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BANASCHEWSKI'S THEOREM FOR GMV-ALGEBRAS

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ABSTRACT. Banaschewski's theorem concerns subdirect product decompositions of lattice ordered groups. In the present paper we deal with the analogous investigation for the case of generalized MV-algebras (GMV-algebras, in short); we apply this notion in the sense studied by Galatos and Tsinakis.

1. Introduction

We are concerned with the notion of generalized MV-algebra (GMV-algebra, in short) in the sense of [6]. Cf. also [1], [3], [5], [11], [12], [13].

The term 'generalized MV-algebra' was used in another meaning in [9] and in some other papers (in fact, the notion of generalized MV-algebra as used in [7] coincides with the notion of pseudo MV-algebra [3]).

Let $\mathbf{A}$ be any algebra. The system of all congruence relations of $\mathbf{A}$ is usually written as $\mathrm{Con}(\mathbf{A})$; in view of the notation used in the relevant papers [11] and [12], we will write $C(\mathbf{A})$ rather than $\mathrm{Con}(\mathbf{A})$. The system $C(\mathbf{A})$ is partially ordered by set inclusion. If $\varrho \in C(\mathbf{A})$, then the corresponding quotient algebra is denoted by $\mathbf{A}/\varrho$.

Let $\mathbf{G}$ be a lattice ordered group and let $C_1(\mathbf{G})$ be the system of all elements ϱ of $C(\mathbf{A})$ such that $\mathbf{G}/\varrho$ is a subdirect product of linearly ordered groups.

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A well-known theorem due tu Banaschewski is concerned with the description of the least element of the system $C_1(\mathbf{G})$ (cf. [1]). In response to a question raised in [1], the author provided an intrinsic description of $C_1(\mathbf{G})$ in [9].

An analogous investigation was performed in [10] for the case of pseudo MV-algebras; we recall that the term 'generalized MV-algebra' was used in [10] (in the sense of [14]).

Let $\mathbf{M}$ be a GMV-algebra. A fundamental result of [6] says that $\mathbf{M}$ can be obtained by a truncation construction from a pair $(\mathbf{G}, \mathbf{G}_1)$ of lattice ordered groups. Moreover, in [4] it was shown that $\mathbf{G}_1$ can be choosen in such a way that it satisfies a certain additional condition (for details, cf. Section 2 below).

In the present paper we prove a result concerning the GMV-algebra $\mathbf{M}$ which is analogous to Banaschewski's theorem (cf. Theorem 2.5 in Section 2 below). In our construction, the elements of $\mathbf{G}$ and of $\mathbf{G}_1$ are used.

In view of [6], each lattice ordered group can be considered as a particular case of GMV-algebra. Thus the result of the present paper is an extension of Banaschewski's result.

2. Preliminaries

If $\mathbf{A}$ is any algebra, then we denote by A the underlying subset of $\mathbf{A}$.

For lattice ordered groups we apply the terminology and the notation as in [4]. We recall that if $\mathbf{G}$ is a lattice ordered group then $G^+ = \{x \in G:\ x \geq e\}$; if $x \in G$, then $|x| = x \vee x^{-1}$. A nonempty subset X of G^+ is a *filter* if $x_1 \wedge x_2 \in X$ for any $x_1, x_2 \in X$, and $x_1 \vee y \in X$ whenever $x_1 \in X$ and $y \in G^+$; if, moreover, $tX = Xt$ for each $t \in G^+$, then the filter X is *normal*.

Let $\mathbf{G}$ be a lattice ordered group and let $W(\mathbf{G})$ be the union of all normal prime filters of the positive cone G^+ of $\mathbf{G}$. Put

$$K_0(\mathbf{G}) = \bigl\{x \in G:\ |x| \notin W(\mathbf{G})\bigr\}.$$

Theorem 2.1. (Cf. [1].) *For each lattice ordered group $\mathbf{G}$, $K_0(\mathbf{G})$ is an ℓ-ideal of $\mathbf{G}$. Let ϱ_0 be the congruence relation of $\mathbf{G}$ which is generated by $K_0(\mathbf{G})$. Then ϱ_0 is the least element of $C_1(\mathbf{G})$.*

In [1] it was remarked that it may be of interest to have the characterizations of $W(\mathbf{G})$ and $K_0(\mathbf{G})$ in terms of elements of $\mathbf{G}$.

By induction we define the subsets K_n and $\overline{K}_n$ of $\mathbf{G}$ by putting $K_1 = \overline{K}_1 = \{e\}$; if $1 < n \in \mathbb{N}$, then let K_n be the set of all $a \in G^+$ such that

$$(x_1 a x_1^{-1}) \wedge (x_2 a x_2^{-1}) \in \overline{K}_{n-1}$$

for some $x_1, x_2 \in G$. Further, let $\overline{K}_n$ be the set of all $b \in G$ which can be expressed in the form $b = a_1 a_2 \dots a_m$ for some $m \in \mathbb{N}$ and $a_1, a_2, \dots, a_m \in K_n$. We put

$$\bigcup_{n=1}^{\infty} \overline{K}_n = K(\mathbf{G}).$$

Theorem 2.2. (Cf. [9].) *For each lattice ordered group* $\mathbf{G}$ *we have*

$$K_0(\mathbf{G}) = K(\mathbf{G}).$$

For the sake of completeness and for fixing the notation we recall the basic definitions concerning GMV-algebras. Let $\mathbf{M} = (M; \wedge, \vee, \cdot, \backslash, /, e)$ be an algebra of type $(2,2,2,2,2,0)$ such that

(i) $(M; \wedge, \vee)$ is a lattice;

(ii) $(M; \cdot, e)$ is a monoid;

(iii) $x \cdot y \leq z \Longleftrightarrow x \leq z/y \Longleftrightarrow y \leq x\backslash z$ for each $x, y, z \in M$;

(iv) $x/((x \vee y)\backslash x) = x \vee y = (x/(x \vee y))\backslash x$ for each $x, y \in M$.

Then $\mathbf{M}$ is a *GMV-algebra*.

Let $\mathbf{G} = (G; \wedge, \vee, \cdot, e)$ be a lattice ordered group and $x, y \in G$; we put

$$x\backslash y = x^{-1}y, \quad y/x = yx^{-1}.$$

Then the algebra $\mathbf{G}^* = (G; \wedge, \vee, \cdot, \backslash, /, e)$ is a GMV-algebra.

Assume that $\mathbf{G}_1$ is a lattice ordered group and that L_γ is a filter of the lattice $(G_1^-; \wedge, \vee)$ such that for each $x \in G_1^-$ there exists the element $\overline{x} = \min\{z \in L_\gamma : z \geq x\}$. For $x, y \in L_\gamma$ we put

$$x \circ y = \overline{xy}, \quad x\backslash_{L_\gamma} y = (x^{-1}y) \wedge e, \quad y/x = (yx^{-1}) \wedge e.$$

Then the algebra

$$\mathbf{L}_\gamma = (L_\gamma; \wedge, \vee, \circ, \backslash_{L_\gamma}, /_{L_\gamma}, e)$$

is a GMV-algebra. We say that $\mathbf{L}_\gamma$ is obtained by a truncation construction from $\mathbf{G}_1$.

A GMV-algebra $\mathbf{A}$ is a *direct sum* of its subalgebras $\mathbf{B}$ and $\mathbf{C}$, in symbols, $\mathbf{A} = \mathbf{B} \oplus \mathbf{C}$, if the map $f\colon B \times C \to A$ defined by $f(x,y) = x \cdot y$ is an isomorphism. The subalgebras $\mathbf{B}$ and $\mathbf{C}$ are *direct summands* of $\mathbf{A}$.

Theorem 2.3. (Cf. [2].) *Let* $\mathbf{M}$ *be a GMV-algebra. Then there exists a pair of lattice ordered groups* $(\mathbf{G}, \mathbf{G}_1)$ *such that* $\mathbf{M}$ *can be expressed in the form*

$$\mathbf{M} = \mathbf{G}^* \oplus \mathbf{L}_\gamma, \tag{1}$$

where $\mathbf{L}_\gamma$ *is a GMV-algebra obtained by a truncation construction* $\mathbf{G}_1$.

Proposition 2.4. (Cf. [4].) *Let* $\mathbf{M}$ *be a* GMV*-algebra. There exist* $\mathbf{G}, \mathbf{G}_1$ *and* $\mathbf{L}_\gamma$ *such that the conditions from* Theorem 2.3 *are satisfied and, moreover, the following condition is valid:*

(m) *For each* $g \in G_1^-$ *there exist elements* $x_1, x_2, \dots, x_n \in L_\gamma$ *with* $g = x_1 x_2 \dots x_n$.

If ϱ is a congruence relation on a GMV-algebra M, then the set $\varrho(e) = \{x \in M : \ x \,\varrho\, e\}$ will be said to be the *kernel* of ϱ.

We denote by $C_1(\mathbf{M})$ the set of all $\varrho \in C(\mathbf{M})$ such that $\mathbf{M}/\varrho$ is a subdirect product of linearly ordered GMV-algebras.

In the present paper we prove

Theorem 2.5. *Assume that* $\mathbf{M}$ *is a* GMV*-algebra and that* $\mathbf{G}, \mathbf{G}_1$ *and* $\mathbf{L}_\gamma$ *are as in* Proposition 2.4. *Put*

$$M_0 = K_0(\mathbf{G}) \cdot (K_0(\mathbf{G}_1) \cap L_\gamma).$$

Then

(i) *there exists a uniquely determined congruence relation* ϱ_m *on* $\mathbf{M}$ *such that* M_0 *is the kernel of* ϱ*;*

(ii) ϱ_m *is the least element of the system* $C_1(\mathbf{M})$.

For an GMV-algebra $\mathbf{M}$ we put

$$\mathbf{M}^- = (M^-; \wedge, \vee, \cdot, \backslash_{M^-}, /_{M^-}, e),$$

where $M^- = \{x \in M : \ x \leq e\}$ and

$$x \backslash_{M^-} y = (x \backslash y) \wedge e, \quad x /_{M^-} y = (x/y) \wedge e.$$

Then $\mathbf{M}^-$ is a GMV-algebra. In the case when $\mathbf{G}$ is as in Theorem 2.3 and $G = \{e\}$, then we obtain the same result if we take $L_\gamma = M^-$ and if we consider the GMV-algebra $\mathbf{L}_\gamma$ as defined in Section 2.

Now, assume that $\mathbf{G}_1$ is a lattice ordered group; applying the definition as above, we obtain the GMV-algebra $(\mathbf{G}_1^*)^-$, where $\mathbf{G}_1^*$ is as in Section 2. In this case we have $M^- = G_1^-$.

In this section we prove that the lattices $C(\mathbf{G}_1)$ and $C((G_1^*)^-)$ are isomorphic. On the other hand, the lattices $C(\mathbf{G}_1)$ and $C(\mathbf{G}_1^-)$ need not be isomorphic in general. (Here, the symbol $\mathbf{G}_1^-$ denotes the algebra $(G_1^-; \wedge, \vee, \cdot, e)$.) We obviously have $C(\mathbf{G}_1) = C(\mathbf{G}_1^*)$.

Theorem 2.6. *The lattices $C(\mathbf{G}_1)$ and $C((\mathbf{G}_1^*)^-$ are isomorphic.*

P r o o f. Let us make use of the characterization of congruences on residuated lattices given in [3]. Residuated lattices are defined by omitting condition (iv) in the definition of GMV-algebras. Congruences in residuated lattices are in bijective correspondence to convex, normal subalgebras: this extends to a lattice isomorphism from $\mathrm{Con}(\mathbf{A})$ to the lattice of convex normal subalgebras of $\mathbf{A}$.

First recall that normality for a subset B of A is defined as closure under the conjugates $\lambda_x(b) = x\backslash bx \wedge e$ and $\varrho_x(b) = xb/x \wedge e$. As for a congruence θ, $[e]_\theta$ is a convex normal subalgebra (CNS) of $\mathbf{A}$; this defines a lattice isomorphism. Also, the negative cone of $[e]_\theta$ is a convex subset of A^- that is a submonoid of A^- and normal in $\mathbf{A}$, i.e. a convex normal submonoid (CNM); this defines another lattice isomorphism.

Now, the assertion of the theorem follows from the isomorphism of the lattices of CNMs of $\mathbf{A}$ and $\mathbf{A}^-$, for an ℓ-group $\mathbf{A}$. Actually, these are not just isomorphic but equal lattices. Let M be a CNM for $\mathbf{A}$. Clearly, then it is a convex submonoid for $\mathbf{A}^-$. It is also normal in $\mathbf{A}^-$: for $b \in M$ and $x \in A^-$, we have $\lambda_x(b) = x\backslash_{\mathbf{A}^-} bx \wedge e = x^{-1}bx \wedge e \wedge e = x\backslash_{\mathbf{A}^-} bx \wedge e$, which is in M by normality in $\mathbf{A}$. Conversely, let N be a CNM for $\mathbf{A}^-$. Clearly, then it is a convex submonoid for $\mathbf{A}$. It is also normal in $\mathbf{A}$: for $b \in N$ and $x \in A$, we have $x = y^{-1}z$, where $y, z \in A^-$. Also, $\lambda_x^{\mathbf{A}^-}(b) = x^{-1}bx \wedge e = z^{-1}yby^{-1}z \wedge e = \lambda_z^{\mathbf{A}}(yby^{-1})$. As $b \in A^-$, we have $yby^{-1} \in A^-$, which is in N by normality in $\mathbf{A}^-$, since $z \in A^-$. □

Let us now deal with the relations between the lattices $C((\mathbf{G}_1^*)^-)$ and $C(\mathbf{L}_\gamma)$, where $\mathbf{G}_1$ and $\mathbf{L}_\gamma$ are as above. We assume that the condition (m) from Preposition 2.4 is satisfied. For each $H \in C((\mathbf{G}_1^*)^-)$ we put $\varphi_2(H) = H \cap L_\gamma$.

Theorem 2.7. *The mapping $\varphi_2(H)$ is an isomorphism of $C((\mathbf{G}_1^*)^-)$ onto $C(\mathbf{L}_\gamma)$.*

P r o o f. Consider the GMV-algebra $\mathbf{G}_1^*$, for some ℓ-group $\mathbf{G}_1$, let $\mathbf{L} = (G_1^*)^-$, and let γ be a nucleus on $\mathbf{L}$ with image $\mathbf{L}_\gamma$. We need to show that the lattices $C(\mathbf{L})$ and $C(\mathbf{L}_\gamma)$ are isomorphic, or, what amounts to the same, the lattices $\mathrm{CN}(\mathbf{L})$ and $\mathrm{CN}(\mathbf{L}_\gamma)$ are isomorphic. Let $\varphi_2\colon \mathrm{CN}(\mathbf{L}) \to \mathrm{CN}(\mathbf{L}_\gamma)$ be the map defined by $\varphi_2(H) = H \cap L_\gamma = \gamma[H]$. One easily checks that φ_2 is well defined; namely, whenever $H \in \mathrm{CN}(L)$, then $H \cap L_\gamma \in \mathrm{CN}(\mathbf{L}_\gamma)$. To prove that φ_2 is injective, consider $H_1, H_2 \in \mathrm{CN}(\mathbf{L})$ such that $H_1 \cap L_\gamma = H_2 \cap L_\gamma$. Let $a \in H_1$. There exist elements $a_1, \dots, a_n \in L_\gamma$ such that $a = a_1 \dots a_n$, where the product takes place in $\mathbf{L}$. Now, each $a_i \geq a$, and so, $a_i \in H_1$. But then, $a_i \in H_2$ by assumption. It follows that $a \in H_2$ is closed under multiplication. Thus, $H_1 \subseteq H_2$, and, by symmetry, $H_1 = H_2$.

It is worth mentioning here that this part of the proof works for arbitrary residuated lattices. However, the proof of the fact that φ_2 is surjective seems to require special assumptions. Let $J \in \mathrm{CN}(\mathbf{L}_\gamma)$, and let $\mathrm{cn}(J)$ denote the convex normal subalgebra of $\mathbf{L}$ generated by J. We claim that $\varphi_2(\mathrm{cn}(J)) = \mathrm{cn}(J) \cap L_\gamma = J$. To prove this, consider $a \in \mathrm{cn}(j) \cap L_\gamma$. By [3, Lemma 5.3] of the paper by Blount and Tsinakis, as well as the assumption that $\mathbf{L}$ is the negative cone of an ℓ-group, there exist elements $k_1, \dots, k_n \in L_\gamma$, and arbitrary elements $a_1, \dots, a_n \in L$ such that $[(a_1^{-1} k_1 a_1) \wedge e] \dots [(a_n^{-1} k_n a_n) \wedge e] \le x$. Since, in addition, $x \in L_\gamma$ and γ is a nucleus, we have

$$\gamma[(a_1^{-1} k_1 a_1) \wedge e] * \dots * \gamma[(a_n^{-1} k_n a_n) \wedge e] \le x,$$

where $*$ denotes the multiplication in $\mathbf{L}_\gamma$. But for each $i \in \{1, \dots, n\}$,

$$\gamma[(a_1^{-1} k_i a_i) \wedge e] = \big[\gamma(a_i^{-1}) * \gamma(k_i) * \gamma(a_i)\big] \wedge e \ge \big[\gamma(a_i)^{-1} * \gamma(k_i) * \gamma(a_i)\big] \wedge e.$$

Thus, again by the same lemma, $x \in J$, showing that $\mathrm{cn}(J) \cap L_\gamma \subseteq J$. Since the reverse inclusion is itrivial, we have the required equality.

Lastly, it is clear that φ_2 is an order-isomorphism, thereby completing the proof of the theorem. □

Concerning the proofs of Theorem 2.6 and Theorem 2.7, cf. "Acknowledgements" at the end of the present paper.

Theorem 2.6 and Theorem 2.7 yield:

Theorem 2.8. *There exists an isomorphism φ^0 of $C(\mathbf{G}_1^*)$ onto $C(\mathbf{L}_\gamma)$ such that if $\varrho \in C(\mathbf{L}_\gamma)$ and X is the kernel of ϱ, then $\varrho \cap L_\gamma$ is the kernel of $\varphi^0(\varrho)$.*

3. Linearly ordered quotients

The following result is well-known.

Lemma 3.1. *Let $\mathbf{H}$ be a lattice ordered group. Then the following conditions are equivalent:*

(i) $\mathbf{H}$ *fails to be linearly ordered.*

(ii) *There are elements x and y in H such that $x > e$, $y > e$ and $x \wedge y = e$.*

(iii) *There are elements x and y in H such that $x < e$, $y < e$ and $x \vee y = e$.*

We will apply the notation as above; in particular, $\mathbf{M}, \mathbf{G}, \mathbf{G}_1$ and $\mathbf{L}_\gamma$ are as in Proposition 2.4.

Lemma 3.2. *The following conditions are equivalent:*

(i) $\mathbf{L}_\gamma$ *fails to be linearly ordered.*

(ii) *There are elements* x_1 *and* y_1 *in* L_γ *such that* $x_1 \neq e \neq y_1$ *and* $x_1 \vee y_1 = e$.

P r o o f. The validity of the implication (ii) $\Longrightarrow$ (i) is obvious. Assume that (i) is satisfied. Hence there are $x, y \in L_\gamma$ such that x is incomparable with y. Put $v = x \vee y$. Hence $v > x$ and $v > y$. Let us consider the elements

$$v_1 = v \backslash_{L_\gamma} v, \quad x_1 = v \backslash_{L_\gamma} x, \quad y_1 \backslash_{L_\gamma} y.$$

Then we have

$$v_1 = e, \quad x_1 = v^{-1}x, \quad y_1 = v^{-1}y.$$

This yields

$$x_1 < e, \quad y_1 < e, \quad x_1 \vee y_1 = e.$$

Hence the condition (ii) hodls. □

Lemma 3.3. *Let* $\varrho \in C(\mathbf{G}_1)$. *The following conditions are equivalent:*

(i) *The quotient* $\mathbf{G}_1/\varrho$ *is linearly ordered.*

(ii) *Whenever* $x, y \in G_1$ *such that* $x < e$, $y < e$ *and* $x \vee y = e$, *then either* $x \,\varrho\, e$ *or* $y \,\varrho\, e$.

P r o o f. This is a consequence of Lemma 3.1. □

Analogously, using Lemma 3.2, we obtain

Lemma 3.3.1. *Let* $\beta \in C(\mathbf{L}_\gamma)$. *The following conditions are equivalent:*

(i) *The quotient* $\mathbf{L}_\gamma/\beta$ *is linearly ordered.*

(ii) *Whenever* $x, y \in L_\gamma$ *such that* $x \neq e \neq y$ *and* $x \vee y = e$, *then either* $x \,\beta\, e$ *or* $y \,\beta\, e$.

Let φ^0 be as in Theorem 2.8.

Lemma 3.4. *Let* $\varrho \in C(\mathbf{G}_1^*)$ *and* $\beta = \varphi^0(\varrho)$. *The following conditions are equivalent:*

(i) *The quotient* $\mathbf{L}_\gamma/\beta$ *is linearly ordered.*

(ii) *The quotient* $\mathbf{G}_1/\varrho$ *is linearly ordered.*

P r o o f. Consider the kernel $\varrho(e)$ of ϱ and the kernel $\beta(e)$ of β. Since $\beta = \varphi^0(\varrho)$, the relation

$$\beta(e) = \varrho(e) \cap L_\gamma \tag{1}$$

is valid.

a) Assume that the condition (i) holds. Let $x, y \in G_1$, $x < e$, $x \vee y = e$. In view of the condition (m) from Proposition 2.4 there exist elements $x_1, x_2, \ldots, x_n \in L_\gamma$ and $y_1, y_2, \ldots, y_m \in L_\gamma$ such that

$$x = x_1 x_2 \ldots x_n, \quad y = y_1 y_2 \ldots y_m.$$

Without loss of generality we can suppose that $x_i < e$ for $i = 1, 2, \ldots, n$ and $y_j < e$ for $j = 1, 2, \ldots, m$.

We have $x_1 \geq x$ for $i = 1, 2, \ldots, n$ and $y_j \geq y$ for $j = 1, 2, \ldots, m$, hence $x_i \vee y_j = e$.

Suppose that x does not belong to $\varrho(e)$. Then there exists $i \in \{1, 2, \ldots, n\}$ such that x_i does not belong to $\varrho(e)$. Thus according to (1), x_i does not belong to $\beta(e)$. Hence in view of Lemma 3.3.1, all elements $y_1, y_2 \ldots, y_m$ belong to $\beta(e)$. Therefore $y_1, y_2, \ldots, y_m \in \varrho(e)$ and then we have $y \in \varrho(e)$. Thus Lemma 3.3 yields that the condition (ii) is satisfied.

b) Assume that the condition (ii) is valid. Let $x, y \in L_\gamma$ such that $x \neq e \neq y$ and $x \vee y = e$.

Suppose that x does not belong to $\beta(e)$. Then in view of (1), x does not belong to $\varrho(e)$. We have $x, y \in G_1$, thus according to (ii) and Lemma 3.3 we infer that y belongs to $\varrho(e)$. Hence in view of (1), y belongs to $\beta(e)$. According to Lemma 3.3.1 we conclude that the condition (i) is satisfied. □

We denote by $C_L(\mathbf{G}_1)$ the set of all $\varrho \in C(\mathbf{G}_1)$ such that the quotient $\mathbf{G}_1/\varrho$ is linearly ordered. The set $C_L(\mathbf{L}_\gamma)$ is defined analogously.

Further, let $C_1(\mathbf{G}_1)$ be the set of all $\varrho \in C(\mathbf{G}_1)$ such that ϱ can be expressed as a meet of some nonempty subset of $C_L(\mathbf{G}_1)$. The meaning of the symbol $C_1(\mathbf{L}_\gamma)$ is analogous.

From the well-known Birkhoff's theorem concerning subdirect product decompositions of algebras we easily obtain that if $\varrho \in C(\mathbf{G}_1)$ then the quotient $\mathbf{G}_1/\varrho$ is a subdirect product of linearly ordered groups if and only if ϱ belongs to $C_1(\mathbf{G}_1)$. An analogous result holds for elements of $C(\mathbf{L}_\gamma)$.

Let us remark that in view of Lemma 3.1, here (and also in Theorem 2.8) we can replace $\mathbf{G}_1$ by $\mathbf{G}_1^*$. According to Theorem 2.8 we obtain

$$\begin{aligned} \varphi^0(C_L(\mathbf{G}_1)) &= C_L(\mathbf{L}_\gamma), \\ \varphi^0(C_1(\mathbf{G}_1)) &= C_1(\mathbf{L}_\gamma). \end{aligned} \tag{2}$$

Let us apply the notation as in Section 2. Consider the ℓ-ideal $K_0(\mathbf{G}_1)$ of $\mathbf{G}_1$ and let ϱ_1 be the congruence relation of $\mathbf{G}_1$ which is generated by $K_0(\mathbf{G}_1)$. Put $\beta_1 = \varphi^0(\varrho_1)$.

Theorem 3.5. *β_1 is the least element of $C(\mathbf{L}_\gamma)$ having the property that the corresponding quotient $\mathbf{L}_\gamma/\beta_1$ is a subdirect product of linearly ordered GMV-algebras. The kernel of β_1 is equal to $K_0(\mathbf{G}_1)\cap \mathbf{L}_\gamma$.*

Proof. This is a consequence of Theorem 2.1, Theorem 2.8 and the relation (2). □

4. The system $C(\mathbf{M})$

Assume that $\mathbf{M}$ is a GMV-algebra; we apply the notation as in Section 2. The condition (m) from Theorem 2.4 is supposed to be satisfied.

It is well-known that each element of $C(\mathbf{G})$ is uniquely determined by its kernel, and similarly for $C(\mathbf{G}_1)$. In view of Theorem 2.8, this holds also for the lattice $C(\mathbf{L}_\gamma)$. From this and from Theorem 2.3 we conclude that an analogous situation occurs for the lattice $C(\mathbf{M})$.

In the present section we prove Theorem 2.5 from Section 2.

If $z \in M, x \in G$, $y \in L_\gamma$ and $z = xy$, then we write $x = z(\mathbf{G})$, $y = z(\mathbf{L}_\gamma)$.

In accordance with the above notation, $C(\mathbf{M})$ is the system of all congruence relations of $\mathbf{M}$. Further, let $C_L(\mathbf{M})$ be the system of all $\tau \in C(M)$ such that the quotient $\mathbf{M}/\tau$ is linearly ordered. The symbol $C_1(\mathbf{M})$ denotes the system of all $\tau \in C(\mathbf{M})$ such that τ is a meet of some nonempty subsystem of $C_L(\mathbf{M})$.

We remark that analogously as in the case of $C(\mathbf{G}_1)$ (cf. Section 3), for any $\tau \in C(\mathbf{M})$, the quotient $\mathbf{M}/\varrho$ is a subdirect product of linearly ordered GMV-algebras if and only if τ belongs to $C_1(\mathbf{M})$.

Let $z_1, z_2 \in M$, $\varrho \in C(\mathbf{G})$ and $\beta \in C(\mathbf{L}_\gamma)$. We put $z_1\, \overline{\varrho}\, z_2$ if $z_1(G)\, \varrho\, z_2(G)$. Similarly, we set $z_1\, \overline{\beta}\, z_2$ if $z_1(\mathbf{L}_\gamma)\, \beta\, z_2(\mathbf{L}_\gamma)$.

Then $\overline{\varrho}, \overline{\beta} \in C(\mathbf{M})$ and for the corresponding kernels we have

$$\overline{\varrho}(e) = \varrho(e)\cdot L_\gamma, \quad \overline{\beta}(e) = G\cdot\beta(e).$$

The following assertion is easy to verify.

Lemma 4.1. *Let $\{\varrho_i\}_{i\in I} \subseteq C(\mathbf{G})$ and $\{\beta_j\}_{j\in J} \subseteq C(\mathbf{L}_\gamma)$. Put*

$$\varrho = \bigwedge_{i\in I} \varrho_i, \quad \beta = \bigwedge_{j\in J} \beta_j.$$

Then

$$\overline{\varrho} = \bigwedge_{i\in I} \overline{\varrho}_i, \quad \overline{\beta} = \bigwedge_{j\in J} \overline{\beta}_j.$$

Also, for any $\varrho \in C(\mathbf{G})$ and $\beta \in C(\mathbf{L}_\gamma)$ we have

$$\mathbf{M}/\overline{\varrho} \simeq \mathbf{G}^*/\varrho, \quad \mathbf{M}/\overline{\beta} \simeq \mathbf{L}_\gamma/\beta. \tag{1}$$

According to Lemma 3.1 and in view of (1), we conclude

Lemma 4.2. *Let $\tau C(\mathbf{M})$. Then τ belongs to $C_L(\mathbf{M})$ if and only if some of the following conditions is satisfied:*

(i) *$\tau = \overline{\varrho}$ for some $\varrho \in C_L(\mathbf{G})$;*

(ii) *$\tau = \overline{\beta}$ for some $\beta \in C_L(\mathbf{L}_\gamma)$.*

Let

$$C_{11}(\mathbf{M}) = \{\tau_i\}_{i \in I_1}$$

be the set of all elements τ of $C(\mathbf{M})$ satisfying the condition (i); similarly, let

$$C_{12}(\mathbf{M}) = \{\tau_j\}_{j \in I_2}$$

be defined analogously with respect to the condition (ii).

For $i \in I_1$ let ϱ_i be the corresponding element of $C_L(\mathbf{G})$ with $\overline{\varrho}_i = \tau_i$. Hence $\{\varrho_i\}_{i \in I_1}$ is the set $C_L(\mathbf{G})$ and thus in view of Theorem 2.1 we have

$$\bigwedge_{i \in I_1} \varrho_i = \varrho_0$$

and the kernel of ϱ_0 is $K_0(\mathbf{G})$. Hence according to Lemma 4.1, we obtain

$$\bigwedge_{i \in I} \overline{\varrho}_i = \overline{\varrho}_0$$

and the kernel of $\overline{\varrho}_0$ is $K_0(\mathbf{G}) \cdot L_\gamma$. Applying the above notation, we get

$$\overline{\varrho}_0 = \bigwedge_{i \in I_1} \tau_i.$$

Further, let us apply Theorem 2.1 for the lattice ordered group $\mathbf{G}_1$; we conclude that the least element ϱ_{01} of $C_1(\mathbf{G}_1)$ has the kernel $K_0(\mathbf{G}_1)$.

For $j \in I_2$ let β_j be the element of $C_L(\mathbf{L}_\gamma)$ with $\tau_j = \overline{\beta}_j$. Let β_1 be as in Theorem 3.5. Then we have

$$\beta_1 = \bigwedge_{j \in I_2} \beta_j$$

and the kernel of β_1 is equal to $K_0(\mathbf{G}_1) \cap L_\gamma$.

Thus in view of Lemma 4.1 we get

$$\overline{\beta}_1 = \bigwedge_{j \in I_2} \overline{\beta}_j$$

and the kernel of $\overline{\beta}_1$ is $G \cdot (K_0(\mathbf{G}_1) \cap L_\gamma)$.

Applying the relation $\overline{\beta}_j = \tau_j$ for each $j \in I_2$ we obtain

$$\overline{\beta}_1 = \bigwedge_{j \in I_2} \tau_j.$$

In view of the definition of $C_1(\mathbf{M})$, the element

$$\varrho_m = \inf C_L(\mathbf{M})$$

is the least element of $C_1(\mathbf{M})$. Hence according to Lemma 4.2 we get

$$\varrho_m = \Big(\bigwedge_{i \in I_1} \tau_i\Big) \wedge \Big(\bigwedge_{j \in I_2} \tau_j\Big) = \overline{\varrho}_0 \wedge \overline{\beta}_1.$$

Then the kernel of ϱ_m is equal to

$$(K_0(\mathbf{G}) \cdot L_\gamma) \cap (G \cdot (K_0(\mathbf{G}_1) \cap L_\gamma)).$$

Thus in view of Theorem 2.3 the kernel of ϱ_m is

$$(K_0(\mathbf{G}) \cap G) \cdot (L_\gamma \cap (K_0(\mathbf{G}_1) \cap L_\gamma)) = K_0(\mathbf{G}) \cdot (K_0(G_1) \cap L_\gamma).$$

We conclude that Theorem 2.5 is valid.

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