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ON FINITE RETRACT LATTICES OF MONOUNARY ALGEBRAS

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ABSTRACT. For a monounary algebra (A, f) we denote $\mathbf{R}^{\emptyset}(A, f)$ the system of all retracts (together with the empty set) of (A, f) ordered by inclusion. This system forms a lattice. We prove that if (A, f) is a connected monounary algebra and $\mathbf{R}^{\emptyset}(A, f)$ is finite, then this lattice contains no diamond. Next distributivity of $\mathbf{R}^{\emptyset}(A, f)$ is studied. We find a representation of a certain class of finite distributive lattices as retract lattices of monounary algebras.

1. Introduction

The present paper can be regarded as a continuation of the authors' article [6] where several properties of the lattice of all retracts of monounary algebras were investigated. Now we focus mainly on finite retract lattices of monounary algebras.

The notion of retract, first introduced for topological spaces, was later generalized to the notion of retract of arbitrary algebraic structures and was intensively studied. We recall that a substructure $\mathcal{B}$ of a structure $\mathcal{A}$ is called a retract of $\mathcal{A}$ if there exists and endomorphism (called a retraction) φ of $\mathcal{A}$ onto $\mathcal{B}$ such that $\varphi(b) = b$ for each element b of $\mathcal{B}$. The study of retracts of monounary algebras [3]–[5] was motivated by the paper [1], where analogous questions for posets were dealt with.

The study of monounary algebras occurs as a useful tool for investigating some properties of algebraic structures of arbitrary type. Novotný [10] proved

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that the constructions of homomorphisms or of retracts of general algebras can be reduced to constructions of homomorphisms or of retracts of some special monounary algebras.

For a monounary algebra (A, f) we denote $\mathbf{R}^{\emptyset}(A, f)$ the system of all retracts (together with the empty set) of (A, f) ordered by inclusion. This system forms a lattice (cf. [6]). In this paper it is proved that if (A, f) is a connected monounary algebra and $\mathbf{R}^{\emptyset}(A, f)$ is finite, then this lattice contains no diamond. Next distributivity of $\mathbf{R}^{\emptyset}(A, f)$ with the connection of the length of $\mathbf{R}^{\emptyset}(A, f)$ is studied. Finally we find a representation of a certain class of finite distributive lattices as retract lattices of monounary algebras.

2. Preliminaries

By a monounary algebra we understand an ordered pair (A, f) where A is a nonempty set and $f\colon A \to A$ is a mapping.

If $\emptyset \neq B \subseteq A$, then we denote $f \restriction B$ the partial operation on B such that $\operatorname{dom}(f \restriction B) = \{b \in B : \ f(b) \in B\}$ and if $b \in \operatorname{dom}(f \restriction B)$, then $(f \restriction B)(b) = f(b)$. Then $(B, f \restriction B)$ is called a *relative subalgebra* of (A, f). If there is no danger of confusion we briefly write (B, f) instead of $(B, f \restriction B)$.

Let (A, f) be a monounary algebra. A nonempty subset M of A is said to be a *retract* of (A, f) if there is a mapping φ of A onto M such that φ is an endomorphism of (A, f) and $\varphi(x) = x$ for each $x \in M$. The mapping φ is then called a *retraction endomorphism* corresponding to the retract M.

A monounary algebra (A, f) is called *connected* if for arbitrary elements $x, y \in A$ there are non-negative integers n, m such that $f^n(x) = f^m(y)$. A maximal connected subalgebra of a monounary algebra is called a *(connected) component.*

For $X \subseteq A$ we denote by $[X]$ subalgebra generated by the set X. If $X = \{x\}$ we use the notation $[x]$ instead of $[\{x\}]$.

The notion of *degree* $s_f(x)$ of an element $x \in A$ was introduced in [9] (cf. also [8]) for describing homomorphisms of monounary algebras as follows. Let us denote by A^{∞} the set of all elements $x \in A$ such that there exists a sequence $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ of elements belonging to A with the property $x_0 = x$ and $f(x_n) = x_{n-1}$ for each $n \in \mathbb{N}$. Further, we put $A^0 = \{x \in A : \ f^{-1}(x) = \emptyset\}$. Now we define a set $A^{(\lambda)} \subseteq A$ for each ordinal λ by induction. Assume that we have defined A^α for each ordinal $\alpha < \lambda$. Then we put

$$A^\lambda = \Big\{x \in A \setminus \bigcup_{\alpha<\lambda} A^{(\alpha)} :\ f^{-1}(x) \subseteq \bigcup_{\alpha<\lambda} A^{(\alpha)}\Big\}.$$

The sets A^λ are pairwise disjoint. For each $x \in A$, either $x \in A^\infty$ or there is an ordinal λ with $x \in A^\lambda$. In the former case we put $s_f(x) = \infty$, in the latter we set $s_f(x) = \lambda$. We put $\lambda < \infty$ for each ordinal λ.

Let (A, f) be a connected monounary algebra. We say that (A, f) is *unbounded*, if

(i) $s_f(x) \neq \infty$ for each $x \in A$,

(ii) if $x \in A$, $n \in \mathbb{N}$, then there is $m \in \mathbb{N}$ such that $f^{-(m+n)}(f^m(x)) \neq \emptyset$.

The connected monounary algebra (A, f) is said to be *bounded* if (A, f) satisfies (i) and does not satisfy (ii).

We will use the following notation for some algebras.

As usual, the symbols $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of all integers or of positive integers, respectively. We denote by $(\mathbb{Z}, \mathrm{suc})$ and $(\mathbb{N}, \mathrm{suc})$ the algebras such that suc is the operation of the successor. Next we denote $Z_{n,\infty} = \mathbb{Z}_n \cup \mathbb{N}$ and for $x \in Z_{n,\infty}$ we set

$$\mathrm{suc}(x) = \begin{cases} x+1 & \text{if } x \in \mathbb{Z}_n \\ x-1 & \text{if } x \in \mathbb{N} \setminus \{1\} \\ 0_n & \text{if } x = 1. \end{cases}$$

Let us notice that in [5, Corollary 1.6] it was proved that a connected monounary algebra (A, f) contains the copy of $(\mathbb{N}, \mathrm{suc})$ as a retract if and only if (A, f) is bounded.

Also, we will use the following notation: Let (A, f) be a connected monounary algebra. For a subset $B \subseteq A$ denote by $B^0 = \big\{x \in B :\ f^{-1}(x) = \emptyset\big\}$. If (A, f) contains no cycle, we denote by $B^{\mathbb{Z}} = \big\{X \subseteq B :\ (X, f) \cong (\mathbb{Z}, \mathrm{suc})\big\}$.

In [3] the following theorem characterizing retracts of connected monounary algebras was proved.

Theorem 2.1. *Let (A, f) be a connected monounary algebra and let (M, f) be a subalgebra of (A, f). Then M is a retract of (A, f) if and only if the following condition is satisfied:*

If $y \in f^{-1}(M)$, then there is $z \in M$ with $f(y) = f(z)$ and $s_f(y) \leq s_f(z)$.

Let (A, f) be a connected monounary algebra. From Theorem 2.1 we obtain the following facts:

(i) If (A, f) contains a cycle C, then C is a retract of (A, f).

(ii) If (A, f) contains a subalgebra (M, f), $(M, f) \cong (\mathbb{Z}, \mathrm{suc})$, then M is a retract of (A, f).

(iii) Suppose that (A, f) contains no cycle. Then every retract M of (A, f) as a subalgebra is equal to a subalgebra generated by the set $\bigcup M^{\mathbb{Z}} \cup M^0$. We note that for any system X of sets, $\bigcup X = \{x : (\exists X' \in X)(x \in X')\}$.

(iv) If M is a retract of (A, f) and $A^{\mathbb{Z}} \neq \emptyset$ then $M^{\mathbb{Z}} \neq \emptyset$.

3. Lattice of retracts of monounary algebras

We recall some results contained in [6]. The system of all retracts of a given monounary algebra (A, f) is denoted $R(A, f)$. This system together with empty set ordered by inclusion is denoted $\mathbf{R}^{\emptyset}(A, f)$. Retracts of monounary algebra (A, f) are closed under arbitrary unions, hence $\mathbf{R}^{\emptyset}(A, f)$ forms a complete lattice with the greatest element A and with the least element $\emptyset$. For $B \in R(A, f) \cup \{\emptyset\}$ denote by $\langle \mathbf{B})$ a sublattice consisting of all retracts of (A, f) greater than B.

Further let (A, f) be a connected monounary algebra, $x \in A$ be an arbitrary noncyclic element. Denote $A^x = \bigcup\limits_{n \in \mathbb{N}_0} f^{-n}(x) \cup \mathbb{N}$ (we may assume that the sets $\mathbb{N}$ and $\bigcup\limits_{n \in \mathbb{N}_0} f^{-n}(x)$ are disjoint) and for $y \in A^x$ we set $\tilde{f}(y) = f(y)$ if $y \in \bigcup\limits_{n \in \mathbb{N}} f^{-n}(x)$, $\tilde{f}(x) = 1$ and $\tilde{f}(y) = y + 1$ for $y \in \mathbb{N}$.

Lemma 3.1. ([6, Lemma 3.3]) *If (A, f) is a connected monounary algebra, $B \in R(A, f)$, then*

$$\langle \mathbf{B}) \cong \prod_{x \in f^{-1}(B) \smallsetminus B} \mathbf{R}^{\emptyset}(A^x, \tilde{f}) \ .$$

Next, we describe some other properties of $\mathbf{R}^{\emptyset}(A, f)$. We say that a lattice L is *atomic* if L has the least element 0, and for every $a \in L$, $a \neq 0$, there is an atom $p \leq a$. It is not difficult to verify that if C is a retract of B and B is a retract of A then C is a retract of A. Hence, if $\mathbf{R}^{\emptyset}(A, f)$ contains an atom, then this atom has no proper retracts and it is isomorphic to a cycle or $(\mathbb{Z}, \mathrm{suc})$ or $(\mathbb{N}, \mathrm{suc})$.

Lemma 3.2. ([6, Lemma 4.2]) *Let (A, f) be a connected monounary algebra. The lattice $\mathbf{R}^{\emptyset}(A, f)$ is atomic if and only if (A, f) is not unbounded.*

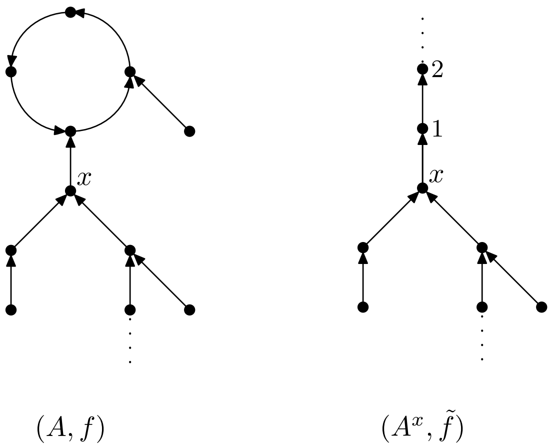


FIGURE 1. An example of an algebra (A, f) and $(A^x, \tilde{f})$ corresponding to the element x of algebra (A, f)

Next two assertions describe covering relation in $\mathbf{R}^{\emptyset}(A, f)$ and semimodularity.

LEMMA 3.3. ([6, Lemma 5.1]) *Let (A, f) be a connected monounary algebra, $B, C \in R(A, f)$, $\emptyset \neq B \subseteq C$. Then C covers B in $\mathbf{R}(A, f)$ if and only if the relative subalgebra $(C \setminus B, f)$ is isomorphic to $\big((-\infty, 0\rangle, \mathrm{suc}\big)$ or $\big(\langle 0, n\rangle, \mathrm{suc}\big)$ for some $n \in \mathbb{N}_0$.*

THEOREM 3.4. ([6, Theorem 5.3]) *If (A, f) is a connected monounary algebra, then $\mathbf{R}^{\emptyset}(A, f)$ forms a semimodular lattice.*

The following theorem give a description of connected monounary algebras with modular retract lattices via forbidden configuration of their elements.

THEOREM 3.5. ([6, Theorem 6.3]) *Let (A, f) be a connected monounary algebra possessing no cycle with $s_f(x) \in \omega \cup \{\infty\}$ for all $x \in A$. Then $\mathbf{R}^{\emptyset}(A, f)$ is not modular if and only if there exist $a, b, c \in A$ satisfying*

(i) $f(b) = f(c)$ *and* $s_f(b) = s_f(c) = \max\big\{s_f(x) : \ x \in f^{-1}(f(b))\big\}$,

(ii) *there is* $n \in \mathbb{N}$ *such that* $f(a) = f^n(b) = f^n(c)$ *and* $s_f(a) < s_f(f^{n-1}(b))$,

(iii) $s_f(x) \leq s_f(f^{m-1}(b))$ *for all* $x \in f^{-1}(f^m(b))$, $m = 1, \dots, n$.

We note that if an algebra (A, f) contains an element of degree ω, then the lattice $\mathbf{R}^{\emptyset}(A, f)$ is not modular.

The last theorem of this section gives a description of completely join-irreducible elements in $\mathbf{R}^{\emptyset}(A, f)$.

Theorem 3.6. ([7, Theorem 3.1]) *Let (A, f) be a connected monounary algebra possessing no cycle. A retract $R \in R(A, f)$ is completely join-irreducible in $\mathbf{R}^{\emptyset}(A, f)$ if and only if the following conditions are satisfied.*

1. *For all $x, y \in R$, $x \neq y$, $f(x) = f(y)$ implies that $s_f(x) \neq s_f(y)$.*
2. *For all $x \in R$, $\left|f^{-1}(x) \cap R\right| \leq 2$.*
3. *For all $x \in R^0 \cup \bigcup R^{\mathbb{Z}}$ there exist at most one $n \in \mathbb{N}$ and $z \in R$ with $f(z) = f^n(x)$ and $s_f(z) < s_f(f^{n-1}(x))$.*

4. Length of retract lattices

In this section we will deal with the length of retract lattices. We prove that for a connected monounary algebra (A, f) the lattice $\mathbf{R}^{\emptyset}(A, f)$ is of finite length if and only if A as a retract is finitely generated.

The *length*, $l(C)$, of a finite chain C is $|C| - 1$. A poset P is said to be of length n, (in formula, $l(P) = n$), where $n \in \mathbb{N} \cup \{0\}$, if there is a chain in P of length n and all chains in P are of length $\leq n$. A poset P is of finite length if it is of length n, for some positive integer n.

Theorem 4.1. *Let (A, f) be a connected monounary algebra possessing no cycle and $n \in \mathbb{N}$. Then $l\big(\mathbf{R}^{\emptyset}(A, f)\big) = n$ if and only if $\left|A^0\right| + \left|A^{\mathbb{Z}}\right| = n$.*

P r o o f. Let (A, f) be a connected algebra possessing no cycle with $\left|A^0\right| + \left|A^{\mathbb{Z}}\right| = n$. Since $\mathbf{R}^{\emptyset}(A, f)$ is semimodular (due 3.4), it satisfies the Jordan-Hölder chain condition (any two maximal chains of $\mathbf{R}^{\emptyset}(A, f)$ are of the same length), hence to prove that $l\big(\mathbf{R}^{\emptyset}(A, f)\big) = n$ it is sufficient to find any maximal chain of length n.

Let $A^0 = \{x_1, x_2, \ldots, x_k\}$ and $A^{\mathbb{Z}} = \{X_1, X_2, \ldots, X_m\}$. Put $C_0 = \emptyset$ and for $i = 1, \ldots, m$, $C_i = \bigcup\limits_{1 \leq j \leq i} X_j$. If $A^{\mathbb{Z}} = \emptyset$ we put $C_1 = R$ where R is any minimal retract isomorphic to $(\mathbb{N}, \mathrm{suc})$ and we put $m = 1$. Further let $m < i < n$ and suppose that C_{i-1} is defined. Let $y \in f^{-1}(C_{i-1}) \smallsetminus C_{i-1}$ be an arbitrary element. An algebra $(A^y, \tilde{f})$ is bounded, thus there is a retract $\tilde{R}$ of $(A^y, \tilde{f})$ such that $(\tilde{R}, \tilde{f}) \cong (\mathbb{N}, \mathrm{suc})$. Let $\tilde{R} = [x_j]$ for some j, $1 \leq j \leq k$. Put $C_i = C_{i-1} \cup [x_j]$. Since $\tilde{R}$ is a retract of $(A^y, \tilde{f})$, C_i is a retract of (A, f). Now it is evident that $C_n = A$. For all i, $1 \leq i \leq n - 1$, the relative subalgebra $(C_{i+1} \smallsetminus C_i, f)$ is isomorphic to $\big((-\infty, 0\rangle, \mathrm{suc}\big)$ or $\big(\langle 0, r\rangle, \mathrm{suc}\big)$ for some $r \in \mathbb{N}_0$, hence according to 3.3, $C_i \prec C_{i+1}$ for all $i = 0, 1, \ldots, n - 1$. The set $\{C_0, C_1, \ldots, C_n\}$ forms a maximal chain in $\mathbf{R}^{\emptyset}(A, f)$, therefore $l\big(\mathbf{R}^{\emptyset}(A, f)\big) = n$.

To complete the proof we show that $|A^0| + |A^{\mathbb{Z}}| \geq \aleph_0$ implies the existence of an infinite chain in $\mathbf{R}^{\emptyset}(A, f)$.

First suppose that $|A^{\mathbb{Z}}| \geq \aleph_0$. Let $\{X_i :\ i \in \mathbb{N}\}$ be an infinite subset of $A^{\mathbb{Z}}$. For $n \in \mathbb{N}$ put $C_n = \bigcup_{1 \leq i \leq n} X_i$. Then the set $\{C_i :\ i \in \mathbb{N}\}$ forms an infinite chain in $\mathbf{R}^{\emptyset}(A, f)$.

If (A, f) is unbounded, then there is no atom in $\mathbf{R}^{\emptyset}(A, f)$, hence there is an infinite chain, too.

Assume that (A, f) is not unbounded and $|A^{\mathbb{Z}}| < \aleph_0$. Denote $R = \bigcup A^{\mathbb{Z}}$. We may assume that for all $y \in f^{-1}(R) \smallsetminus R$ the algebra $(A^y, \tilde{f})$ is bounded, otherwise $\mathbf{R}^{\emptyset}(A, f)$ contains an infinite chain, since according to 3.1 $\langle \mathbf{R} \rangle \cong \prod_{y \in f^{-1}(R) \smallsetminus R} \mathbf{R}^{\emptyset}(A^y, \tilde{f})$. In the case that each $(A^y, \tilde{f})$ is bounded put $C_0 = R$ and for $n \in \mathbb{N}$ put $C_n = C_{n-1} \cup [x]$, where the element x generates a retract of $(A^y, \tilde{f})$, $y \in f^{-1}(C_{n-1}) \smallsetminus C_{n-1}$ isomorphic to $(\mathbb{N}, \mathrm{suc})$. Obviously, the set $\{C_i :\ i \in \mathbb{N}_0\}$ forms an infinite chain in $\mathbf{R}^{\emptyset}(A, f)$. $\square$

If a connected monounary algebra (A, f) possesses a cycle C then we obtain that $l(\mathbf{R}^{\emptyset}(A, f)) = n + 1$ if and only if $|A^0| + |A^{\mathbb{Z}_C}| = n$ where $A^{\mathbb{Z}_C} = \{X \subseteq A : (X, f) \cong (Z_{m,\infty}, \mathrm{suc}), m = |C|\}$

Suppose that (A, f) is a connected monounary algebra such that $\mathbf{R}^{\emptyset}(A, f)$ is finite. In general it is not possible to find a finite connected monounary algebra (B, g) such that $\mathbf{R}^{\emptyset}(B, g) \cong \mathbf{R}^{\emptyset}(A, f)$. Consider a connected monounary algebra (A, f) having at least two minimal retracts. Since (B, g) is finite and connected, it possesses a cycle which is the only atom in $\mathbf{R}^{\emptyset}(B, g)$, thus $\mathbf{R}^{\emptyset}(B, g) \not\cong \mathbf{R}^{\emptyset}(A, f)$.

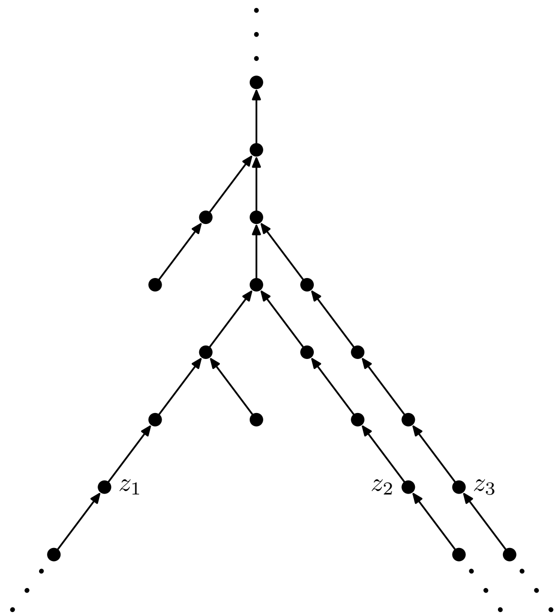
This problem can be eliminated if we allow that $\mathbf{R}^{\emptyset}(B, g) \cong \mathbf{R}^{\emptyset}(A, f)$ or $\mathbf{R}(B, g) \cong \mathbf{R}^{\emptyset}(A, f)$, where $\mathbf{R}(B, g)$ denote the system of all retracts of (B, g) without the emptyset.

First suppose that (A, f) possesses no cycle and $A^{\mathbb{Z}} = \{X_1, \dots, X_m\}$, $m \in \mathbb{N}$. In this case we "cut off" the copies of $(\mathbb{Z}, \mathrm{suc})$. Let $\{z_1, \dots, z_m\}$ be a subset of A with the following properties:

(i) $z_i \in X_i$, for all $i = 1, \dots, m$,

(ii) if $f^k(z_i) = f^l(z_j)$ then $k = l$,

(iii) if $f(y) = f^{k+1}(z_i)$ and $y \notin X_j$ for $j = 1, \dots, m$ then $s_f(y) < k$.



Let (B', f) be a subalgebra generated by the set $A^0 \cup \{z_1, \dots, z_m\}$. According to the properties (ii) and (iii), for each $i = 1, \dots, m$, $[z_i]$ forms a retract. These retracts are the only atoms in $\mathbf{R}^{\emptyset}(B', f)$. For $R \in R^{\emptyset}(B', f)$ put

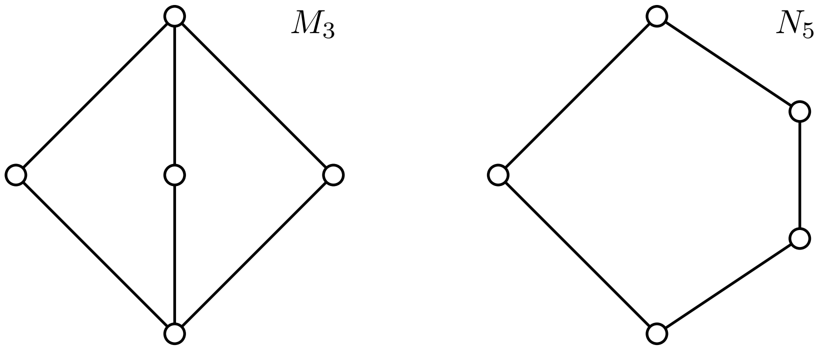
$$\varphi(R) = [R^0 \cap A^0] \cup \bigcup_{\{i\colon\ z_i \in R^0\}} X_i \ .$$

It is not difficult to verify that φ is isotone with isotone inverse. Hence $\mathbf{R}^{\emptyset}(A, f) \cong \mathbf{R}^{\emptyset}(B', f)$. Futher let $z \in A$ be such an element that for all $n \in \mathbb{N}_0$ we have: $\left|f^{-1}(f^n(z))\right| = 1$. Denote $B = B' \setminus \bigcup_{n \in \mathbb{N}} f^n(z)$ and define an operation g as follows: $g(z) = z, g(x) = f(x)$ otherwise. The mapping $\varphi\colon \mathbf{R}^{\emptyset}(B', f) \to \mathbf{R}(B, g)$ defined $\varphi(R) = [R^0]$ if $R \in R(B', f)$ and $\varphi(\emptyset) = \{z\}$ is a lattice isomorphism. Therefore we obtain that $\mathbf{R}^{\emptyset}(A, f) \cong \mathbf{R}(B, g)$.

If the algebra (A, f) possesses a cycle we construct a finite algebra (B, g) in a similar way. In this case $\mathbf{R}^{\emptyset}(A, f) \cong \mathbf{R}^{\emptyset}(B, g)$.

5. Diamonds in $\mathbf{R}^{\emptyset}(A, f)$

A subset L' of a lattice L is called a *diamond* or *pentagon* if L' is a sublattice isomorphic to M_3 or N_5 respectively.



First we recall some facts. The lattice M_3 is simple, i.e., has only trivial congruences, thus if a direct product $\prod_{i\in I} L_i$ contains a diamond, there is $j \in I$ such that L_j contains a diamond, too. It follows from the fact that all projections are homomorphisms and at least one distinguishes the elements of M_3. It means that there is at least one $j \in I$ such that projection $p_j\colon M_3 \subseteq \prod_{i\in I} L_i \to L_j$ has not one element range.

In [6, Corollary 6.2] it was proved that for a connected monounary algebra (A, f) the following holds: if $\mathbf{R}^{\emptyset}(A, f)$ is modular then it is distributive, thus if it contains a diamond then it contains a pentagon. The purpose of this part is to show that there are no diamonds in finite retract lattices.

Lemma 5.1. *Let (A, f) be a connected monounary algebra. If $\mathbf{R}^{\emptyset}(A, f)$ contains a diamond with the bottom element $\emptyset$, then (A, f) is unbounded.*

P r o o f. By the way of contradiction, suppose that there exists a connected not unbounded algebra (A, f), such that $\mathbf{R}^{\emptyset}(A, f)$ contains a diamond with the bottom element $\emptyset$. Obviously, (A, f) possesses no cycle. Let $X, Y, Z \in R(A, f)$ be a retracts in the middle of this diamond. According to 3.2 the lattice $\mathbf{R}^{\emptyset}(A, f)$ is atomic, hence there exists an atom $X_1 \subseteq X$ satisfying $X_1 \nsubseteq Y$ and $X_1 \nsubseteq Z$.

Since X_1 is an atom, there exists an element $x \in X_1$, such that $x \notin Y$, $x \notin Z$. If $(X_1, f) \cong (\mathbb{Z}, \mathrm{suc})$, then there exists $y, z \in X_1$ with $y \notin Y$ and $z \notin Z$. In this case we can take arbitrary $x \in X_1$, x bellow y, z. If $(X_1, f) \cong (\mathbb{N}, \mathrm{suc})$ take $x = x_1$, where $[x_1] = X_1$.

Since $x \in X \subseteq Y \cup Z$, we obtain that $x \in Y$ or $x \in Z$, which is a contradiction. $\square$

Now we show that a finite retract lattice contains no diamond.

Theorem 5.2. *Let (A, f) be a connected monounary algebra. If $\mathbf{R}^{\emptyset}(A, f)$ is finite, then there is no diamond in $\mathbf{R}^{\emptyset}(A, f)$.*

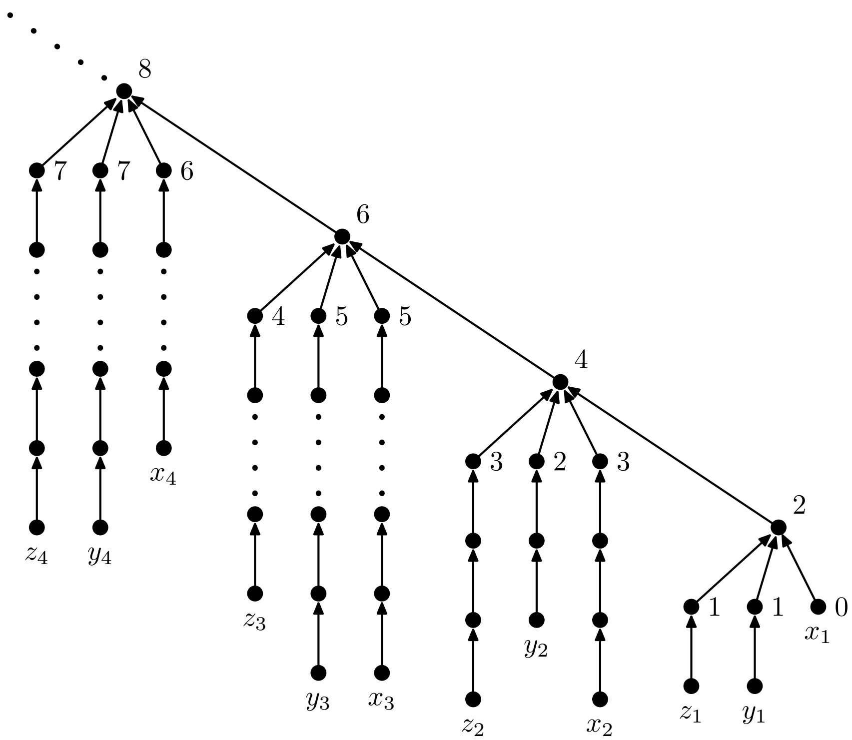
P r o o f. Suppose that $\mathbf{R}^{\emptyset}(A, f)$ contains a diamond with the bottom element V. Since (A, f) is not unbounded we obtain that $V \neq \emptyset$. In this case

$$\langle \mathbf{V} \rangle \cong \prod_{y \in f^{-1}(V) \smallsetminus V} \mathbf{R}^{\emptyset}(A^y, \tilde{f}) \ ,$$

thus there is an element $x \in f^{-1}(V) \smallsetminus V$, such that $\mathbf{R}^{\emptyset}(A^x, \tilde{f})$ contains a diamond with the bottom element $\emptyset$. Since $(A^x, \tilde{f})$ is not unbounded, due to Lemma 5.1 it is impossible. □

At the end of this subsection we give an example of an unbounded monounary algebra, such that $\mathbf{R}^{\emptyset}(A, f)$ contains a diamond.

Example 5.3.



Let (A, f) be the algebra as on the figure. Denote by X the subalgebra generated by the set $\{x_i : \ i \in \mathbb{N}\} \cup \{y_i : \ i \in \mathbb{N}\}$, by Y the subalgebra generated by $\{y_i : \ i \in \mathbb{N}\} \cup \{z_i : \ i \in \mathbb{N}\}$ and by Z the subalgebra generated by $\{z_i : i \in \mathbb{N}\} \cup \{x_i : \ i \in \mathbb{N}\}$. All these subalgebras are retracts of (A, f). Obviously, $X \cup Y = X \cup Z = Y \cup Z = A$. Since $X \cap Y = \big[\{y_i : \ i \in \mathbb{N}\}\big]$ we obtain that

$X \wedge Y = \emptyset$. Similarly, the same holds for $X \wedge Z$ and $Y \wedge Z$. Hence we obtain that $\emptyset, A, X, Y, Z$ forms a diamond.

6. Finite distributive retract lattices

In this section we will deal with finite distributive retract lattices. First we show the connection between length and distributivity of finite retract lattices. The rest of this section is devoted to find a representation of finite distributive lattices as a retract lattices of monounary algebras.

For a lattice L, let $\mathrm{J}(L)$ denote the set of all nonzero join-irreducible elements, regarded as a poset under the partial ordering of L.

Theorem 6.1. *Let L be a finite lattice of length n, $n \in \mathbb{N}$ such that $L \cong \mathbf{R}^{\emptyset}(A, f)$ for some connected monounary algebra (A, f). Then the lattice L is distributive if and only if $|\mathrm{J}(L)| = n$.*

Proof. It is well known that if a finite distributive lattice is of length n, then it contains precisely n join-irreducible elements.

To prove the converse, we may assume that the lattice L is isomorphic to $\mathbf{R}^{\emptyset}(A, f)$, where an algebra (A, f) possesses no cycle and also $A^{\mathbb{Z}} = \emptyset$. Hence $\left|A^0\right| = n$. We show that if $\mathbf{R}^{\emptyset}(A, f)$ is not distributive, then it contains more then n join-irreducible elements.

First, we show that there exists an injection from the set A^0 to the set $\mathrm{J}(\mathbf{R}^{\emptyset}(A, f))$. Let $x \in A^0$ be an arbitrary element. Denote by m the least integer, such that $f^m(x) \in R$, where R is an atom of $\mathbf{R}^{\emptyset}(A, f)$. If $m = 0$ then $[x]$ is an atom. Further suppose that $m \geq 1$. Denote by J a set $J = \left\{j \in \{0, \dots, m-1\} : \ (\exists\, y \in A)\big(f^{j+1}(x) = f(y) \ \&\ s_f(f^j(x)) < s_f(y)\big)\right\}$. For each $j \in J$ denote by x_{j+1} such element that $[x_{j+1}]$ is an atom of the lattice $\mathbf{R}^{\emptyset}(A^{f^{j+1}(x)}, \tilde{f})$. Put $R_x = \left[\{x_{j+1} : \ j \in J\} \cup \{x\}\right]$. It is not difficult to verify that R_x is a retract of the algebra (A, f) and that satisfy the conditions (1), (2) and (3) of the Theorem 3.6. For each $y \in R_x^0$ assign the value $p(y) = \min\left\{k \in \mathbb{N} : \ f^k(y) \in \left[\{y' : \ y' \in R_x^0\}\right], y' \neq y\right\}$. The element x is the only element of R_x^0 satisfying the condition $p(x) = \min\left\{p(y) : \ y \in R_x^0\right\}$. Thus for each $x, z \in A^0$, $R_x^0 = R_z^0$ implies $x = z$.

Suppose that $\mathbf{R}^{\emptyset}(A, f)$ is not distributive. According to the Theorem 3.5 there exists a triple $a, b, c \in A$ satisfying the conditions (i), (ii) and (iii). Let $f(a) = f^l(b) = f^l(c)$ for some $l \in \mathbb{N}$. Let a_1 be such an element that $[a_1]$ is an atom of $\mathbf{R}^{\emptyset}(A^a, \tilde{f})$. Since $s_f(a) < s_f(f^{l-1}(b))$, there exists an element $a' \in R_{a_1}^0$

such that $[a']$ is an atom in $\mathbf{R}^{\emptyset}(A^{f(a)}, \tilde{f})$. Denote by b_1 and c_1 the elements such that $[b_1]$ is an atom in $\mathbf{R}^{\emptyset}(A^{b}, \tilde{f})$ and $[c_1]$ is an atom in $\mathbf{R}^{\emptyset}(A^{c}, \tilde{f})$. Due to the condition (iii) of 3.5, $[b_1]$, $[c_1]$ are atoms also in $\mathbf{R}^{\emptyset}(A^{f(a)}, \tilde{f})$. Hence the sets $(R_{a_1} \smallsetminus [a']) \cup [b_1]$ and $(R_{a_1} \smallsetminus [a']) \cup [c_1]$ are join-irreducible retracts of (A, f) and we obtain that $\left|\mathrm{J}(\mathbf{R}^{\emptyset}(A, f))\right| \geq n + 1$. $\square$

We notice that the assertion of this theorem fails to hold in general for finite lattices (the lattice N_5).

For a poset P, let $\mathrm{H}(P)$ denote the set of all hereditary subsets partially ordered by set inclusion. Note that $S \subseteq P$ is hereditary if $x \in S$ and $y \leq x$ imply that $y \in S$.

There is the following well known result of representation of finite distributive lattices (see [2]).

Let L be a finite distributive lattice. Then $L \cong \mathrm{H}(\mathrm{J}(L))$. The isomorphism is given by $a \mapsto r(a)$, where $r(a) = \{x : \ x \leq a, x \in \mathrm{J}(L)\} = (a\rangle \cap \mathrm{J}(L)$. Note that $(a\rangle = \{x \in L : \ x \leq a\}$.

Theorem 6.2. *Let L be a finite nontrivial distributive lattice. If $(a\rangle$ forms a chain for all $a \in \mathrm{J}(L)$, then there exists a connected monounary algebra (A, f) such that $L \cong \mathbf{R}^{\emptyset}(A, f)$.*

Proof. Let L be a finite distributive lattice satisfying the condition of the theorem. Then each nonzero x, $x \leq a$, $a \in \mathrm{J}(L)$ is join-irreducible. Denote $n = |\mathrm{J}(L)|$. Define a function $h\colon \mathrm{J}(L) \to \mathbb{N}_0$, $h(a) = 2\big(n - |r(a)|\big)$. Obviously, if $a < b$ then $h(a) \geq h(b) + 2$. Put

$$A = \mathbb{N}_0 \cup \bigcup_{a \in \mathrm{J}(L)} \{(a, i) : \ 0 \leq i \leq h(a)\}$$

and define an operation f as follows:
for any $a \in \mathrm{J}(L)$ put

$$f\big((a, i)\big) = \begin{cases} (a, i+1) & \text{if } 0 \leq i < h(a) \\ (b, h(b)) & \text{if } i = h(a) \text{ and } b \prec a \\ 0 & \text{if } i = h(a) \text{ and } a \text{ is atom} \end{cases}$$

for $x \in \mathbb{N}_0$ put $f(x) = x + 1$.

The algebra (A, f) is connected and for all $a \in \mathrm{J}(L)$, $i = 0, 1, \dots, h(a)$ we have $s_f((a, i)) = i$. Moreover, $f\big((a, i)\big) = f\big((b, j)\big)$ if and only if a, b cover the same element and $i = j = h(a) = h(b)$ or $b \prec a$ and $i = h(a)$, $j = h(b) - 1$ or $a \prec b$ and $i = h(a) - 1$, $j = h(b)$.

Suppose that $f\big((b,i)\big) = f\big((c,j)\big)$, $s_f((b,i)) = s_f((c,j))$ (in this case b, c cover the same element and $i = j = h(a) = h(b)$) and there is $m \in \mathbb{N}$, $(a,k) \in A$ with $f\big((a,k)\big) = f^m\big((b,i)\big)$. Then either $a \leq b$ or $a \not\leq b$ and a is atom. Since $f^m\big((b,h(b))\big)$ is equal to some $(d,h(d))$, $d < b$ or equal to 0, we obtain that $s_f((a,k)) = k \geq s_f(f^m((b,h(b))))$.

Hence there is no triple satisfying the conditions of Theorem 3.5 and we obtain that $\mathbf{R}^{\emptyset}(A,f)$ is distributive.

Define $\varphi\colon \mathrm{J}(L) \to \mathbf{R}^{\emptyset}(A,f)$ as follows: $\varphi(a) = \big[\big\{(x,0) :\ x \in r(a)\big\}\big]$. Since $a \leq b$ if and only if $r(a) \subseteq r(b)$, we obtain that $a \leq b$ if and only if $\varphi(a) \subseteq \varphi(b)$.

First we show that for all $a \in \mathrm{J}(L)$, $\varphi(a)$ is a retract of (A,f). Let $y' \in f^{-1}(\varphi(a))$ be an arbitrary element with $f(y') = f(x')$ for some $x' \in \varphi(a)$, $x' \neq y'$. Then $x' = (x,i)$, $x \in r(a)$, $0 \leq i \leq h(x)$ and $y' = (y,j)$, $y \in \mathrm{J}(L)$, $0 \leq j \leq h(y)$. If x, y cover the same element, then $s_f(x') = i = j = s_f(y')$. In the case that $x \prec y$, $s_f(x') = i = h(x) - 1 > h(y) = j = s_f(y')$. If $y \prec x$ then $y \in r(a)$ and $y' \in \varphi(a)$. Hence, according to Theorem 2.1, $\varphi(a)$ is a retract of (A,f).

Next we prove that $\varphi(a)$ satisfies the conditions of Theorem 3.6 and thus $\varphi(a)$ is join-irreducible. The only elements x of (A,f) with $\left|f^{-1}(x)\right| \geq 2$ are of the form $(y,h(y))$ or $x = 0$. Since $r(a)$ forms a chain, for all $b \in r(a)$ there is at most one $c \in r(a)$ such that $b \prec c$ and thus $f\big((c,h(c))\big) = (b,h(b))$. Then $f\big((c,h(c))\big) = f\big((b,h(b)-1)\big)$ and $h(c) < h(b) - 1$. Since $f^{-1}(0) \cap \varphi(a) = (a',h(a'))$, $a' \leq a$, a' is an atom, we obtain that that $\varphi(a)$ satisfies (1), (2) and (3) of 3.6.

The lattice $\mathbf{R}^{\emptyset}(A,f)$ is distributive and $\left|A^0\right| = n$. According to Theorem 4.1 $l(\mathbf{R}^{\emptyset}(A,f)) = n$ and thus $\left|\mathrm{J}(\mathbf{R}^{\emptyset}(A,f))\right| = n$. Since φ is injective, we obtain that $\mathrm{J}(L) \cong \mathrm{J}(\mathbf{R}^{\emptyset}(A,f))$, what completes the proof. □

The condition in Theorem 6.2 is also necessary (see [7]). Hence, finite distributive lattice L is representable as a retract lattice of a connected monounary algebra if and only if $(a\rangle$ forms a chain for all $a \in \mathrm{J}(L)$.

REFERENCES

[1] DUFFUS, D.—RIVAL, I.: *A structure theory for ordered sets*, Discrete Math. **35** (1981), 53–118.

[2] GRÄTZER, G.: *General Lattice Theory*, Birkhäuser, Basel, 1998.

[3] JAKUBÍKOVÁ-STUDENOVSKÁ, D.: *Retract irreducibility of connected monounary algebras I*, Czechoslovak Math. J. **46(121)** (1996), 291–308.

[4] JAKUBÍKOVÁ-STUDENOVSKÁ, D.: *Retract varieties of monounary algebras*, Czechoslovak Math. J. **47(122)** (1997), 701–716.
[5] JAKUBÍKOVÁ-STUDENOVSKÁ, D.: *Antiatomic retract varieties of monounary algebras*, Czechoslovak Math. J. **48(123)** (1998), 793–808.
[6] JAKUBÍKOVÁ-STUDENOVSKÁ, D.—PÓCS, J.: *Lattice of retracts of monounary algebras*, Math. Slovaca **61** (2011), 107–125.
[7] JAKUBÍKOVÁ-STUDENOVSKÁ, D.—PÓCS, J.: *Some properties of retract lattices of monounary algebras*, Math. Slovaca **62** (2012), 169–186.
[8] NOVOTNÝ, M.: *Mono-unary algebras in the work of Czechoslovak mathematicians*, Arch. Math. (Brno) **26** (1990), 155–164.
[9] NOVOTNÝ, M.: *Über Abbildungen von Mengen*, Pacific J. Math. **13** (1963), 1359–1369.
[10] NOVOTNÝ, M.: *Retracts of Algebras*, Fund. Inform. **75** (2007), 375–384.

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