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A NOTE ON REPRESENTATION OF PSEUDOVARIANT MAPS

Ruchi Das — Tarun Das

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ABSTRACT. We define and study the notions of pseudovariant maps and pseudogeny maps on G-spaces. We prove that the set of all pseudogenies of a locally comapct G-space is a group. We further obtain representation of pseudovariant maps in terms of pseudogenies. Finally, we obtain Tietze type extension result for pseudovariant homotopies defined on locally compact second countable G-spaces.

1. Introduction

The notion of equivariant maps between two G-spaces is defined and studied in detail (e.g. [2], [6]). We recall that equivariant maps serve as morphisms in the category of objects as G-spaces. In [4], we have defined the notion of pseudoequivarince, weaker than equivariance. We have observed that some interesting results regarding equivariant maps still hold true under the weaker condition of pseudoequivariance. For example pseudoequivariant maps also serve as morphisms in the category of objects as G-spaces. Using this notion we have also obtained several results for G-expansive homeomorphisms ([4], [5]). Recently Choi and Kim in [3] have used this concept to generalize topological decomposition theorem in [1] due to Aoki for compact metric G-spaces.

In [6], the notions of equivalence of two G-spaces, isovariant maps between two G-spaces and an isogeny on a G-space X using equivariance are defined and studied. In this paper, we define some new concepts involving pseudoequivariance, provide examples and study properties related to these notions. In Section 2, we define pseudoequivalency and observe that if f is a pseudoequivalence then it induces unique homeomorphism on orbit spaces commuting with orbit maps. We further introduce the notion of pseudovariancy and show that

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if induced map f_G of a given pseudovariant map f is a homeomorphism, then f is a pseudoequivalence. In Section 3, we define pseudogeny maps on a G-space X and show that if X is a locally compact G-space with G compact then the collection $P_I(X)$ of all pseudogeny maps of X forms a group. We also prove that if X and Y are G-spaces with G compact and X locally compact then for any pseudovariance f from X to Y, $P_I(X)$ completely represents the class of pseudovariancies of X into Y having f_G as the induced map on the orbit spaces. In Section 4, we define pseudovariant homotopy, obtain Tietze type extension result for pseudogeny homotopies defined on normal G-spaces and use this result to obtain Tietze type extension theorem for pseudovariant homotopies defined on locally compact second countable G-spaces.

Throughout G denotes a topological group. By an action of G on X, we mean a continuous map $\theta\colon G \times X \to X$ satisfying $\theta(e,x) = x$ and $\theta(g_1, \theta(g_2, x)) = \theta(g_1 g_2, x)$, where $x \in X$, $g_1, g_2 \in G$ and e is the identity of G. By a *G-space* we mean a triple (X, G, θ), where X is a topological space, G is a topological group and θ is a continuous action of G on X. If (X, G, θ) is a G-space then for $x \in X$, the set $G(x) = \{gx \mid g \in G\}$, where gx denotes $\theta(g, x)$, is called the *G-orbit* of x in X. The set $X/G = \{G(x) \mid x \in X\}$ endowed with the quotient topology induced by the quotient map $\pi_X\colon X \to X/G$ defined by $\pi_X(x) = G(x)$, $x \in X$, is called the *orbit space* of X. The map π_X is called the *orbit map*. We recall that π_X is continuous and open. If G is compact then π_X is closed also. A continuous map f from a G-space X to a G-space Y is called an *equivariant map* if $f(gx) = gf(x)$, for each $x \in X$ and $g \in G$ ([2], [6]). A continuous map f from a G-space X to a G-space Y is a called a *pseudoequivariant map* if $f(G(x)) = G(f(x))$, for each $x \in X$ ([4]). Clearly every equivariant map is pseudoequivariant but following example shows that every pseudoequivariant map need not be equivariant.

Example. Consider the space $X \equiv Y \equiv \mathbb{R}^2$ with the usual metric. Let θ be the action of $G \equiv \mathbb{R}$ on $\mathbb{R}^2$ defined by $\theta(x, (y, z)) = (x + y, z)$; $x, y, z \in \mathbb{R}$. Then the homeomorphism $h_a\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h_a(u) = au$, where $u \in \mathbb{R}^2$ and $a \in \mathbb{R} - \{-1, 0, 1\}$ is fixed, is pseudoequivariant but not equivariant.

2. Pseudovariant maps and pseudoequivalence of G-spaces

Throughout X and Y are G-spaces and $f\colon X \to Y$ is a continuous map. We recall the following result proved in [6].

Proposition 2.1. *If f is an equivariant map then there exists a unique map $f_G\colon X/G \to Y/G$ such that $f_G \circ \pi_X = \pi_Y \circ f$, where $\pi_X\colon X \to X/G$ and $\pi_Y\colon Y \to Y/G$ are orbit maps.*

Remark 1. The above result remains true if equivariancy of f is replaced by pseudoequivariancy (see [5]).

Definition 2.1. f is called a *pseudoequivalence* of X with Y if f is a pseudoequivariant homeomorphism.

Following is a consequence of Proposition 2.1 and Remark 1.

Theorem 2.2. *If f is a pseudoequivalence of X with Y, then the naturally induced map $f_G\colon X/G \to Y/G$ defined by $f_G(G(x)) = G(f(x))$ is a homeomorphism satisfying $f_G \circ \pi_X = \pi_Y \circ f$. Moreover, such an f_G is unique.*

Definition 2.2. A pseudoequivariant map which is one-one on each G-orbit is called a *pseudovariant* map.

Following example justifies that a pseudovariant map need not be a pseudoequivalence.

Example. For each $n \in N$, let X_n be the circle in the plane centered at origin of radius $\frac{1}{n}$. Let $X = \bigcup\limits_{n=1}^{\infty} X_n \cup \{(0,0)\}$ with the usual subspace topology from Euclidean plane $\mathbb{R}^2$. Define a self map f on X fixing poins of $X_1 \cup \{(0,0)\}$ and sending any point z in X_n, $n > 1$ to the point z_1 which is the point of intersection of X_{n-1} with the line joining z and $(0,0)$. Let the special orthogonal group $SO(2)$ act naturally on X by matrix multiplication. Since f preserves angle so it is pseudovariant. It is easy to verify that f is not a pseudoequivalence.

We have following result related to converse of Theorem 2.2.

Theorem 2.3. *Let X be a locally compact G-space, G compact. Suppose f is a pseudovariant map such that the induced map f_G is a homeomorphism. Then f is a pseudoequivalence of X with Y.*

Proof. Since f is pseudovariant, it is pseudoequivariant and one-one on each G-orbit. We need to show that f is bijective and f^{-1} is continuous.

Let $x, y \in X$, $x \neq y$. If $G(x) = G(y)$ then $x, y \in G(x)$ and f being one-one on each G-orbit, $f(x) \neq f(y)$. If $G(x) \neq G(y)$ then f_G being one-one on X/G, $f_G(G(x)) \neq f_G(G(y))$ i.e. $f_G(\pi_X(x)) \neq f_G(\pi_X(y))$. Since $f_G \circ \pi_X = \pi_Y \circ f$, we get $\pi_Y f(x) \neq \pi_Y f(y))$ i.e. $G(f(x)) \neq G(f(y))$ which implies $f(x) \neq gf(y)$ for every $g \in G$ and hence $f(x) \neq f(y)$. Thus f is one-one on X. Let $y \in Y$. Then $G(y) \in Y/G$. Since f_G is onto there exists $G(x) \in X/G$ such that $f_G(G(x)) = G(y)$ i.e. $G(f(x)) = G(y)$. Since f is pseudoequivariant, we get $f(G(x)) = G(y)$ which implies $y = f(gx)$ for some $g \in G$ and hence f is onto.

For $x \in X$, consider a compact G-invariant neighbourhood K of x. Since π_X is open, $\pi_X(K)$ is a neighbourhood of $\pi_X(x)$ so $f_G(\pi_X(K))$ is a neighbourhood of $f_G(\pi_X(x)) = \pi_Y(f(x))$. Hence $\pi_Y^{-1} f_G(\pi_X(K))$ is a neighbourhood of $f(x)$ which implies $\pi_Y^{-1}(\pi_Y(f(K)))$ is a neighbourhood of $f(x)$. Since $f(K)$ is G-invariant therefore we get that $\pi_Y^{-1}(\pi_Y(f(K))) = f(K)$ is a neighbourhood of $f(x)$.

Since K is compact, $f|_K\colon K \to f(K)$ is a homeomorphism. Let U_x be any neighbourhood of x. Then there exists $V_{f(x)}$, an open set of Y containing $f(x)$ such that $(f|_K)^{-1}\big(V_{f(x)} \cap f(K)\big) \subseteq U_x \cap K$. Since $f(K)$ and $V_{f(x)}$ are neighbourhoods of $f(x)$ therefore $V_{f(x)} \cap f(K) \equiv H_{f(x)}$ is a neighbourhood of $f(x)$. Thus $f^{-1}\big(H_{f(x)}\big) = \big(f|_K\big)^{-1}\big(H_{f(x)}\big) \subseteq U_x \cap K \subseteq U_x$. Consequently f^{-1} is continuous at $f(x)$. $\square$

3. Pseudogeny maps

Recall that an equivariant map between G-spaces is called *isovariant* if it is one-one on each G-orbit. An isovariant self-map is called an *isogeny* of a G-space X if the induced map on the orbit space is identity. It is known that the set $I(X)$ of all isogenies of a locally compact G-space is a group. Further, if X, Y are G-spaces with X locally compact, G compact and $f\colon X \to Y$ is an isovariant map then there is a one-one onto map of $I(X)$ into the set of isovariant maps of X into Y having induced map f_G ([6]). In this section we define pseudogenies and observe that the set $P_I(X)$ of all pseudogenies is a group. We further obtain representation of pseudovariant maps in terms of pseudogenies.

Definition 3.1. A pseudovariant map $f\colon X \to X$ is called a *pseudogeny* if its induced map $f_G\colon X/G \to X/G$ is identity.

Theorem 3.1. *Let X be a locally compact G-space, G compact, then $P_I(X)$, the set of pseudogenies of X, forms a group.*

Proof. By Theorem 2.3 members of $P_I(X)$ are pseudoequivalences of X. One can now verify that $P_I(X)$ is a group under composition. □

Theorem 3.2. *Let X and Y be G-spaces, G compact and let $f\colon X \to Y$ be a pseudovariant map. Then $T \mapsto f \circ T$ is a one-one map of $P_I(X)$ into the set of pseudovariant maps of X into Y with induced map f_G. If X is locally compact then this correspondence is onto i.e. any pseudovariant map g of X into Y with induced map f_G is of the form $f \circ T$ for a unique $T \in P_I(X)$.*

Proof. For any $T \in P_I(X)$, $f \circ T$ is a pseudovariance from X to Y satisfying $(f \circ T)_G = f_G \circ T_G = f_G$. Using the fact that any $T \in P_I(X)$ is a self-homeomorphism of $G(x)$, $x \in X$, we get that if $T_1, T_2 \in P_I(X)$ satisfy $f \circ T_1 = f \circ T_2$ then $T_1 = T_2$.

Now suppose X is locally compact and g is a pseudovariant map of X into Y with induced map f_G. We have to show that there exits a unique map $T \in P_I(X)$ satisfying $g = f \circ T$. For any $G(x) \in X/G$, since f and g are pseudoequivariant, one-one on $G(x)$ and $G(x)$ is compact, each of $f|_{G(x)}$ and $g|_{G(x)}$ is a pseudoequivalence of $G(x)$ with $f_G(G(x))$. Define $T\colon X \to X$ by $T(G(x)) = \big(f|_{G(x)}\big)^{-1} \circ \big(g|_{G(x)}\big)$. Then $g = f \circ T$, $\pi_X \circ T = \pi_X$ and $T(kx) = k_1 x$, for some $k_1 \in G$. By definition, T is one-one on each G-orbit (since each of $f|_{G(x)}$ and $g|_{G(x)}$ is one-one on $G(x)$). So it is sufficient to show that T is continuous. Let $\{x_\alpha\}$ be a net converging to x in X. We have to show that $T(x_\alpha)$ converges to $T(x)$. Since X is locally compact, we can assume that $x_\alpha \in V$, where V is a compact neighbourhood of x. Then $T(x_\alpha) \in GV$ (by definition of T). Note that G being compact GV is also compact. If $\{T(x_\alpha)\}$ does not converge to $T(x)$ then by passing to a subnet we can assume that $\{T(x_\alpha)\}$ coverges to $x_1 \neq T(x)$. Using $\pi_X = \pi_X \circ T$, we get that x_1 and $T(x)$ are in the same G-orbit and $g = f \circ T$ gives $f(x_1) = f(T(x))$, which implies $x_1 = T(x)$, a contradiction. □

4. Pseudovariant homotopy

If X is a G-space, then $X \times I$, the product of X with the closed unit interval I is also a G-space under the action of G on $X \times I$ defined by $g(x,t) = (gx,t)$. If $f\colon X \times I \to Y$ then for each $t \in I$, we denote by f_t the map from X into Y defined by $f_t(x) = f(x,t)$.

Definition 4.1. If Y is also a G-space and f is pseudovariant then we call f a pseudovariant homotopy *connecting* f_0 *and* f_1.

We recall the following definition and result from [6].

Definition 4.2. A mapping f from a G-space X into a topological space Y is called *invariant* if $f(gx) = x$ for all $g \in G$ and $x \in X$.

Proposition 4.1. *Let X be a normal G-space and K, F be disjoint closed invariant subspaces of X then there is an invariant map f of X into the closed unit interval I such that $f|_K \equiv 0$ and $f|_F \equiv 1$.*

Next, we prove a Tietze type extension result for pseudogeny homotopies defined on normal G-spaces and use this result to obtain Tietze type extension theorem for pseudovariant homotopies defined on locally compact second countable G-spaces.

Lemma 4.1. *Let X be a normal G-space, C a closed invariant subspace of X and U an invariant neighborhood of C. If S is a pseudogeny of $U \times I$ satisfying $S(u,0) = u$ for all $u \in U$ then there exists a pseudogeny T of $X \times I$ such that $T(x,0) = x$ for all $x \in X$ and $T|_{C\times I} = S|_{C\times I}$.*

Proof. Since C and $X - U$ are disjoint closed invariant subsets of X, by Proposition 4.1, there is an invariant map $f\colon X \to I$ such that $f|_C \equiv 1$ and $f|_{(X-U)} \equiv 0$. Define $T\colon X \times I \to X \times I$ such that $T(x,t) = S\bigl(x, f(x)t\bigr)$ for $x \in U$ and $T(x,t) = x$ for $x \in X - U$. Since $f|_C \equiv 1$ therefore $T|_{C\times I} = S|_{C\times I}$. That $T(x,0) = x$ for all $x \in X$ follows from the definition of T and the fact that $S(u,0) = u$ for all $u \in U$. For pseudoequivariancy of T, let us first take $x \in U$. Using the fact that f is invariant and S is pseudoequivariant one obtains

$$T(g(x,t)) = S\bigl(gx, f(gx)t\bigr) = g'S\bigl(x, f(x)t\bigr) = g'T(x,t)$$

for some $g' \in G$. In case $x \in X - U$,

$$T\bigl(g(x,t)\bigr) = T(gx,t) = gx = gT(x,t).$$

Thus $T(G(x,t)) = G(T(x,t))$ for all $x \in X$ and $t \in I$. This proves T is pseudoequivariant. For one-oneness of T on each G-orbit, take $x \in U$ and let $T(g(x,t)) = T(g'(x,t))$. Then invariancy of f implies

$$S\bigl(gx, f(x)t\bigr) = S\bigl(g'x, f(x)t\bigr)$$

and since S is one one on each G-orbit,we have $gx = g'x$ and hence $g(x,t) = g'(x,t)$. In case $x \in X - U$, invariancy of $X - U$ and definition of T gives $T(g(x,t)) = T(g'(x,t))$ implies $g(x,t) = g'(x,t)$. Thus T is pesudovariant.

Finally, using arguments similar as above and the fact that the induced map S_G on $\frac{U\times I}{G}$ is identity, it follows that the map $T_G\colon \frac{X\times I}{G} \to \frac{X\times I}{G}$ defined by $T_G(G(x,t)) = G(T(x,t))$ is identity. □

Theorem 4.2. *Let X be a locally compact G-space satisfying second axiom of countability and let $f\colon X\times I \to Y$ be a pseudovariant homotopy. Let C be a closed invariant subspace of X and let U be an invariant neighborhood of C. If there exists a pseudovariant homotopy $f^*\colon U\times I \to Y$ such that $f_0^* = f_0|_U$ and $f_G^* = f_G|_{\pi_X(U)\times I}$ then there exists a pseudovariant homotopy $h\colon X\times I \to Y$ with induced map f_G such that $h|_{C\times I} = f^*|_{C\times I}$.*

Proof. By Theorem 3.2, $f^* = \big(f|_{U\times I}\big)T^*$, where T^* is uniquely determined pseudogeny of $U\times I$. Since $f_0^* = f_0|_U$, we have $T^*(u,0) = u$ for $u\in U$. Observe that X being locally compact and second countable, is metrizable and hence is a normal space. Thus by Lemma 4.1, we can find a pseudogeny T of $X\times I$ such that $T|_{C\times I} = T^*|_{C\times I}$. Set $h = f\circ T$. Using Theorem 3.2 one can check that h has desired properties. □

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Department of Mathematics
Faculty of Science
The M. S. University of Baroda
Vadodara-390002
INDIA
E-mail: ruchid99@gmail.com
tarukd@gmail.com