that $\mu \subseteq\vee q\ \nu$. Let $x \in X$. If $\nu(x) < M(\mu(x), 0.5)$, then there exists r such that $\nu(x) < r < M(\mu(x), 0.5)$, that is, $r_x \in \mu$ but $r_x\ \overline{\in\vee q}\ \nu$, a contradiction. Hence $\nu(x) \geq M(\mu(x), 0.5)$.

Conversely, assume that $\nu(x) \geq M(\mu(x), 0.5)$ for all $x \in X$. If $\mu\ \overline{\subseteq\vee q}\ \nu$, then there exists $r_x \in \mu$ but $r_x\ \overline{\in\vee q}\ \nu$, and so $\mu(x) \geq r$, $\nu(x) < r$ and $\nu(x) < 0.5$, which contradicts $\nu(x) \geq M(\mu(x), 0.5)$. □

Lemma 2.3. *Let $\mu, \nu, \omega \in \mathcal{F}(X)$ be such that $\mu \subseteq \vee q\ \nu \subseteq\vee q\ \omega$. Then $\mu \subseteq\vee q\ \omega$.*

P r o o f. It is straightforward by Lemma 2.2. □

Lemma 2.2 gives that $\mu \approx \nu$ if and only if $M(\mu(x), 0.5) = M(\nu(x), 0.5)$ for all $x \in X$ and $\mu, \nu \in \mathcal{F}(X)$, and it follows from Lemmas 2.2 and 2.3 that $\approx$ is an equivalence relation on $\mathcal{F}(X)$.

Next, we introduce a fuzzy n-ary hyperoperation on an n-ary hypergroupoid as follows.

Definition 2.4. Let (H, f) be an n-ary hypergroupoid and $\mu_1, \ldots, \mu_n \in \mathcal{F}(H)$. We define a fuzzy n-ary hyperoperation $F\colon \underbrace{\mathcal{F}(H) \times \cdots \times \mathcal{F}(H)}_{n} \to \mathcal{F}(H)$ by

$$F(\mu_1, \ldots, \mu_n)(x) = \bigvee_{x \in f(y_1^n)} M(\mu_1(y_1), \ldots, \mu_n(y_n))$$

for all $x \in H$. In particular, for any $x_1^n \in H$ and $i \in \{1, \ldots, n\}$, define

$$F(\mu_1, \ldots, \mu_i, x_{i+1}^n)(x) = \bigvee_{x \in f(y_1^i, x_{i+1}^n)} M(\mu_1(y_1), \ldots, \mu_i(y_i))$$

for all $x \in H$.

Note that for any $r_1^n \in (0, 1]$ and $x_1^n \in H$, by Definition 2.4, $F(r_{1_{x_1}}, \ldots, r_{n_{x_n}}) = M(r_1^n)_{f(x_1^n)}$.

In the following, we shall denote the sequence $\mu_i, \mu_{i+1}, \ldots, \mu_j$ by μ_i^j. For $j < i$, μ_i^j is the zero fuzzy set. The following elementary facts follow easily from the definition.

Lemma 2.5. *Let (H, f) be an n-ary hypergroupoid, and $\mu_1^{2n-1}, \nu_1^n \in \mathcal{F}(H)$. Then:*

(1) *If (H, f) is an n-ary semihypergroup, then*

$$F\big(\mu_1^{i-1}, F(\mu_i^{n+i-1}), \mu_{n+i}^{2n-1}\big) = F\big(\mu_1^{j-1}, F(\mu_j^{n+j-1}), \mu_{n+j}^{2n-1}\big)$$

for all $i, j \in \{1, \ldots, n\}$.

(2) *If (H, f) is commutative, then the value of $F(\mu_1, \ldots, \mu_n)$ does not depend on the permutation of $\mu_1, \ldots, \mu_n$.*

(3) *If $\mu_i \subseteq\vee q\ \nu_i$ for all $i \in \{1, \ldots, n\}$, then $F(\mu_1^n) \subseteq\vee q\ F(\nu_1^n)$.*

Lemma 2.5 indicates that for an n-ary semihypergroup (H, f), $(\mathcal{F}(H), F)$ is an n-ary semigroup and the relation $\approx$ is a congruence relation on $(\mathcal{F}(H), F)$.

3. (Invertible) $(\in, \in\vee q)$-fuzzy n-ary subhypergroups of a commutative n-ary hypergroup

In this section, we will introduce and investigate the concepts of $(\in, \in\vee q)$-fuzzy n-ary subhypergroups of a commutative n-ary hypergroup. Let us start by giving the following concept.

Definition 3.1. Let $\mu \in \mathcal{F}(H)$. Then μ is called an $(\in, \in\vee q)$-*fuzzy n-ary subhypergroup* of H if it satisfies the following conditions:

(F1a) for all $x \in H$, $\mu(e) \geq M(\mu(x), 0.5)$;

(F2a) for all $x_1^n \in H$, $F(\mu(x_1)_{x_1}, \ldots, \mu(x_n)_{x_n}) \subseteq\vee q\ \mu$;

(F3a) for all $y, y_1^{n-1} \in H$, there exists $x \in H$ such that

$$M\big(\mu_{y_1}^{y_{n-1}}, \mu(y)\big)_y \subseteq\vee q\ F\big(\mu(x)_x, \mu(y_1)_{y_1}, \ldots, \mu(y_{n-1})_{y_{n-1}}\big).$$

An $(\in, \in\vee q)$-fuzzy n-ary subhypergroup μ of H is said to be *invertible* if it satisfies

(F4a) for all $x, y \in H$ and $r \in (0, 1]$,

$$r_x \in F(r_y, \underbrace{\mu, \ldots, \mu}_{n-1}) \implies r_y \in\vee q\ F(r_x, \underbrace{\mu, \ldots, \mu}_{n-1}).$$

Next let us first provide some auxiliary lemmas as follows.

Lemma 3.2. *Let $\mu \in \mathcal{F}(H)$. Then* (F2a) *holds if and only if one of the following conditions holds:*

(F2b) *for all $r_{i_{x_i}} \in \mu$ $(i \in \{1,\dots,n\})$, $F(r_{1_{x_1}},\dots,r_{n_{x_n}}) \subseteq\vee q\ \mu$.*

(F2c) $\bigwedge\limits_{x\in f(x_1^n)} \mu(x) \geq M(\mu_{x_1}^{x_n}, 0.5)$ *for all $x_1^n \in H$.*

(F2d) $F(\underbrace{\mu,\dots,\mu}_{n}) \subseteq\vee q\ \mu$.

Proof.

(F2a) $\Longleftrightarrow$ (F2b) This is straightforward.

(F2b) $\Longrightarrow$ (F2c) Let $x \in H$ and $x_1^n \in H$ be such that $x \in f(x_1^n)$. Then, by Lemma 2.2 and (F2b), we have

$$\begin{aligned}\mu(x) &\geq M(F(\mu(x_1)_{x_1},\dots,\mu(x_n)_{x_n})(x), 0.5)\\ &= M((M(\mu_{x_1}^{x_n})_{f(x_1^n)})(x), 0.5) = M(\mu_{x_1}^{x_n}, 0.5).\end{aligned}$$

This implies $\bigwedge\limits_{x\in f(x_1^n)} \mu(x) \geq M(\mu_{x_1}^{x_n}, 0.5)$ for all $x_1^n \in H$ and so (F2c) holds.

(F2c) $\Longrightarrow$ (F2d) Assume that (F2c) holds. Then for any $x \in H$, we have

$$\begin{aligned}M(F(\underbrace{\mu,\dots,\mu}_{n})(x), 0.5) &= M\Bigg(\bigvee_{x\in f(y_1^n)} M(\mu_{y_1}^{y_n}), 0.5\Bigg)\\ &= \bigvee_{x\in f(y_1^n)} M(\mu_{y_1}^{y_n}, 0.5) \leq \mu(x)\end{aligned}$$

and so $F(\underbrace{\mu,\dots,\mu}_{n}) \subseteq\vee q\ \mu$ by Lemma 2.2.

(F2d) $\Longrightarrow$ (F2a) Let $x_1^n \in H$. Since $\mu(x_i)_{x_i} \in \mu$ for all $i \in \{1,2,\dots,n\}$, we have

$$F(\mu(x_1)_{x_1}, \mu(x_2)_{x_2},\dots,\mu(x_n)_{x_n}) \subseteq F(\underbrace{\mu,\dots,\mu}_{n}) \subseteq\vee q\ \mu,$$

and so $F(\mu(x_1)_{x_1}, \mu(x_2)_{x_2},\dots,\mu(x_n)_{x_n}) \subseteq\vee q\ \mu$. □

Lemma 3.3. *Let $\mu \in \mathcal{F}(H)$. Then*

(1) (F3a) *holds if and only if the following condition holds:*
(F3c) *for all $y, y_1^{n-1} \in H$, there exists $x \in H$ such that*

$$y \in f(x, y_1^{n-1}) \quad \text{and} \quad \mu(x) \geq M(\mu_{y_1}^{y_{n-1}}, \mu(y), 0.5).$$

(2) *If* (F3a) *holds, then the following condition holds:*
(F3d) *for all $y_1^{n-1} \in H$,*

$$\mu \cap M(\mu_{y_1}^{y_{n-1}})_H \subseteq\vee q\ F(\mu, \mu(y_1)_{y_1},\dots,\mu(y_{n-1})_{y_{n-1}})$$

and (F3d) *implies* (F3a) *if μ has the sup-property or H is finite.*

Proof. (F3a) $\Longrightarrow$ (F3c) Let $y, y_1^{n-1} \in H$. By (F3a), there exists $x \in H$ such that

$$M(\mu_{y_1}^{y_{n-1}}, \mu(y))_y \subseteq\vee q\ F(\mu(x)_x, \mu(y_1)_{y_1}, \dots, \mu(y_{n-1})_{y_{n-1}}).$$

It follows from Lemma 2.2 that

$$\begin{aligned} F(\mu(x)_x, \mu(y_1)_{y_1}, \dots, \mu(y_{n-1})_{y_{n-1}})(y) &= \left(M(\mu_{y_1}^{y_{n-1}}, \mu(x))_{f(x,y_1^{n-1})}\right)(y) \\ &\geq M((M(\mu_{y_1}^{y_{n-1}}, \mu(y))_y)(y), 0.5) \\ &= M(\mu_{y_1}^{y_{n-1}}, \mu(y), 0.5). \end{aligned}$$

This implies $y \in f(x, y_1^{n-1})$ and $M(\mu_{y_1}^{y_{n-1}}, \mu(x)) \geq M(\mu_{y_1}^{y_{n-1}}, \mu(y), 0.5)$. Hence

$$\mu(x) \geq M(\mu_{y_1}^{y_{n-1}}, \mu(x)) \geq M(\mu_{y_1}^{y_{n-1}}, \mu(y), 0.5)$$

and so (F3c) holds.

(F3c) $\Longrightarrow$ (F3a) Let $y, y_1^{n-1} \in H$. By (F3c), there exists $x \in H$ such that

$$y \in f(x, y_1^{n-1}) \qquad \text{and} \qquad \mu(x) \geq M(\mu_{y_1}^{y_{n-1}}, \mu(y), 0.5).$$

This implies $M(\mu_{y_1}^{y_{n-1}}, \mu(x)) \geq M(\mu_{y_1}^{y_{n-1}}, \mu(y), 0.5)$ and so

$$\left(M(\mu_{y_1}^{y_{n-1}}, \mu(x))_{f(x,y_1^{n-1})}\right)(y) \geq M(\mu_{y_1}^{y_{n-1}}, \mu(y), 0.5).$$

Thus it follows from the above proof that

$$M(\mu_{y_1}^{y_{n-1}}, \mu(y))_y \subseteq\vee q\ F(\mu(x)_x, \mu(y_1)_{y_1}, \dots, \mu(y_{n-1})_{y_{n-1}}).$$

and so (F3a) holds.

(F3a) $\Longrightarrow$ (F3d) This is straightforward.

In the following, assume that μ has the sup-property or H is finite. We show (F3d) $\Longrightarrow$ (F3a). Let $y_1^{n-1} \in H$. If for any $x \in H$ such that $y \in f(x, y_1^{n-1})$, we have

$$\mu(x) < M(\mu_{y_1}^{y_{n-1}}, \mu(y), 0.5) \leq M(\mu_{y_1}^{y_{n-1}}).$$

Then since μ has the sup-property or H is finite, we have

$$\begin{aligned} &F(\mu, \mu(y_1)_{y_1}, \dots, \mu(y_{n-1})_{y_{n-1}})(y) \\ &= \bigvee_{y \in f(x,y_1^{n-1})} M(\mu(x), \mu_{y_1}^{y_{n-1}}) = \bigvee_{y \in f(x,y_1^{n-1})} \mu(x) \\ &< M(\mu_{y_1}^{y_{n-1}}, \mu(y), 0.5) = M((\mu \cap M(\mu_{y_1}^{y_{n-1}})_H)(y), 0.5), \end{aligned}$$

which contradicts $\mu \cap M(\mu_{y_1}^{y_{n-1}})_H \subseteq\vee q\ F(\mu, \mu(y_1)_{y_1}, \dots, \mu(y_{n-1})_{y_{n-1}})$ by Lemma 2.2. Hence (F3c) holds and so (F3a) is valid. □

Lemma 3.4. *Let $\mu \in \mathcal{F}(H)$. Then* (F4a) *holds if and only if the following condition holds:*

(F4c) *for all $x, y \in H$, $M(F(y, \underbrace{\mu, \dots, \mu}_{n-1})(x), 0.5) = M(F(x, \underbrace{\mu, \dots, \mu}_{n-1})(y), 0.5)$.*

P r o o f. (F4a) $\Longrightarrow$ (F4c) Let $x, y \in H$. If there $r \in (0,1]$ such that $F(y, \underbrace{\mu, \dots, \mu}_{n-1})(x) < r < M(F(x, \underbrace{\mu, \dots, \mu}_{n-1})(y), 0.5)$. Then

$$F(r_y, \underbrace{\mu, \dots, \mu}_{n-1})(x) = M(F(y, \underbrace{\mu, \dots, \mu}_{n-1})(x), r) < r$$
$$\leq M(F(x, \underbrace{\mu, \dots, \mu}_{n-1})(y), r) = F(r_x, \underbrace{\mu, \dots, \mu}_{n-1})(y)$$

and $r < 0.5$. This implies $r_y \in F(r_x, \underbrace{\mu, \dots, \mu}_{n-1})$ but $r_x \ \overline{\in\vee q}\ F(r_y, \underbrace{\mu, \dots, \mu}_{n-1})$, a contradiction. Hence $F(y, \underbrace{\mu, \dots, \mu}_{n-1})(x) \geq M(F(x, \underbrace{\mu, \dots, \mu}_{n-1})(y), 0.5)$ and so $M(F(y, \underbrace{\mu, \dots, \mu}_{n-1})(x), 0.5) \geq M(F(x, \underbrace{\mu, \dots, \mu}_{n-1})(y), 0.5)$. In a similar way, we have $M(F(x, \underbrace{\mu, \dots, \mu}_{n-1})(y), 0.5) \geq M(F(y, \underbrace{\mu, \dots, \mu}_{n-1})(x), 0.5)$. Thus (F4c) holds.

(F4c) $\Longrightarrow$ (F4a) Let $x, y \in H$. If there exists $r \in (0, 1]$ such that $r_y \in F(r_x, \underbrace{\mu, \dots, \mu}_{n-1})$ but $r_x \ \overline{\in\vee q}\ F(r_y, \underbrace{\mu, \dots, \mu}_{n-1})$. Then

$$F(x, \underbrace{\mu, \dots, \mu}_{n-1})(y) \geq F(r_x, \underbrace{\mu, \dots, \mu}_{n-1})(y) \geq r,$$

but $F(r_y, \underbrace{\mu, \dots, \mu}_{n-1})(x) < r$ and $F(r_y, \underbrace{\mu, \dots, \mu}_{n-1})(x) + r \leq 1$. Then

$$M(F(y, \underbrace{\mu, \dots, \mu}_{n-1})(x), r) = F(r_y, \underbrace{\mu, \dots, \mu}_{n-1})(x) < r \quad \text{gives} \quad F(y, \underbrace{\mu, \dots, \mu}_{n-1})(x) < r$$

and so

$$2F(y, \underbrace{\mu, \dots, \mu}_{n-1})(x) < M(F(y, \underbrace{\mu, \dots, \mu}_{n-1})(x), r) + r \leq 1,$$

that is, $F(y, \underbrace{\mu, \dots, \mu}_{n-1})(x) < 0.5$. Thus we have

$$F(y, \underbrace{\mu, \dots, \mu}_{n-1})(x) < M(F(x, \underbrace{\mu, \dots, \mu}_{n-1})(y), 0.5),$$

a contradiction. Hence (F4a) holds. □

Two examples of invertible $(\in, \in\vee q)$-fuzzy subhypergroups are provided as follows.

Example 3.5. Consider [28, Example 1.1(2)]. Let $(G,\cdot)$ be a commutative group with identity e in which $|G|>2$ and S a subgroup of H. Define $f(x_1^n)=Gx_1\dots x_n$. Then (G,f) is a commutative n-ary hypergroup with identity e and r_S is an invertible $(\in,\in\vee q)$-fuzzy subhypergroup of G for all $r\in(0,1]$.

Example 3.6. Let $(H;\wedge,\vee)$ be a complete lattice, where $H=[0,1]$. For all $x_1^n\in H$ and $i\in\{1,\dots,n\}$, we denote $A_n^{(i)}=M(x_1^{i-1},x_{i+1}^n)$ and $A_n=M(x_1^n)$. Define the following n-ary hyperoperation on H:

$$f(x_1^n)=\big\{x\in H\mid A_n=M(x,A_n^{(i)})\ \text{ for all }\ i\in\{1,\dots,n\}\big\}.$$

Then (H,f) is a commutative and invertible n-ary hypergroup with an identity 1. Now, we define an fuzzy subset μ of H as follows:

$$\mu(1)=1\qquad\text{and}\qquad \mu(z)=M(\mu(x),\mu(y))\ \text{ if }\ z=M(x,y)\quad\text{for all }\ z\in H.$$

Note that for any $x\in x_1^n$, we have

$$M(x_1^n)=M(x,x_2^n)=M(x,x_1,x_3^n).$$

Thus

$$M(x_1^n)=M(M(x,x_2^n),M(x,x_1,x_3^n))=M(x,x_1^n).$$

According to the definition of μ, we have

$$M(\mu_{x_1}^{x_n})=M(\mu(x),\mu_{x_1}^{x_n})\le\mu(x).$$

This implies $F(\underbrace{\mu,\dots,\mu}_{n})\subseteq\mu$.

Now for any $y\in f(x,y_1^{n-1})$, we have $M(x,y_1^{n-1})=M(y,y_1^{n-1})$. Thus

$$M(y,y_1^{n-1})=M(M(y,y_1^{n-1}),M(x,y_1^{n-1}))=M(x,y,y_1^{n-1}).$$

According to the definition of μ, we have

$$\mu(x)\ge M(\mu(x),\mu(y),\mu_{y_1}^{y_{n-1}})=M(\mu(y),\mu_{y_1}^{y_{n-1}}).$$

Thus, by Lemmas 3.2 and 3.3, μ is an $(\in,\in\vee q)$-fuzzy n-ary subhypergroup of (H,f). Moreover, μ is invertible. In fact, by the definition, it is clear that $x\in f(y,a_1^n)$ if and only if $y\in f(x,a_1^n)$, this implies $F(x,\underbrace{\mu,\dots,\mu}_{n-1})(y)=F(y,\underbrace{\mu,\dots,\mu}_{n-1})(x)$ for all $x,y\in H$ and so μ is invertible.

The next theorem provides the relationships between (invertible) $(\in,\in\vee q)$-fuzzy n-ary subhypergroups of H and crisp (invertible) n-ary subhypergroups of H.

Theorem 3.7. *Let $\mu \in \mathcal{F}(H)$. Then:*

(1) *μ is an (invertible) $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H if and only if $\mu_r(\mu_r \neq \emptyset)$ is an (invertible)n-ary subhypergroup of H with identity e for all $r \in (0, 0.5]$.*

(2) *μ is an (invertible) $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H if and only if $[\mu]_r([\mu]_r \neq \emptyset)$ is an (invertible)n-ary subhypergroup of H with identity e for all $r \in (0, 1]$.*

Proof. The proof is straightforward by Lemmas 3.2–3.4. □

Lemma 3.8. *Let $\mu \in \mathcal{F}(H)$. Then $\mu \subseteq F(\underbrace{e, \dots, e}_{i-1}, \mu, \underbrace{e, \dots, e}_{n-i})$ for all $x \in H$ and $i \in \{1, \dots, n\}$.*

Proof. It is straightforward. □

Lemma 3.9. *Let μ be an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H. Then*

(1) *$F(\underbrace{\mu, \dots, \mu}_{i}, e*) \approx \mu$ for all $i \in \{1, \dots, n\}$.*

(2) *$F(x, \underbrace{\mu, \dots, \mu}_{i}, e*) \approx F(x, \underbrace{\mu, \dots, \mu}_{n-1})$ for all $i \in \{1, \dots, n\}$.*

Proof.

(1) Let μ be an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H and $i \in \{1, \dots, n\}$. Since $\mu(e) \geq M(\mu(x), 0.5)$ for all $x \in H$, we have

$$\begin{aligned}\mu \subseteq F(\mu, e*) &= F(\mu, \underbrace{1_e, \dots, 1_e}_{n-1}) \\ &\subseteq\vee q\, F(\underbrace{\mu, \dots, \mu}_{i}, \underbrace{1_e, \dots, 1_e}_{n-i}) \subseteq\vee q\, F(\underbrace{\mu, \dots, \mu}_{n}) \subseteq\vee q\, \mu.\end{aligned}$$

Therefore, $F(\underbrace{\mu, \dots, \mu}_{i}, e*) = F(\underbrace{\mu, \dots, \mu}_{i}, \underbrace{1_e, \dots, 1_e}_{n-i}) \approx \mu$.

(2) Let $x \in H$ and $i \in \{1, \dots, n\}$. By (1), we have

$$\begin{aligned}F(x, \underbrace{\mu, \dots, \mu}_{n-1}) &\approx F(x, \underbrace{\mu, \dots, \mu}_{n-2}, F(\mu, e*)) = F(x, \underbrace{\mu, \dots, \mu}_{i-1}, F(\underbrace{\mu, \dots, \mu}_{n-i}, e*), e*) \\ &\approx F(x, \underbrace{\mu, \dots, \mu}_{i-1}, \mu, e*) = F(x, \underbrace{\mu, \dots, \mu}_{i}, e*),\end{aligned}$$

as required. □

Lemma 3.10. *Let μ be an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H. Then μ is invertible if and only if*

$$M(F(x, \underbrace{\mu, \ldots, \mu}_{i}, e*)(y), 0.5) = M(F(y, \underbrace{\mu, \ldots, \mu}_{i}, e*)(x), 0.5)$$

for all $x, y \in H$ and $i \in \{1, \ldots, n\}$.

P r o o f. It is straightforward by Lemmas 3.4 and 3.9. □

Lemma 3.11. *Let μ and μ_1^i $(i \in \{2, \ldots, n\})$ be invertible $(\in, \in\vee q)$-fuzzy n-ary subhypergroups of H. Then:*

(1) *$M(F(x, \mu_1^i, e*)(y), 0.5) = M(F(y, \mu_1^i, e*)(x), 0.5)$ for all $x, y \in H$.*

(2) *For all $x, y, y_1^{n-1} \in H$ such that $y \in f(x, y_1^{n-1})$, we have $\mu(x) \geq M(\mu_{y_1}^{y_{n-1}}, \mu(y), 0.5)$.*

P r o o f.

(1) Let $x, y \in H$. Then, by Lemma 3.9, we have

$$\begin{aligned} F(x, \mu_1^i, e*) = F(x, \mu_1, \mu_2, \mu_3^i, e*) &\approx F(x, \mu_1, F(\underbrace{e, \ldots, e}_{n-2}, \mu_2, e), \mu_3^i, e*) \\ &= F(F(x, \mu_1, e*), \mu_2, e, \mu_3^i, e*)y) \approx F(F(x, \mu_1, e*), \mu_2^i, e*) \\ &\approx F(F(x, \mu_1, e*), F(\mu_2^i, e*), e*). \end{aligned}$$

It follows that

$$\begin{aligned} &F(x, \mu_1^i, e*)(y) \\ &\geq M(F(F(x, \mu_1, e*), F(\mu_2^i, e*), e*)(y), 0.5) \\ &= M\Bigg(\bigvee_{y \in f(r_1, r_2, e*)} M(F(x, \mu_1, e*)(r_1), F(\mu_2^i, e*)(r_2)), 0.5\Bigg) \\ &= M\Bigg(\bigvee_{z_1 \in H} M(F(x, \mu_1, e*)(z_1), F(z_1, \mu_2^i, e*)(y)), 0.5\Bigg) \\ &= M\Bigg(\bigvee_{z_1, z_2 \in H} M(F(x, \mu_1, e*)(z_1), M(F(z_1, \mu_2, e*)(z_2), F(z_2, \mu_3^i, e*)(y))), 0.5\Bigg) \\ &= M\Bigg(\bigvee_{z_1, z_2 \in H} M(F(x, \mu_1, e*)(z_1), F(z_1, \mu_2, e*)(z_2), F(z_2, \mu_3^i, e*)(y)), 0.5\Bigg) \\ &\quad \vdots \\ &= M\Bigg(\bigvee_{z_1^{i-1} \in H} M(F(x, \mu_1, e*)(z_1), F(z_1, \mu_2, e*)(z_2), \ldots, F(z_{i-1}, \mu_i, e*)(y)), 0.5\Bigg) \end{aligned}$$

$$\geq \bigvee_{z_1^{i-1}\in H} M(F(z_1,\mu_1,e*)(x), F(z_2,\mu_2,e*)(z_1),\dots,F(y,\mu_i,e*)(z_{i-1}),0.5)$$
$$= M(F(y,\mu_i,\mu_{i-1},\dots,\mu_1,e*)(x),0.5) = M(F(y,\mu_1^i,e*)(x),0.5).$$

This implies $F(x,\mu_1^i,e*)(y) \geq M(F(y,\mu_1^i,e*)(x),0.5)$. In a similar way, we have $F(y,\mu_1^i,e*)(x) \geq M(F(x,\mu_1^i,e*)(y),0.5)$. Hence $M(F(x,\mu_1^i,e*)(y),0.5) = M(F(y,\mu_1^i,e*)(x),0.5)$.

(2) Let $x,y,y_1^{n-1}\in H$ be such that $y\in f(x,y_1^{n-1})$. Then

$$F(x,\underbrace{\mu,\dots,\mu}_{n-1})(y) \geq M(\mu_{y_1}^{y_{n-1}}).$$

Thus, we have

$$\begin{aligned}\mu(x) =&F(\mu,e*)(x) = F(\underbrace{\mu,\dots,\mu}_{n})(x) \geq M(\mu(y),F(y,\underbrace{\mu,\dots,\mu}_{n-1})(x))\\ \geq& M(\mu(y),M(F(x,\underbrace{\mu,\dots,\mu}_{n-1})(y),0.5)) \geq M(\mu_{y_1}^{y_{n-1}},\mu(y),0.5),\end{aligned}$$

as required. □

Theorem 3.12. *Let μ_1^n be invertible $(\in,\in\vee q)$-fuzzy n-ary subhypergroups of H and $i\in\{2,\dots,n\}$. Then $F(\mu_1^i,e*)$ is an invertible $(\in,\in\vee q)$-fuzzy n-ary subhypergroup of H.*

Proof. Since μ_1^n are invertible $(\in,\in\vee q)$-fuzzy n-ary subhypergroups of H. It is clear that $F(\mu_1^i,e*)(e) \geq M(F(\mu_1^i,e*)(x),0.5)$. From Lemma 3.9, we have

$$\begin{aligned}F(\underbrace{F(\mu_1^i,e*),\dots,F(\mu_1^i,e*)}_{j},e*) =&F(F(\underbrace{\mu_1,\dots,\mu_1}_{j},e*),\dots,F(\underbrace{\mu_i,\dots,\mu_i}_{j},e*),e*)\\ \approx& F(\mu_1^i,e*).\end{aligned}$$

In particular, we have

$$F(\underbrace{F(\mu_1^i,e*),\dots,F(\mu_1^i,e*)}_{n}) \approx F(\mu_1^i,e*).$$

Now we show that

$$\begin{aligned}&M(F(x,\underbrace{F(\mu_1^i,e*),\dots,F(\mu_1^i,e*)}_{n-1})(y),0.5)\\ =& M(F(y,\underbrace{F(\mu_1^i,e*),\dots,F(\mu_1^i,e*)}_{n-1})(x),0.5)\end{aligned}$$

for all $x, y \in H$, that is, $F(\mu_1^i, e*)$ is invertible. In fact, by Lemmas 3.9 and 3.11(1), we have

$$\begin{aligned} &M(F(x, \underbrace{F(\mu_1^i, e*), \ldots, F(\mu_1^i, e*)}_{n-1})(y), 0.5) \\ &= M(F(x, \mu_1^i, e*)(y), 0.5) = M(F(y, \mu_1^i, e*)(x), 0.5) \\ &= M(F(y, \underbrace{F(\mu_1^i, e*), \ldots, F(\mu_1^i, e*)}_{n-1})(x), 0.5). \end{aligned}$$

Next, let $x, y, y_1^{n-1} \in H$ be such that $y \in f(x, y_1^{n-1})$. Then by Lemma 3.11(2), we have $F(\mu_1^i, e*)(x) \geq M(F(\mu_1^i, e*)_{y_1}^{y_{n-1}}, F(\mu_1^i, e*)(y), 0.5)$.

Summing up the above arguments, $F(\mu_1^i, e*)$ is an invertible $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H. □

Next, we will construct an n-ary quotient hypergroup by an $(\in, \in\vee q)$ fuzzy n-ary subhypergroup of a commutative n-ary hypergroup. Let μ be an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H. Denote by H/μ the set of all $F(x, \mu, e*)$, where $x \in H$. Before proceeding, let us first provide an useful lemma.

Lemma 3.13. *Let μ be an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H. Then:*

(1) $M(\mu(e), 0.5) \geq M(F(x, \mu, e*)(y), 0.5)$ *for all* $x, y \in H$.

(2) $F(x, \mu, e*) \approx F(y, \mu, e*) \iff M(\mu(e), 0.5) = M(F(x, \mu, e*)(y), 0.5) = M(F(y, \mu, e*)(x), 0.5)$. *In particular, if* $0.5 \in \mathrm{Im}(\mu)$, *then*

$$F(x, \mu, e*) = F(y, \mu, e*) \iff M(F(x, \mu, e*)(y), F(y, \mu, e*)(x)) \geq 0.5.$$

Proof. (1) This is straightforward.

(2) Let $x, y \in H$ be such that $F(x, \mu, e*) \approx F(y, \mu, e*)$. Then

$$\begin{aligned} M(F(x, \mu, e*)(y), 0.5) &= M(F(y, \mu, e*)(y), 0.5) \\ &= M\Bigg(\bigvee_{y \in f(y, a, e*)} \mu(a), 0.5\Bigg) = M(\mu(e), 0.5). \end{aligned}$$

In a similar way, we have $M(F(y, \mu, e*)(x), 0.5) = M(\mu(e), 0.5)$.

Conversely, assume that

$$M(\mu(e), 0.5) = M(F(x, \mu, e*)(y), 0.5) = M(F(y, \mu, e*)(x), 0.5)$$

for some $x, y \in H$. Then for any $z \in H$, by (1), we have

$$\begin{aligned} M(F(x, \mu, e*)(z), 0.5) &= M\left(M(F(x, \mu, e*)(z), 0.5), M(\mu(e), 0.5)\right) \\ &= M\left(M(F(x, \mu, e*)(z), 0.5), M(F(y, \mu, e*)(x), 0.5)\right) \\ &= M\left(F(x, \mu, e*)(z), F(y, \mu, e*)(x), 0.5\right) \end{aligned}$$

$$\begin{aligned}
&= M\Big(\bigvee_{z\in f(x,a,e*)} \mu(a), \bigvee_{x\in f(y,b,e*)} \mu(b), 0.5\Big)\\
&= M\Big(\bigvee_{z\in f(x,a,e*), x\in f(y,b,e*)} M(\mu(a),\mu(b)), 0.5\Big)\\
&\le M\Big(\bigvee_{z\in f(y,b,a,e*)} M(\mu(a),\mu(b)), 0.5\Big)\\
&= M(F(y,\mu,\mu,e*)(z), 0.5) = M(F(y,\mu,e*)(z), 0.5).
\end{aligned}$$

Hence $M(F(x,\mu,e*)(z),0.5) \le M(F(y,\mu,e*)(z),0.5)$. In a similar way, we have $M(F(y,\mu,e*)(z),0.5) \le M(F(x,\mu,e*)(z),0.5)$. Therefore,

$$F(x,\mu,e*) \approx F(y,\mu,e*).$$

If $0.5 \in \mathrm{Im}(\mu)$, then $\mu(e) \ge 0.5$. Thus it is clear that

$$F(x,\mu,e*) \approx F(y,\mu,e*) \iff M(F(x,\mu,e*)(y), F(y,\mu,e*)(x)) \ge 0.5.$$

This completes the proof. □

Theorem 3.14. *Let H be a commutative n-ary hypergroup with an identity e such that $f(x_1^n)$ is finite for all $x_1^n \in H$ and μ an $(\in,\in\vee q)$-fuzzy n-ary subhypergroup of H. Then $(H/\mu, g)$ is an n-ary hypergroup with an identity $F(e,\mu,e*)$, called the quotient hypergroup of H by μ under the relation $\approx$, where the n-ary hyperoperation $g\colon \underbrace{H/\mu\times\cdots\times H/\mu}_{n} \to H/\mu$ is defined by*

$$g(F(x_1,\mu,e*),\ldots,F(x_n,\mu,e*)) = \big\{F(x,\mu,e*) \mid x\in f(x_1^n)\big\}$$

for all $x, x_1^n \in H$.

Proof. We shall first show that g is well-defined. Let $x_1^n, y_1^n \in H$ be such that $F(x_i,\mu,e*) \approx F(y_i,\mu,e*)$ for all $i\in\{1,\ldots,n\}$. Then we have

$$\begin{aligned}
F(f(x_1^n),\mu,e*) &\approx F(f(x_1^n), F(\underbrace{\mu,\ldots,\mu}_{n}), e*) \approx F(F(x_1,\mu,e*),\ldots,F(x_n,\mu,e*))\\
&\approx F(F(y_1,\mu,e*),\ldots,F(y_n,\mu,e*)) \approx F(f(y_1^n),\mu,e*).
\end{aligned}$$

Now, for any $F(x,\mu,e*) \in g(F(x_1,\mu,e*),\ldots,F(x_n,\mu,e*))$, there exists $a \in f(x_1^n)$ such that $F(x,\mu,e*) \approx F(a,\mu,e*)$. Thus we have

$$\begin{aligned}
M(\mu(e),0.5) &= M(F(a,\mu,e*)(x),0.5)\\
&\le M(F(f(x_1^n),\mu,e*)(x),0.5)
\end{aligned}$$

$$= M(F(f(y_1^n), \mu, e*)(x), 0.5)$$
$$= M\Bigg(\bigvee_{b \in f(y_1^n)} F(b, \mu, e*)(x), 0.5\Bigg) \leq M(\mu(e), 0.5).$$

By the assumption, $f(y_1^n)$ is finite and so there exists $y \in f(y_1^n)$ such that $M(\mu(e), 0.5) = M(F(y, \mu, e*)(x), 0.5)$. Hence $F(x, \mu, e*) \approx F(y, \mu, e*)$ by Lemma 3.13, this gives $F(x, \mu, e*) \in g(F(y_1, \mu, e*), \dots, F(y_n, \mu, e*))$, that is,

$$g(F(x_1, \mu, e*), \dots, F(x_n, \mu, e*)) \subseteq g(F(y_1, \mu, e*), \dots, F(y_n, \mu, e*)).$$

In a similar way, we have

$$g(F(y_1, \mu, e*), \dots, F(y_n, \mu, e*)) \subseteq g(F(x_1, \mu, e*), \dots, F(x_n, \mu, e*)).$$

Hence g is well defined. Now it is easy to verify that $(H/\mu, g)$ is an n-ary hypergroup and that $F(e, \mu, e*)$ is an identity of $(H/\mu, g)$. □

Theorem 3.15. *Let H be a commutative n-ary hypergroup with an identity e such that $f(x_1^n)$ is finite for all $x_1^n \in H$ and μ an n-ary subhypergroup of H. Define a fuzzy subset ν/μ of H/μ and a fuzzy n-ary hyperoperation*

$$G\colon \underbrace{\mathcal{F}(H)/\mu \times \cdots \times \mathcal{F}(H)}_{n}/\mu \to \mathcal{F}(H)/\mu$$

by

$$\nu/\mu(F(x, \mu, e*)) = \bigvee_{F(x,\mu,e*) \approx F(y,\mu,e*)} \nu(y)$$

and

$$G(\nu_1/\mu, \dots, \nu_n/\mu)(F(x, \mu, e*))$$
$$= \bigvee_{F(x,\mu,e*) \in g(F(x_1,\mu,e*),\dots,F(x_n,\mu,e*))} M(\nu_1/\mu(F(x_1, \mu, e*), \dots, \nu_n/\mu(F(x_n, \mu, e*)),$$

respectively, for all $x \in H$ and $\nu, \nu_1^n \in \mathcal{F}(H)$. If ν is an (invertible) $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H which has the sup-property, then ν/μ is also an (invertible) $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of $(H/\mu, g)$.

Proof. Let ν be an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H. We show that ν/μ is an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of $(H/\mu, g)$.

(1) It is clear that $\nu/\mu(F(e, \mu, e*)) \geq M(\nu/\mu(F(x, \mu, e*)), 0.5)$ for all $x \in H$.

(2) Let $x, x_1^n \in H$ be such that $F(x, \mu, e*) \in g(F(x_1, \mu, e*), \dots, F(x_n, \mu, e*))$. Since ν has the sup-property, there exist $y_1^n \in H$ such that $F(x_i, \mu, e*) \approx F(y_i, \mu, e*)$ and $\nu/\mu(F(x_i, \mu, e*)) = \nu(y_i)$ for all $i \in \{1, \dots, n\}$. This gives

$F(x,\mu,e*) \in g(F(y_1,\mu,e*),\ldots,F(y_n,\mu,e*))$ and so there exists $z \in f(y_1^n)$ such that $F(z,\mu,e*) \approx F(x,\mu,e*)$. Hence we have

$$\begin{aligned}\nu/\mu(F(x,\mu,e*)) &= \bigvee_{F(x,\mu,e*)\approx F(y,\mu,e*)} \nu(y) \geq \nu(z) \geq M(\nu_{y_1}^{y_n}, 0.5)\\ &= M(\nu/\mu(F(x_1,\mu,e*)),\ldots,\nu/\mu(F(x_n,\mu,e*)),0.5).\end{aligned}$$

(3) Let $y, y_1^{n-1} \in H$. Since ν has the sup-property, there exist $z, z_1^{n-1} \in H$ such that $F(y,\mu,e*) \approx F(z,\mu,e*)$, $\nu/\mu(F(y,\mu,e*)) = \nu(z)$, $F(y_i,\mu,e*) \approx F(z_i,\mu,e*)$ and $\nu/\mu(F(y_i,\mu,e*)) = \nu(z_i)$ for all $i \in \{1,\ldots,n-1\}$. For $z, z_1^{n-1} \in H$, since ν is an $(\in,\in\vee q)$-fuzzy n-ary subhypergroup of H, there exists $x \in H$ such that $z \in f(x, z_1^{n-1})$ and $\nu(x) \geq M(\nu_{z_1}^{z_{n-1}}, \nu(z), 0.5)$. Thus we have

$$\begin{aligned}&\nu/\mu(F(x,\mu,e*))\\ &= \bigvee_{F(x,\mu,e*)\approx F(a,\mu,e*)} \nu(a) \geq \nu(x) \geq M(\nu_{z_1}^{z_{n-1}}, \nu(z), 0.5)\\ &= M(\nu/\mu(F(y_1,\mu,e*)),\ldots,\nu/\mu(F(y_{n-1},\mu,e*)),\nu/\mu(F(y,\mu,e*)),0.5).\end{aligned}$$

Summing up the above arguments, ν/μ is an $(\in,\in\vee q)$-fuzzy n-ary subhypergroup of H/μ.

Now assume that ν is an invertible $(\in,\in\vee q)$-fuzzy n-ary subhypergroup of H. Let $x, y \in H$. Then

$$\begin{aligned}&M(G(F(x,\mu,e*),\underbrace{\nu/\mu,\ldots,\nu/\mu}_{n-1})(F(y,\mu,e*)),0.5)\\ &= M\Big(\bigvee_{F(x,\mu,e*)\approx F(a,\mu,e*),F(y,\mu,e*)\approx F(b,\mu,e*),b\in f(a,a_1^{n-1})} M(\nu_{a_1}^{a_{n-1}}),0.5\Big)\\ &= M\Big(\bigvee_{F(x,\mu,e*)\approx F(a,\mu,e*),F(y,\mu,e*)\approx F(b,\mu,e*)} F(a,\underbrace{\nu,\ldots,\nu}_{n-1})(b),0.5\Big)\\ &= \bigvee_{F(x,\mu,e*)\approx F(a,\mu,e*),F(y,\mu,e*)\approx F(b,\mu,e*)} M(F(a,\underbrace{\nu,\ldots,\nu}_{n-1})(b),0.5)\end{aligned}$$

In a similar way, we have

$$\begin{aligned}&M(G(F(y,\mu,e*),\underbrace{\nu/\mu,\ldots,\nu/\mu}_{n-1})(F(x,\mu,e*)),0.5)\\ &= \bigvee_{F(x,\mu,e*)\approx F(a,\mu,e*),F(y,\mu,e*)\approx F(b,\mu,e*)} M(F(b,\underbrace{\nu,\ldots,\nu}_{n-1})(a),0.5).\end{aligned}$$

Since ν is invertible, we have

$$M(F(a,\underbrace{\nu,\ldots,\nu}_{n-1})(b),0.5) = M(F(b,\underbrace{\nu,\ldots,\nu}_{n-1})(a),0.5)$$

and so

$$\begin{aligned} &M(G(F(x,\mu,e*),\underbrace{\nu/\mu,\ldots,\nu/\mu}_{n-1})(F(y,\mu,e*)),0.5)\\ &= M(G(F(y,\mu,e*),\underbrace{\nu/\mu,\ldots,\nu/\mu}_{n-1})(F(x,\mu,e*)),0.5).\end{aligned}$$

Therefore, ν/μ is an invertible $(\in,\in\vee q)$-fuzzy n-ary subhypergroup of H/μ. □

4. The homomorphism properties of (invertible) $(\in,\in\vee q)$-fuzzy n-ary subhypergroups

Let φ be a mapping from a non-empty set X to a non-empty set X'. Let $\mu\in\mathcal{F}(X)$ and $\mu'\in\mathcal{F}(X')$. Then the *inverse image* $\varphi^{-1}(\mu')$ of μ' is the fuzzy subset of X defined by $\varphi^{-1}(\mu')(x)=\mu'(\varphi(x))$ for all $x\in X$. The *image* $\varphi(\mu)$ of μ is the fuzzy subset of X' defined by

$$\varphi(\mu)(x') = \begin{cases} \bigvee\limits_{x\in\varphi^{-1}(x')} \mu(x) & \text{if } \varphi^{-1}(x')\neq\emptyset,\\ 0, & \text{otherwise,}\end{cases}$$

for all $x'\in X'$. It is no difficult to see that the following assertions hold:

Lemma 4.1. *Let φ be a mapping from a non-empty set X to a non-empty set X', $\mu\in\mathcal{F}(X)$ and $\mu'\in\mathcal{F}(X')$. Then:*

1. *$\mu\subseteq\vee q\ \nu$ implies $\varphi(\mu)\subseteq\vee q\ \varphi(\nu)$.*
2. *$\mu'\subseteq\vee q\ \nu'$ implies $\varphi^{-1}(\mu')\subseteq\vee q\ \varphi^{-1}(\nu')$.*
3. *$\mu\subseteq\varphi^{-1}(\varphi(\mu))$. If φ is injective, then $\mu=\varphi^{-1}(\varphi(\mu))$.*
4. *$\varphi(\varphi^{-1}(\mu'))\subseteq\mu'$. If φ is surjective, then $\mu'=\varphi(\varphi^{-1}(\mu'))$.*

Definition 4.2. Let (H,f) and (H',f') be two n-ary hypergroups with identities e and e', respectively, and φ a mapping from H to H'. Then φ is called a *homomorphism* if $\varphi(e)=e'$ and $\varphi(f(x_1^n))=f'(\varphi(x_1),\ldots,\varphi(x_n))$ for all $x_1^n\in H$. If such a homomorphism is surjective, injective or bijective, it is called an *epimorphism*, a *monomorphism* or an *isomorphism*.

Lemma 4.3. *Let (H,f) and (H',f') be two n-ary hypergroups with identities e and e', respectively, and φ a homomorphism from H to H'. Then $\varphi(F(\mu_1^n)) = F'(\varphi(\mu_1),\ldots,\varphi(\mu_n))$ for all $\mu_1^n\in\mathcal{F}(H)$.*

Proof. Let $\mu_1^n \in \mathcal{F}(H)$ and $x' \in H'$. If $\varphi^{-1}(x') = \emptyset$, then $\varphi(F(\mu_1^n))(x') = 0 = F'(\varphi(\mu_1), \ldots, \varphi(\mu_n))(x')$. Otherwise, we have

$$\begin{aligned}
&F'(\varphi(\mu_1), \ldots, \varphi(\mu_n))(x') \\
&= \bigvee_{x' \in f'(x_1', \ldots, x_n')} M(\varphi(\mu_1)(x_1'), \ldots, \varphi(\mu_n)(x_n')) \\
&= \bigvee_{x_1', \ldots, x_n' \in \mathrm{Im}(\varphi), x' \in f'(x_1', \ldots, x_n')} M\left(\bigvee_{\varphi(x_1) = x_1'} \mu_1(x_1), \ldots, \bigvee_{\varphi(x_n) = x_n'} \mu_n(x_n)\right) \\
&= \bigvee_{x' \in f'(\varphi(x_1), \ldots, \varphi(x_n))} M(\mu_1(x_1), \ldots, \mu_n(x_n)) \\
&= \bigvee_{x' \in \varphi(f(x_1, \ldots, x_n))} M(\mu_1(x_1), \ldots, \mu_n(x_n)) \\
&= \bigvee_{x \in \varphi^{-1}(x'), x \in f(x_1, \ldots, x_n)} M(\mu_1(x_1), \ldots, \mu_n(x_n)) \\
&= \bigvee_{x \in \varphi^{-1}(x')} F(\mu_1^n) = \varphi(F(\mu_1^n))(x').
\end{aligned}$$

This completes the proof. □

Theorem 4.4. *Let (H, f) and (H', f') be two commutative n-ary hypergroups with identities e and e', respectively, and φ a homomorphism from H to H'. Let μ be an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H which has the sup-property. Then $\varphi(\mu)$ is an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H'. If φ is an epimorphism from H onto H' and μ is invertible, then $\varphi(\mu)$ is also invertible.*

Proof. Assume that μ is an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H which has the sup-property. Then we have:

(1) It is clear that $\varphi(\mu)(e') \geq M(\varphi(\mu)(x'), 0.5)$ for all $x' \in H$.

(2) By Lemmas 4.1 and 4.3, we have,

$$F'(\underbrace{\varphi(\mu), \ldots, \varphi(\mu)}_{n}) = \varphi(F(\underbrace{\mu, \ldots, \mu}_{n})) \subseteq\vee q\ \varphi(\mu).$$

(3) Let $y', y_1', \ldots, y_{n-1}' \in H'$. If $\varphi^{-1}(y_j') = \emptyset$ ($j \in \{1, \ldots, n-1\}$) or $\varphi^{-1}(y') = \emptyset$, then $\varphi(\mu)(x') \geq 0 = M(\varphi(\mu)_{y_1'}^{y_{n-1}'}, \varphi(\mu)(y'), 0.5)$ for $y' \in f'(x', y_1', \ldots, y_{n-1}')$. Otherwise, there exist $y, y_1^{n-1} \in H$ such that $\varphi(y) = y', \varphi(y_i) = y_i'$, $\varphi(\mu)(y') = \mu(y)$ and $\varphi(\mu)(y_i') = \mu(y_i)$ for all $i \in \{1, \ldots, n-1\}$ since μ has the sup-property. Now for $y, y_1^{n-1} \in H$, since μ is an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H, there exists $x \in H$ such that $y \in f(x, y_1^{n-1})$ and $\mu(x) \geq M(\mu_{y_1}^{y_{n-1}}, \mu(y), 0.5)$.

Hence we have

$$\begin{aligned} y' = \varphi(y) \in \varphi(f(x, y_1^{n-1})) &= f'(\varphi(x), \varphi(y_1), \ldots, \varphi(y_{n-1})) \\ &= f'(\varphi(x), y_1', \ldots, y_{n-1}') \end{aligned}$$

and

$$\begin{aligned} \varphi(\mu)(\varphi(x)) = &\bigvee_{\varphi(a)=\varphi(x)} \mu(a) \geq \mu(x) \\ &\geq M(\mu_{y_1}^{y_{n-1}}, \mu(y), 0.5) = M(\varphi(\mu)_{y_1'}^{y_{n-1}'}, \varphi(\mu)(y'), 0.5). \end{aligned}$$

Combing (1), (2) and (3), by Lemmas 3.2 and 3.3, $\varphi(\mu)$ is an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H'.

Next assume that φ is an epimorphism from H onto H' and that μ is an invertible $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H. Then for any $x', y' \in H'$, we have

$$\begin{aligned} &M(F'(x', \underbrace{\varphi(\mu), \ldots, \varphi(\mu)}_{n-1})(y'), 0.5) \\ &= M\Big(\bigvee_{\varphi(x)=x'} F'(\varphi(x), \underbrace{\varphi(\mu), \ldots, \varphi(\mu)}_{n-1})(y'), 0.5\Big) \\ &= M\Big(\bigvee_{\varphi(x)=x'} \varphi(F(x, \underbrace{\mu, \ldots, \mu}_{n-1}))(y'), 0.5\Big) \\ &= M\Big(\bigvee_{\varphi(x)=x', \varphi(y)=y'} F(x, \underbrace{\mu, \ldots, \mu}_{n})(y), 0.5\Big) \\ &= \bigvee_{\varphi(x)=x', \varphi(y)=y'} M(F(x, \underbrace{\mu, \ldots, \mu}_{n-1})(y), 0.5) \\ &= \bigvee_{\varphi(x)=x', \varphi(y)=y'} M(F(y, \underbrace{\mu, \ldots, \mu}_{n-1})(x), 0.5) \\ &= M\Big(\bigvee_{\varphi(x)=x', \varphi(y)=y'} F(y, \underbrace{\mu, \ldots, \mu}_{n-1})(x), 0.5\Big) \\ &= M(F'(y', \underbrace{\varphi(\mu), \ldots, \varphi(\mu)}_{n-1})(x'), 0.5). \end{aligned}$$

This implies that $\varphi(\mu)$ is invertible. □

Theorem 4.5. *Let (H, f) and (H', f') be two commutative n-ary hypergroups with identities e and e', respectively, and φ a homomorphism from H to H'. Let μ' be an invertible $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H'. Then $\varphi^{-1}(\mu')$ is an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H. If φ is an isomorphism from H to H' and μ' is an (invertible) $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H', then $\varphi^{-1}(\mu')$ is an (invertible)$(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H.*

Proof. Assume that μ' is an invertible $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H'. Then we have:

(1) For any $x \in H$,

$$\varphi^{-1}(\mu')(e) = \mu'(\varphi(e)) \geq M(\mu'(\varphi(x)), 0.5) = M(\varphi^{-1}(\mu')(x), 0.5).$$

(2) By Lemmas 4.1 and 4.3, we have,

$$\begin{aligned}\varphi(F(\underbrace{\varphi^{-1}(\mu'), \ldots, \varphi^{-1}(\mu')}_{n})) &= F'(\underbrace{\varphi(\varphi^{-1}(\mu')), \ldots, \varphi(\varphi^{-1}(\mu'))}_{n})) \\ &\subseteq F'(\underbrace{\mu', \ldots, \mu'}_{n}) \subseteq\vee q\ \mu'.\end{aligned}$$

Thus we have

$$F(\underbrace{\varphi^{-1}(\mu'), \ldots, \varphi^{-1}(\mu')}_{n}) \subseteq \varphi^{-1}(\varphi(F(\underbrace{\varphi^{-1}(\mu'), \ldots, \varphi^{-1}(\mu')}_{n}))) \subseteq\vee q\ \varphi^{-1}(\mu').$$

(3) Let $x, y, y_1^{n-1} \in H$ be such that $y \in f(x, y_1^{n-1})$. Then

$$\varphi(y) \in f'(\varphi(x), \varphi(y_1), \ldots, \varphi(y_{n-1})).$$

Since μ' is invertible, by Lemma 3.11(2), we have

$$\begin{aligned}\varphi^{-1}(\mu')(x) = \mu'(\varphi(x)) &\geq M(\mu'(\varphi(y_1)), \ldots, \mu'(\varphi(y_{n-1})), \mu'(\varphi(y)), 0.5) \\ &= M(\varphi^{-1}(\mu')_{y_1}^{y_{n-1}}, \varphi^{-1}(\mu')(y), 0.5).\end{aligned}$$

Combing (1), (2) and (3), $\varphi^{-1}(\mu')$ is an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H. If φ is an isomorphism from H to H' and μ' is an (invertible) $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H', it is easy to check that $\varphi^{-1}(\mu')$ is an (invertible) $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H. □

Next, let us consider the relationships among $(\in, \in\vee q)$-fuzzy n-ary subhypergroups, quotient n-ary hypergroups and homomorphism. Before proceeding, we first give an auxiliary lemma.

Lemma 4.6. *Let μ be an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H. Define $\mu_H = \{x \in H \mid M(\mu(e), 0.5) = M(\mu(x), 0.5)\}$, then:*

(1) *μ_H is an n-ary subhypergroup of H with an identity e. If μ has the sup-property and μ is invertible, so is μ_H.*

(2) *if* μ *has the sup-property, then*

$$F(x,\mu,e*) \approx F(y,\mu,e*) \iff f(x,\mu_H,e*) = f(y,\mu_H,e*).$$

Proof.

(1) We first show that μ_H is an n-ary subhypergroup of H. Let $x \in H$ and $x_1^n \in \mu_H$ be such that $x \in f(x_1^n)$. Then

$$M(\mu(e),0.5) \geq M(\mu(x),0.5) \geq M(M(\mu_{x_1}^{x_n},0.5),0.5) = M(\mu(e),0.5),$$

this implies $M(\mu(x),0.5) = M(\mu(e),0.5)$ and so $x \in \mu_H$. Similarly, we may show that for all $y, y_1^{n-1} \in \mu_H$, there exists $x \in \mu_H$ such that $y \in f(x, y_1^{n-1})$. Hence μ_H is an n-ary subhypergroup of H. Clearly, $e \in \mu_H$. Now assume that μ has the sup-property and that μ is invertible. If $x \in f(y,\underbrace{\mu_H,\dots,\mu_H}_{n-1})$, then there exists $z \in \mu_H$ such that $x \in f(y,z,e*)$ since $e \in \mu_H$. Thus we have

$$\begin{aligned} M\Big(\bigvee_{y\in f(x,a,e*)} \mu(a),0.5\Big) &= M(F(x,\mu,e*)(y),0.5) = M(F(y,\mu,e*)(x),0.5) \\ &= M\Big(\bigvee_{x\in f(y,b,e*)} \mu(b),0.5\Big) \geq M(\mu(z),0.5) \\ &= M(\mu(e),0.5). \end{aligned}$$

Since μ has the sup-property, there exists $a \in H$ such that $y \in f(x,a,e*)$ and $M(\mu(a),0.5) = M(\mu(e),0.5)$, that is, $a \in \mu_H$ and so $y \in f(x,\mu_H,e*) \subseteq f(x,\underbrace{\mu_H,\dots,\mu_H}_{n-1})$. Hence μ_H is invertible.

(2) Assume that μ has the sup-property. Let $x, y \in H$ be such that $F(x,\mu,e*) \approx F(y,\mu,e*)$. By Lemma 3.13, we have $M(\mu(e),0.5) = M(F(y,\mu,e*)(x),0.5) = M(F(x,\mu,e*)(y),0.5)$, that is, $M(\mu(e),0.5) = M\Big(\bigvee_{x\in f(y,a,e*)} \mu(a),0.5\Big)$. Since μ has the sup-property, there exists $z \in S$ such that $y \in f(x,z,e*)$ and $M(\mu(e),0.5) = M(\mu(z),0.5)$, that is, $z \in \mu_H$ and so $f(y,\mu_H,e*) \subseteq f(f(x,z,e*),\mu_H,e*) = f(x,f(z,\mu_H,e*),e*) \subseteq f(x,\mu_H,e*)$. In a similar way, we have $f(x,\mu_H,e*) \subseteq f(y,\mu_H,e*)$. Hence $f(x,\mu_H,e*) = f(y,\mu_H,e*)$. Conversely, assume that $f(x,\mu_H,e*) = f(y,\mu_H,e*)$. Then $x \in f(x,e*) \subseteq f(x,\mu_H,e*)$ and so there exists $z \in \mu_H$ such that $x \in f(y,z,e*)$. Thus we have

$$M(\mu(e),0.5) \geq M(F(y,\mu,e*)(x),0.5) \geq M(\mu(z),0.5) = M(\mu(e),0.5).$$

This implies $M(\mu(e),0.5) = M(F(y,\mu,e*)(x),0.5)$. In a similar way, we have $M(\mu(e),0.5) = M(F(x,\mu,e*)(y),0.5)$. By Lemma 3.13, we have $F(x,\mu,e*) \approx F(y,\mu,e*)$. This completes the proof. □

In the sequel, unless otherwise stated, (H,f) and (H',f') always denote any two given commutative n-ary hypergroups with identities e and e', respectively, such that both $f(x_1^n)$ and $f'(x_1^{'n})$ are finite for all $x_1^n \in H$ and $x_1^{'n} \in H'$.

Theorem 4.7. *Let μ be an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H. Define $\varphi(x) = F(x,\mu,e*)$ for all $x \in H$. Then φ is an epimorphism from H onto H/μ with* $\mathrm{Ker}(\varphi) = \mu_H$ *under the relation $\approx$. Moreover, if μ has the sup-property, then $H/\mu_H \cong H/\mu$.*

P r o o f. It is clear that φ is surjective. Let $x_1^n \in H$. Then

$$\begin{aligned}\varphi(f(x_1^n)) &= \big\{F(x,\mu,e*) \mid x \in f(x_1^n)\big\}\\ &= g(F(x_1,\mu,e*),\dots,F(x_n,\mu,e*))\\ &= g(\varphi(x_1),\dots,\varphi(x_n)).\end{aligned}$$

Hence φ is an epimorphism. From Lemma 4.6, $x \in \mathrm{Ker}(\varphi) \iff \varphi(x) = F(x,\mu,e*) \approx F(e,\mu,e*) \iff M(\mu(x),0.5) = M(\mu(e),0.5) \iff x \in \mu_H$, hence $\mathrm{Ker}(\varphi) = \mu_H$. Now assume that μ has the sup-property. Then, by Lemma 4.6, $F(x,\mu,e*) = F(y,\mu,e*)$ implies $f(x,\mu_H,e*) = f(y,\mu_H,e*)$ for all $x,y \in H$, that is, φ is injective. Hence $H/\mu_H \cong H/\mu$. This completes the proof. □

Theorem 4.8. *Let φ be an epimorphism from H onto H', and μ an $(\in,\in\vee q)$-fuzzy n-ary subhypergroup of H such that $\mu_H \subseteq$* $\mathrm{Ker}(\varphi)$ *and that μ has the sup-property. Then there exists a unique epimorphism ψ from $(H/\mu, g)$ onto (H',f') such that $\varphi = \psi \circ \eta$ under the relation $\approx$, where $\eta(x) = F(x,\mu,e*)$ for all $x \in H$.*

P r o o f. Define a mapping $\psi\colon H/\mu \to H'$ by $\psi(F(x,\mu,e*)) = \varphi(x)$ for all $x \in H$. Then ψ is well defined. In fact, if $F(x,\mu,e*) \approx F(y,\mu,e*)$ for some $x,y \in H$, it follows from Lemma 4.6 that $f(x,\mu_H,e*) = f(y,\mu_H,e*)$. Hence $x \in f(x,e,e*) \subseteq f(y,\mu_H,e*)$ and $y \in f(y,e,e*) \subseteq f(x,\mu_H,e*)$. Since $\mu_H \subseteq \mathrm{Ker}(\varphi)$, we have $f(x,\mathrm{Ker}(\varphi),e*) \subseteq f(f(y,\mu_H,e*),\ \mathrm{Ker}(\varphi),e*) = f(y,f(\mu_H,\mathrm{Ker}(\varphi),e*),e*) \subseteq f(y,\mathrm{Ker}(\varphi),e*)$. In a similar way, we have $f(y,\mathrm{Ker}(\varphi),e*) \subseteq f(x,\mathrm{Ker}(\varphi),e*)$. Hence $f(x,\mathrm{Ker}(\varphi),e*) = f(y,\mathrm{Ker}(\varphi),e*)$ and so

$$\begin{aligned}\varphi(f(x,\mathrm{Ker}(\varphi),e*)) &= f'(\varphi(x),\varphi(\mathrm{Ker}(\varphi)),\varphi(e)*) = f'(\varphi(x),e'*)\\ &= \varphi(x) = \varphi(y) = \varphi(f(y,\mathrm{Ker}(\varphi),e*)).\end{aligned}$$

Now it is easy to check that ψ is a homomorphism.

Further, since φ is onto, ψ is also onto. On the other hand, $\varphi(x) = \psi(F(x,\mu,e*)) = \psi(\eta(x)) = (\psi\circ\eta)(x)$ for all $x \in H$. Finally, we show that ψ is unique. If there exists another epimorphism ϕ from $(H/\mu,g)$ onto (H',f') such that $\varphi = \phi\circ\eta$. Then $\psi(F(x,\mu,e*)) = \varphi(x) = (\phi\circ\eta)(x) = \phi(F(x,\mu,e*))$ for all $x \in H$. This implies $\psi = \phi$. □

Theorem 4.9. *Let φ be a homomorphism from H to H' and let μ and μ' be $(\in, \in\vee q)$-fuzzy n-ary subhypergroups of H and H', respectively, such that $\varphi(\mu) \subseteq\vee q\ \mu'$ and $\mu(e) = \mu'(e')$. Then there exists a homomorphism $\psi\colon H/\mu \to H'/\mu'$ such that the diagram*

$$\begin{array}{ccc} H & \xrightarrow{\ \varphi\ } & H' \\ \big\downarrow & & \big\downarrow \\ H/\mu & \xrightarrow[\ \psi\ ]{} & H'/\mu' \end{array}$$

is commutative. Moreover, if φ is an isomorphism and $\mu' \approx \varphi(\mu)$, then so is ψ.

Proof. Define a mapping $\psi\colon H/\mu \to H'/\mu'$ by $\psi(F(x,\mu,e*)) = F'(\varphi(x),\mu',e'*)$ for all $x \in H$. We first show that ψ is well defined. In fact, if $F(x,\mu,e*) \approx F(y,\mu,e*)$ for some $x, y \in H$, it follows from Lemma 3.13 that $M(\mu(e),0.5) = M((F(x,\mu,e*))(y),0.5) = M((F(y,\mu,e*))(x),0.5)$. To show $F'(\varphi(x),\mu',e'*) \approx F'(\varphi(y),\mu',e'*)$, it needs only to show

$$M(\mu'(e'),0.5) = M(F'(\varphi(x),\mu',e'*)(\varphi(y)),0.5) = M(F'(\varphi(y),\mu',e'*)(\varphi(x)),0.5)$$

by Lemma 3.13. Now, by the assumption, $\varphi(\mu) \subseteq\vee q\ \mu'$, we have

$$F'(\varphi(x),\varphi(\mu),e'*) \subseteq\vee q\ F'(\varphi(x),\mu',e'*)$$

and so

$$\begin{aligned} F'(\varphi(x),\mu',e'*)(\varphi(y)) &\geq M(F'(\varphi(x),\varphi(\mu),e'*)(\varphi(y)),0.5) \\ &= M(\varphi(F(x,\mu,e*))(\varphi(y)),0.5) \\ &\geq M(F(x,\mu,e*)(y),0.5) \\ &= M(\mu(e),0.5) = M(\mu'(e'),0.5). \end{aligned}$$

On the other hand, by Lemma 3.13(1),

$$M(\mu'(e'),0.5) \geq M(F'(\varphi(x),\mu',e'*)(\varphi(y)),0.5) \geq M(\mu(e'),0.5)$$

and so $M(\mu(e'),0.5) = M(F'(\varphi(x),\mu',e'*)(\varphi(y)),0.5)$. In a similar way, we have $M(\mu(e'),0.5) = M(F'(\varphi(y),\mu',e'*)(\varphi(x)),0.5)$. Hence ψ is well defined. Now it is easy to check that ψ is a homomorphism and the diagram is commutative.

Next assume that φ is an isomorphism and $\mu' \approx \varphi(\mu)$. Since φ is onto, it is clear that ψ is also onto. Let $x, y \in H$ be such that $F'(\varphi(x),\mu',e'*) \approx F'(\varphi(y),\mu',e'*)$, then $F'(\varphi(x),\varphi(\mu),e'*) \approx F'(\varphi(y),\varphi(\mu),e'*)$ and so

$$M(F'(\varphi(x),\varphi(\mu),e'*)(\varphi(y)),0.5) = M(F'(\varphi(y),\varphi(\mu),e'*)(\varphi(x)),0.5).$$

Since φ is injective, we have

$$M(F'(\varphi(x),\varphi(\mu),e'*)(\varphi(y)),0.5) = M\Bigg(\bigvee_{\varphi(y)\in f'(\varphi(x),a',e'*)} \varphi(\mu)(a'),0.5\Bigg)$$

$$= M\Bigg(\bigvee_{\varphi(y)\in \varphi(f(x,a,e*))} \varphi(\mu)(\varphi(a)),0.5\Bigg) = \bigvee_{y\in f(x,a,e*)} \mu(a) = M(F(x,\mu,e*)(y),0.5).$$

In a similar way, we have

$$M(F'(\varphi(y),\varphi(\mu),e'*)(\varphi(x)),0.5) = M(F(y,\mu,e*)(x),0.5).$$

Hence $M(\mu(e),0.5) = M(F(x,\mu,e*)(y),0.5) = M(F(y,\mu,e*)(x),0.5)$. It follows from Lemma 3.13 that $F(x,\mu,e*) \approx F(y,\mu,e*)$. Hence ψ is also injective. Therefore, ψ is an isomorphism. This completes the proof. □

As two special cases of 4.9, we have the following results.

Theorem 4.10. *Let φ be an epimorphism from H onto H' and μ an $(\in,\in\vee q)$-fuzzy n-ary subhypergroup of H. Define*

$$\psi(F(x,\mu,e*)) = F'(\varphi(x),\varphi(\mu),e'*) \qquad \textit{for all} \quad x\in H.$$

Then ψ is an epimorphism from H/μ onto $H'/\varphi(\mu)$. Moreover, if φ is an isomorphism, then $H/\mu \cong H'/\varphi(\mu)$.

Theorem 4.11. *Let φ be an isomorphism from H to H' and μ' an $(\in,\in\vee q)$-fuzzy n-ary subhypergroup of H'. Then $H/\varphi^{-1}(\mu') \cong H'/\mu'$.*

Lemma 4.12. *Let μ and ν be two invertible $(\in,\in\vee q)$-fuzzy n-ary subhypergroups of H. Then $\mu\cap\nu$ $(\mu\cap\nu\neq\emptyset)$ is an $(\in,\in\vee q)$-fuzzy n-ary subhypergroup of H.*

Proof. Let μ and ν be two invertible $(\in,\in\vee q)$-fuzzy n-ary subhypergroups of H. Then we have:

(1) $(\mu\cap\nu)(e) = M(\mu(e),\nu(e)) \geq M(M(\mu(x),0.5),M(\nu(x),0.5)).$
$\qquad = M((\mu\cap\nu)(x),0.5) \qquad$ for all $\quad x\in H$

(2) By Lemma 3.2, we have

$$F(\underbrace{\mu\cap\nu,\dots,\mu\cap\nu}_{n}) \subseteq F(\underbrace{\mu,\dots,\mu}_{n}) \cap F(\underbrace{\nu,\dots,\nu}_{n}) \subseteq\vee q\ \mu\cap\nu.$$

(3) Let $x,y,y_1^{n-1}\in H$ be such that $y\in f(x,y_1^{n-1})$. Since both μ and ν are invertible, by Lemma 3.11,

$$\mu(x) \geq M(\mu_{y_1}^{y_{n-1}},\mu(y),0.5) \quad \text{and} \quad \nu(x) \geq M(\nu_{y_1}^{y_{n-1}},\nu(y),0.5).$$

Hence $(\mu\cap\nu)(x) = M(\mu(x),\nu(x)) \geq M(M(\mu_{y_1}^{y_{n-1}},\mu(y),0.5),M(\nu_{y_1}^{y_{n-1}},\nu(y),0.5))$
$= M((\mu\cap\nu)_{y_1}^{y_{n-1}},(\mu\cap\nu)(y),0.5)$.

Summing up the above arguments, $\mu \cap \nu$ is an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H. □

Theorem 4.13. *Let μ and ν be two invertible $(\in, \in\vee q)$-fuzzy n-ary subhypergroups of H such that $\mu(e) = \nu(e)$. Then $\mu_H/\mu \cap \nu \cong f(\mu_H, \nu_H, e*)/\nu$.*

Proof. It follows from Lemmas 4.6 and 4.12 that both $(\mu_H/\mu \cap \nu, g)$ and $(f(\mu_H, \nu_H, e*)/\nu, g)$ are n-ary hypergroups.

Define a mapping $\varphi\colon \mu_H/\mu \cap \nu \to f(\mu_H, \nu_H, e*)/\nu$ by $\varphi(F(x, \mu \cap \nu, e*)) = F(x, \nu, e*)$ for all $x \in \mu_H$. We first show that φ is well defined. In fact, let $x, y \in \mu_H$ be such that $F(x, \mu \cap \nu, e*) \approx F(y, \mu \cap \nu, e*)$. It follows from Lemma 3.13 that $M((\mu \cap \nu)(e), 0.5) = M(F(x, \mu \cap \nu, e*)(y), 0.5) = M(F(y, \mu\cap\nu, e*)(x), 0.5)$. Thus we have $M(\nu(e), 0.5) \geq M(F(x, \nu, e*)(y), 0.5) \geq M(F(x, \mu\cap\nu, e*)(y), 0.5) = M((\mu\cap\nu)(e), 0.5) = M(\nu(e), 0.5)$ and so $M(\nu(e), 0.5) = M(F(x, \nu, e*)(y), 0.5)$.

In a similar way, we have $M(\nu(e), 0.5) = M(F(y, \nu, e*)(x), 0.5)$. Hence $F(x, \nu, e*) \approx F(y, \nu, e*)$ by Lemma 3.13. Therefore, φ is well defined. Now it is easy to check that φ is a homomorphism.

Next let x be any element of $f(\mu_H, \nu_H, e*)$. Then there exist $y \in \mu_H, z \in \nu_H$ such that $x \in f(y, z, e*)$. Since $z \in \nu_H$ and ν is an invertible $(\in, \in\vee q)$-fuzzy n-ary subhypergroup of H, by Lemma 3.11, we have $M(\nu(e), 0.5) = M(\nu(z), 0.5)$ and

$$M(\nu(e), 0.5) \geq M(F(z, \nu, e*)(e), 0.5) = M\Bigg(\bigvee_{e\in f(z,a,e*)} \nu(a), 0.5\Bigg)$$
$$\geq M\Bigg(\bigvee_{e\in f(z,a,e*)} M(\nu(e), \nu(z), 0.5), 0.5\Bigg) = M(\nu(e), 0.5),$$

which implies $M(\nu(e), 0.5) = M(F(z, \nu, e*)(e), 0.5)$. Hence $F(z, \nu, e*) \approx \nu$ by Lemma 3.13. Thus $F(x, \nu, e*) \subseteq F(f(y, z, e*), \nu, e*) = F(y, F(z, \nu, e*), e*) \approx F(y, \nu, e*)$. Hence $F(x, \nu, e*) \subseteq\vee q\ F(y, \nu, e*)$. In a similar way, we have $F(y, \nu, e*) \subseteq\vee q\ F(x, \nu, e*)$ since ν is invertible. Thus

$$F(x, \nu, e*) = \varphi(F(y, \mu \cap \nu, e*)).$$

This implies φ is onto.

To show φ is injective, let $x', y' \in f(\mu_H, \nu_H, e*)$ be such that $F(x', \nu, e*) \approx F(y', \nu, e*)$. Then it follows from the above proof that there exist $x, y \in \mu_H$ such that $F(x, \nu, e*) \approx F(x', \nu, e*) \approx F(y', \nu, e*) \approx F(y, \nu, e*)$. Thus $M(\nu(e), 0.5) = M(F(x, \nu, e*)(y), 0.5) = M(F(y, \nu, e*)(x), 0.5)$. On the other hand, since μ and ν are invertible and $x, y \in \mu_H$, by Lemma 3.11, we have

$$\begin{aligned}
&M(F(x,\mu\cap\nu,e*)(y),0.5)\\
&= M\Big(\bigvee_{y\in f(x,a,e*)} M(\mu(a),\nu(a)),0.5\Big)\\
&\geq M\Big(\bigvee_{y\in f(x,a,e*)} M(M(\mu(x),\mu(y),0.5),M(\nu(x),\nu(y),0.5)),0.5\Big)\\
&= M(M(\mu(x),0.5),M(\mu(y),0.5),M(\nu(x),0.5),M(\nu(y),0.5))\\
&= M((\mu\cap\nu)(e),0.5).
\end{aligned}$$

In a similar way, we have $M(F(y,\mu\cap\nu,e*)(x),0.5) = M((\mu\cap\nu)(e),0.5)$. Thus $F(x,\mu\cap\nu,e*) \approx F(y,\mu\cap\nu,e*)$ by Lemma 3.13. This completes the proof. □

Theorem 4.14. *Let μ and ν be two $(\in,\in\vee q)$-fuzzy n-ary subhypergroups of H such that $\mu \subseteq\vee q\ \nu$ and $\mu(e)=\nu(e)$. Then $(H/\mu)/(\nu_H/\mu) \cong H/\nu$.*

Proof. By Theorem 3.15, ν_H/μ is an n-ary subhypergroup of H/μ. Define a mapping $\varphi\colon H/\mu \to H/\nu$ by $\varphi(F(x,\mu,e*)) = F(x,\nu,e*)$ for all $x\in S$. We first show that φ is well defined. In fact, if $F(x,\mu,e*) \approx F(y,\mu,e*)$ for some $x,y\in H$, then $M(\mu(e),0.5) = M(F(x,\mu,e*)(y),0.5) = M(F(y,\mu,e*)(x),0.5)$ by Lemma 3.13. Since $\mu\subseteq\vee q\ \nu$, we have $M(\nu(e),0.5) \geq M(F(x,\nu,e*)(y),0.5)$ $\geq M(M(F(x,\mu,e*)(y),0.5),0.5) = M(\mu(e),0.5) = M(\nu(e),0.5)$ and so $M(\nu(e),0.5) = M(F(x,\nu,e*)(y),0.5)$. In a similar way, we have $M(\nu(e),0.5) = M(F(y,\nu,e*)(x),0.5)$. Hence $F(x,\nu,e*) \approx F(y,\nu,e*)$ by Lemma 3.13. Therefore, φ is well defined. Then it is easy to check that φ is a homomorphism.

Further, it is clear that φ is onto. Now we show that $\mathrm{Ker}(\varphi) = \nu_H/\mu$. Before proceeding, we first show that $F(x,\mu,e*) \approx F(y,\mu,e*)$ for some $x\in\nu_H, y\in S$ implies $y\in\nu_H$. In fact, it follows from $F(x,\mu,e*)\approx F(y,\mu,e*)$ that $M(\mu(e),0.5) = M(F(x,\mu,e*)(y),0.5)$ by Lemma 3.13. Now, if $y\in f(x,a,e*)$ for some $a\in H$, we have

$$\nu(y) \geq M(\nu(x),\nu(a),0.5) = M(\nu(e),\nu(a),0.5) \geq M(\mu(a),0.5)$$

and so

$$\begin{aligned}
\nu(y) \geq \bigvee_{y\in f(x,a,e*)} M(\mu(a),0.5) &= M(F(x,\mu,e*)(y),0.5)\\
&= M(\mu(e),0.5) = M(\nu(e),0.5).
\end{aligned}$$

This implies $M(\nu(y),0.5) = M(\nu(e),0.5)$ and so $y\in\nu_H$. Thus $x\in\mathrm{Ker}(\varphi) \iff F(x,\nu,e*) \approx \nu \iff M(\nu(x),0.5) = M(\nu(e),0.5) \iff x\in\nu_H \iff F(x,\mu,e*)\in\nu_H/\mu$. Hence $(H/\mu)/(\nu_H/\mu) = (H/\mu)/\mathrm{Ker}(\varphi) \cong H/\nu$. □

5. Conclusions

Since Marty [30] introduced the notion of hyperstructure in 1934, his ideas have been studied in the following decades and nowadays by many researchers. By a new ideal, we introduced and studied (invertible) $(\in, \in\vee q)$-fuzzy n-ary subhypergroups of a commutative n-ary hypergroup. We presented and investigated a kind of n-ary quotient hypergroup by an $(\in, \in\vee q)$-fuzzy n-ary subhypergroup. We also considered the relationships among $(\in, \in\vee q)$-fuzzy n-ary subhypergroups, n-ary quotient hypergroups and homomorphism and provided several isomorphism theories of n-ary hypergroups. It is worth noting that in an (α, β)-fuzzy n-ary subhypergroups of a commutative n-ary hypergroup, the values of α, β can be chosen as any one of $\{\in, q, \in\vee q, \in\wedge q\}$ with $\alpha \neq \in\wedge q$. In fact, there are twelve different types of such structures resulting from three choices of α and four choices of β, however, in this paper, we only consider the $(\in, \in\vee q)$-type. In our future study of fuzzy structure of, we will focus on considering other types (α, β)-fuzzy n-ary subhypergroups with relations among them. The results presented in this paper can hopefully provide more insight into and a full understanding of fuzzy set theory and algebraic hyperstructures.

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