VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-011-0077-3
Math. Slovaca **62** (2012), No. 1, 123–136

CONNECTEDNESS OF THE SETS OF WEAK EFFICIENT SOLUTIONS FOR GENERALIZED VECTOR EQUILIBRIUM PROBLEMS

Qing-You Liu* — Xian-Jun Long** — Nan-jing Huang***

(*Communicated by Lajos Molnar*)

ABSTRACT. In this paper, a generalized vector equilibrium problem is introduced and studied. A scalar characterization of weak efficient solutions for the generalized vector equilibrium problem is obtained. By using the scalarization result, the existence of the weak efficient solutions and the connectedness of the set of weak efficient solutions for the generalized vector equilibrium problem are proved in locally convex spaces.

1. Introduction

In recent years, the vector equilibrium problem has received much attention by many authors due to the fact that it provides a unified model including vector optimization problems, vector variational inequality problems, vector complementarity problems and vector saddle point problems as special cases. A great deal of papers have been devoted to the existence of solutions for various kinds

2010 Mathematics Subject Classification: Primary 47J20, 49J40; Secondary 90C29.
Keywords: generalized vector equilibrium problem, weak efficient solution, scalarization, existence, connectedness.
This work was supported by the National Natural Science Foundation of China (50874096, 11001287, 70831005), the Open Fund (PLN0904) of State Key Laboratory of Oil and Gas Reservoir Geology and Exploitation (Southwest Petroleum University), the Natural Science Foundation Project of Chongqing (CSTC 2010BB9254, 2009BB3372), the Education Committee Project Research Foundation of Chongqing (No. KJ100711) and the Research Fund of Chongqing Technology and Business University (09-56-06).

of vector equilibrium problems (see, for example, [1, 3, 5, 6, 9, 10, 13, 15, 19] and the references therein).

It is well known that one of the most important problems of vector variational inequalities and vector equilibrium problems is to investigate the topological properties of the solutions set. Among the topological properties of the solutions set, the connectedness is of interest, as it provides the possibility of continuously moving from one solution to any other solution. Lee et al. [14] discussed the path-connectedness of the set of weakly efficient solutions and the set of efficient solutions for vector variational inequalities in finite-dimensional spaces. Cheng [7] obtained the connectedness of the set of weakly efficient solutions for weak vector variational inequalities in finite-dimensional spaces by using scalarization method. Gong [10] discussed the connectedness of the set of Henig efficient solutions and the set of weak efficient solutions to the vector-valued Hartman-Stampacchia variational inequality in normed spaces. Recently, Gong [11] introduced the concepts of f-efficient solution, Henig efficient solution, globally efficient solution, weakly efficient solution and superefficient solution, and discussed the connectedness of the Henig efficient solution set, globally efficient solution set, weakly efficient solution and superefficient solution set for mixed vector equilibrium problems in locally convex spaces. Very recently, by virtue of a density result and scalarization technique, Gong and Yao [12] discussed the connectedness of the set of efficient solutions for mixed vector equilibrium problems in locally convex spaces. Concerned with the connectedness and path-connectedness of the solution sets for symmetric vector equilibrium problems, we refer to the recent work of Zhong, Huang and Wong [21].

Motivated and inspired by the works mentioned above, the purpose of this paper is to discuss the connectedness of the set of weakly efficient solutions for a generalized vector equilibrium problem by using scalarization method in locally convex spaces due to Gong [11]. The results presented in this paper generalize and improve some corresponding results due to Gong [10].

2. Preliminaries

Throughout this paper, let X and Y be two real Hausdorff topological vector spaces, and let Z be a real locally convex Hausdorff topological vector space. Let K be a nonempty closed convex subset of X and D be a nonempty subset of Y. Let $C \subset Z$ be a pointed closed convex cone with its interior $\operatorname{int} C \neq \emptyset$. Let Z^* be the topological dual space of Z and $C^* = \{f \in Z^* : \ f(x) \geq 0, \ \text{for all} \ x \in C\}$

be the dual cone of C. We also suppose that $T\colon K \to 2^D$ is a set-valued mapping and $F\colon K \times K \times D \to Z$ is a vector-valued mapping.

In this paper, we consider the following generalized vector equilibrium problem (for short, GVEP): finding $x \in K$ such that there exists $z \in T(x)$ satisfying

$$F(x, y, z) \notin -\operatorname{int} C \qquad \text{for all} \quad y \in K.$$

We call this x a weak efficient solution for (GVEP). Denote by $S_w(K, F)$ the set of all weak efficient solutions to (GVEP).

Some special cases of (GVEP):

(1) Let $\varphi\colon K \times K \to Z$ and $\psi\colon K \to Z$ be two vector-valued mappings. Let $F(x, y, z) = \varphi(x, y) + \psi(y) - \psi(x)$. Then (GVEP) reduces to the mixed vector equilibrium problem considered in [11, 12].

(2) Let $X = Y$, $K = D$ and $L(X, Z)$ be the space of all bounded linear mappings from X into Z. Let $T\colon K \to Z$ and $q\colon K \to Z$ be two vector-valued mappings. Let $F(x, y, T(x)) = \langle T(x), y - x\rangle + q(y) - q(x)$. Then (GVEP) reduces to the mixed vector variational inequality problem considered in [10].

(3) Let $X = Y = R^n$, $K = D$, $Z = R^p$ and $C = R^p_+$. Let $T_i\colon K \to R^n$ be vector-valued mappings for $i \in \{1, 2, \dots, p\}$. Let

$$F(x, y, T(x)) = \big(\langle T_1(x), y - x\rangle, \langle T_2(x), y - x\rangle\big), \dots, \langle T_p(x), y - x\rangle),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in the Euclidean space. Then (GVEP) reduces to the vector variational inequality problem considered in [7].

Let $f \in C^*\backslash\{0\}$. A vector $x \in K$ is called a f-efficient solution to (GVEP) if there exists $z \in T(x)$ satisfying

$$f(F(x, y, z)) \geq 0 \qquad \text{for all} \quad y \in K.$$

Denote by $S_f(K, F)$ the set of all f-efficient solutions to (GVEP).

We now recall some definitions and lemmas which will be used in the sequel.

Definition 2.1. ([8]) A set-valued mapping $G\colon X \to 2^X$ is called KKM-mapping if for any finite subset $\{x_1, x_2, \dots, x_n\}$ of X, $\operatorname{co}\{x_1, x_2, \dots, x_n\}$ is contained in $\bigcup_{i=1}^{n} G(x_i)$, where $\operatorname{co} A$ denotes the convex hull of the set A.

Lemma 2.1. ([8]) *Let M be a nonempty subset of X. Let $G\colon M \to 2^X$ be a KKM-mapping such that $G(x)$ is closed for any $x \in M$ and is compact for at least one $x \in M$. Then $\bigcap_{y \in M} G(y) \neq \emptyset$.*

Definition 2.2. A vector-valued mapping $h\colon K \to Z$ is said to be C-convex on K if, for any $x_1, x_2 \in K$ and $\lambda \in [0,1]$, one has

$$\lambda h(x_1) + (1-\lambda)h(x_2) \in h(\lambda x_1 + (1-\lambda)x_2) + C.$$

Remark 2.1.

(i) It is easy to see that h is C-convex on K if and only if for any $x_i \in K$ and $\lambda_i \in [0,1]$ $(i \in \{1,2,\ldots,n\})$ with $\sum_{i=1}^{n} \lambda_i = 1$ holds

$$\sum_{i=1}^{n} \lambda_i h(x_i) \in h\bigg(\sum_{i=1}^{n} \lambda_i x_i\bigg) + C.$$

(ii) If h is C-convex on K, then $F(K) + C$ is a convex set.

(iii) h is said to be C-concave on K if, $-h$ is C-convex on K.

(iv) If $f \in C^*\backslash\{0\}$ and h is C-convex on K, then $f \circ h\colon K \to R$ is convex.

Definition 2.3. ([16]) A vector-valued mapping $h\colon K \to Z$ is said to be C-lower (C-upper) semicontinuous at $x_0 \in K$ if, for any neighborhood U of 0, there exists a neighborhood $U(x_0)$ of x_0 such that

$$h(x) \in h(x_0) + U + C \qquad \text{for all} \quad x \in U(x_0) \cap K.$$
$$(h(x) \in h(x_0) + U - C \qquad \text{for all} \quad x \in U(x_0) \cap K.)$$

h is said to be C-lower (C-upper) semicontinuous on K if it is C-lower (C-upper) semicontinuous at each $x_0 \in K$.

Remark 2.2. ([3]) If $f \in C^*\backslash\{0\}$ and h is C-lower (C-upper) semicontinuous on K, then $f \circ h\colon K \to R$ is lower (supper) semicontinuous on K.

Definition 2.4. Let $f \in C^*\backslash\{0\}$. $F(x,y,\cdot)$ is said to be f-hemicontinuous with respect to T if, for any $x, y \in K$, $\alpha \in [0,1]$, the mapping $\alpha \to f(F(x,y,t_\alpha))$ is upper semicontinuous at 0^+, where $t_\alpha \in T(x + \alpha(y-x))$, i.e.,

$$\lim_{\alpha \downarrow 0} f(F(x,y,t_\alpha)) = f(F(x,y,z)) \qquad \text{for all} \quad z \in T(x).$$

Definition 2.5. F is said to be pseudomonotone with respect to T if, for any $x, y \in K$, $f \in C^*\backslash\{0\}$ and for any $z \in T(x)$, $t \in T(y)$, one has

$$f(F(x,y,z)) \geq 0 \implies f(F(x,y,t)) \geq 0.$$

Definition 2.6. A set-valued mapping $h\colon K \to 2^Z$ is said to be

(i) upper semicontinuous at $x \in K$ if, for any open set V containing $h(x)$, there exists an open set U containing x such that, for all $t \in U$, $h(t) \subset V$; h is said to be upper semicontinuous on K if it is upper semicontinuous at each $x \in K$.

(ii) closed if $\mathrm{Graph}(h) = \{(x,y) : \ x \in K \ \text{ and } \ y \in h(x)\}$ is a closed set in $K \times Z$.

Lemma 2.2. ([2]) *Let $h\colon K \to 2^Z$ be a set-valued mapping. If h is closed and Z is compact, then h is upper semicontinuous.*

Lemma 2.3. ([17]) *If A is a nonempty compact convex subset of a topological vector space, B is a nonempty convex subset of a vector space and the function $f\colon A \times B \to R$ is concave-convex on $A \times B$ and upper semicontinuous on A for every $b \in B$. Then,*

$$\max_{a\in A} \inf_{b\in B} f(a,b) = \inf_{b\in B} \max_{a\in A} f(a,b).$$

Lemma 2.4. ([20]) *Let X and Y be two topological vector spaces, S be a connected subset of X, $F\colon S \to 2^Y$ be a set-valued mapping. If F is upper semicontinuous on S and $F(x)$ is connected subset of Y for each $x \in S$, then, $F(S) = \bigcup\limits_{x\in S} F(x)$ is a connected subset of Y.*

Next, we establish the following scalarization result for the set of weakly efficient solutions to (GVEP).

Lemma 2.5. *For any $x \in K$ and $z \in T(x)$, $F(x,K,z) + C$ is a convex set. Then,*

$$S_w(K,F) = \bigcup_{f\in C^*\setminus\{0\}} S_f(K,F).$$

P r o o f. Let $x \in \bigcup\limits_{f\in C^*\setminus\{0\}} S_f(K,F)$. Then there exist $f \in C^*\setminus\{0\}$ and $z \in T(x)$ such that

$$f(F(x,y,z)) \geq 0 \qquad \text{for all} \quad y \in K. \tag{2.1}$$

Now, we claim that

$$F(x,y,z) \notin -\operatorname{int} C \qquad \text{for all} \quad y \in K.$$

In fact, if there exists some $y \in K$ such that

$$F(x,y,z) \in -\operatorname{int} C.$$

Then, for $f \in C^*\setminus\{0\}$, we have

$$f(F(x,y,z)) < 0,$$

which contradicts (2.1). Hence, $x \in S_w(K,F)$ and so $\bigcup\limits_{f\in C^*\setminus\{0\}} S_f(K,F) \subseteq S_w(K,F)$.

Conversely, let $x \in S_w(K,F)$. Then there exists $z \in T(x)$ such that

$$F(x,y,z) \notin -\operatorname{int} C \qquad \text{for all} \quad y \in K.$$

It follows that

$$F(x,K,z) \cap (-\operatorname{int} C) = \emptyset,$$

and so

$$(F(x,K,z) + C) \cap (-\operatorname{int} C) = \emptyset.$$

Since $F(x,K,z) + C$ is a convex set, by the separation theorem of convex sets [18], there exists some $f \in Z^* \backslash \{0\}$ such that

$$\inf\big\{f(F(x,y,z)+c) : \; y \in K, \;\; c \in C\big\} \geq \sup\big\{f(\overline{c}) : \; \overline{c} \in -\operatorname{int} C\big\}. \tag{2.2}$$

Since C is a cone, $f(\overline{c}) \leq 0$ for all $\overline{c} \in -\operatorname{int} C$. Hence, $f(\overline{c}) \geq 0$ for all $\overline{c} \in C$, that is $f \in C^*$. This fact together with (2.2) yield $f \in C^* \backslash \{0\}$ and

$$f(F(x,y,z)) \geq 0 \qquad \text{for all} \quad y \in K.$$

This means $x \in S_f(K,F)$. It follows that $S_w(K,F) \subseteq \bigcup_{f \in C^* \backslash \{0\}} S_f(K,F)$. This completes the proof. □

3. Connectedness of the solutions set

In this section, we discuss the connectedness of the set of weakly efficient solutions to the generalized vector equilibrium problem by the scalarization result. First, we have the following existence results for (GVEP).

Proposition 3.1. *Suppose that the following conditions are satisfied:*

(i) *for any $x \in K$, $F(x,x,\cdot) = 0$;*

(ii) *for any $x \in K$ and $z \in T(x)$, $F(x,\cdot,z)$ is C-convex on K;*

(iii) *F is pseudomonotone with respect to T;*

(iv) *for any $y \in K$, $t \in D$, $F(\cdot,y,t)$ is C-upper semicontinuous and C-concave on K;*

(v) *F is f-hemicontinuous with respect to T;*

(vi) *for any $x,y \in K$, $F(x,y,\cdot)$ is C-upper semicontinuous and C-concave on D;*

(vii) *there exist a nonempty compact convex subset E of K and $y_0 \in E$ such that for any $x \in K \backslash E$ and $t \in T(x)$ satisfying*

$$F(x,y_0,t) \in -\operatorname{int} C.$$

(viii) *for any* $x \in K$, $T(x)$ *is a nonempty compact convex set.*

Then, for any $f \in C^*\backslash\{0\}$, $S_f(F, K)$ *is a nonempty convex subset of* E.

P r o o f. Let $f \in C^*\backslash\{0\}$. Define two set-valued mappings $A, B\colon K \to 2^K$ as follows:

$$A(y) = \big\{x \in K:\ \big(\exists z \in T(x)\big)\big(f(F(x, y, z)) \geq 0\big)\big\}.$$
$$B(y) = \big\{x \in K:\ \big(\forall t \in T(y)\big)\big(f(F(x, y, t)) \geq 0\big)\big\}.$$

For any $y \in K$, by condition (i), we have $y \in A(y)$. This means $A(y) \neq \emptyset$ for any $y \in K$. The proof of the Theorem is divided into the following four steps.

(I) $A\colon K \to 2^K$ is a KKM-mapping.

Indeed, suppose by contradiction that there exist a finite subset $\{y_1, y_2, \dots, y_n\}$ of K and $\lambda_i \geq 0$, $i \in \{1, 2, \dots, n\}$ with $\sum\limits_{i=1}^{n} \lambda_i = 1$ such that $x = \sum\limits_{i=1}^{n} \lambda_i y_i \notin \bigcup\limits_{i=1}^{n} A(y_i)$. Then, $x \notin A(y_i)$, $i \in \{1, 2, \dots, n\}$. It follows that for any $z \in T(x)$

$$f(F(x, y_i, z)) < 0, \qquad i \in \{1, 2, \dots, n\}. \tag{3.1}$$

Since $F(x, \cdot, z)$ is C-convex on K,

$$\sum_{i=1}^{n} \lambda_i F(x, y_i, z) \in F\bigg(x, \sum_{i=1}^{n} \lambda_i y_i, z\bigg) + C = F(x, x, z) + C = C.$$

Therefore, for any $f \in C^*\backslash\{0\}$, we have

$$f\bigg(\sum_{i=1}^{n} \lambda_i F(x, y_i, z)\bigg) \geq 0,$$

which contradicts (3.1). Thus, A is a KKM-mapping.

(II) $\bigcap\limits_{y \in K} A(y) = \bigcap\limits_{y \in K} B(y)$.

Since F is pseudomonotone with respect to T, one has $A(y) \subset B(y)$ for all $y \in K$. It follows that $\bigcap\limits_{y \in K} A(y) \subset \bigcap\limits_{y \in K} B(y)$. Now, we prove that $\bigcap\limits_{y \in K} B(y) \subset \bigcap\limits_{y \in K} A(y)$. Let $x \in \bigcap\limits_{y \in K} B(y)$. Then, for any $y \in K$, $x \in B(y)$. And so for any $y \in K$ and $t \in T(y)$, we have

$$f(F(x, y, t)) \geq 0.$$

Let $x_\varepsilon = x + \varepsilon(y_0 - x)$ and $\varepsilon \in (0, 1)$. Then, $x_\varepsilon \in K$ and for any $t_\varepsilon \in T(x_\varepsilon)$,

$$f(F(x, x_\varepsilon, t_\varepsilon)) \geq 0.$$

Since $F(x,\cdot,z)$ is C-convex on K, one has

$$\varepsilon f(F(x,y,t_\varepsilon)) + (1-\varepsilon) f(F(x,x,t_\varepsilon)) \geq 0.$$

This fact together with condition (i) yields

$$f(F(x,y,t_\varepsilon)) \geq 0.$$

By condition (vi), passing to the limit when $\varepsilon \downarrow 0$, we get

$$f(F(x,y,z)) \geq 0 \qquad \text{for all} \quad z \in T(x).$$

This means that $x \in A(y)$. By the arbitrary of y, we have $x \in \bigcap_{y\in K} A(y)$; i.e.,

$$\bigcap_{y\in K} A(y) \supset \bigcap_{y\in K} B(y).$$

(III) $S_f(K,F) \neq \emptyset$.
We now show that for any $y \in K$, $A(y)$ is closed. Since $\bigcap_{y\in K} A(y) = \bigcap_{y\in K} B(y)$, we need only to prove that for any $y \in K$, $B(y)$ is closed. In fact, for any fixed $y \in K$, let $\{x_\alpha\} \subset B(y)$ such that $x_\alpha \to x_0$. By the closedness of K, one has $x_0 \in K$. Since $\{x_\alpha\} \subset B(y)$,

$$f(F(x_\alpha,y,t)) \geq 0 \qquad \text{for all} \quad t \in T(y).$$

By the C-upper semicontinuity of F with respect to the first argument,

$$f(F(x_0,y,t)) \geq \limsup_{\alpha} f(F(x_\alpha,y,t)) \geq 0 \qquad \text{for all} \quad t \in T(y).$$

This means $x_0 \in B(y)$. Thus, for any $y \in K$, $B(y)$ is closed and so is $A(y)$. From condition (vii), we have $A(y_0)$ is closed, and $A(y_0) \subset E$. Since E is compact, $A(y_0)$ is compact. By Lemma 2.1, we get $\bigcap_{y\in K} A(y) \neq \emptyset$. Therefore, there exists $x \in \bigcap_{y\in K} A(y)$. It follows that for any $y \in K$ there exists $z \in T(x)$ such that $f(F(x,y,z)) \geq 0$. Hence,

$$\inf_{y\in K} \max_{z\in T(x)} f(F(x,y,z)) \geq 0.$$

By conditions (vi), (viii) and Lemma 2.3,

$$\max_{z\in T(x)} \inf_{y\in K} f(F(x,y,z)) \geq 0.$$

It follows that there exist $z \in T(x)$ such that

$$f(F(x,y,z)) \geq 0 \qquad \text{for all} \quad y \in K.$$

Then $x \in S_f(K,F)$. This fact gives that $\bigcap\limits_{y\in K} A(y) \subset S_f(K,F)$. Noting that $S_f(K,F) \subset \bigcap\limits_{y\in K} A(y)$. Therefore, $S_f(K,F) = \bigcap\limits_{y\in K} A(y)$. It is easy to see that $S_f(K,F) \subset E$.

(IV) $S_f(K,F)$ is a convex subset.
Since $S_f(K,F) = \bigcap\limits_{y\in K} A(y) = \bigcap\limits_{y\in K} B(y)$, we need only to prove that for any $y \in K$, $B(y)$ is convex. In fact, for any fixed $y \in K$, let $x_1, x_2 \in B(y)$ and $\lambda \in [0,1]$. Then, $\lambda x_1 + (1-\lambda)x_2 \in K$ and for any $t \in T(y)$,

$$f(F(x_1,y,t)) \geq 0, \qquad f(F(x_2,y,t)) \geq 0.$$

By the C-concavity of F with respect to the first argument,

$$f(F(\lambda x_1 + (1-\lambda)x_2, y, t)) \geq f(\lambda F(x_1,y,t)) + f((1-\lambda)F(x_2,y,t)) \geq 0,$$

It follows that $\lambda x_1 + (1-\lambda)x_2 \in B(y)$. Therefore, for any $y \in K$, $B(y)$ is convex and so does $S_f(K,F)$. This completes the proof. □

Now we establish the connectedness of the set of weak efficient solutions to (GVEP).

Theorem 3.1. *Assume that conditions* (i)–(vii) *of* Proposition 3.1 *hold. If*

$$W = \{F(x,y,z) : \ x,y \in K, \ \ z \in D\}$$

is a bounded subset of Z, then $S_w(K,F)$ is a connected subset of K.

P r o o f. Define a set-valued mapping $H\colon C^*\backslash\{0\} \to 2^E$ by

$$H(f) = S_f(K,F) \qquad \text{for all} \quad f \in C^*\backslash\{0\}.$$

By Proposition 3.1, for any $f \in C^*\backslash\{0\}$, $S_f(K,F) \subset E$ is a nonempty convex subset. It follows that for any $f \in C^*\backslash\{0\}$, $H(f)$ is a connected subset. It is easy to see that $C^*\backslash\{0\}$ is convex, so it is a connected subset.

Now we show that $H(f)$ is upper semicontinuous on $C^*\backslash\{0\}$. Since E is compact, by Lemma 2.2, we need only prove that H is closed. Let $\{(f_\alpha, x_\alpha) : \alpha \in I\}$ be a net such that

$$\{(f_\alpha, x_\alpha) : \ \alpha \in I\} \subset \mathrm{Graph}(H) = \{(f,x) \in C^*\backslash\{0\} \times E : \ x \in H(f)\}$$

and

$$(f_\alpha, x_\alpha) \to (f,x) \in C^*\backslash\{0\} \times E,$$

where $f_\alpha \to f$ means that $\{f_\alpha\}$ converges to f with respect to the strong topology $\beta(Z^*, Z)$ in Z^*. From $x_\alpha \in H(f_\alpha)$, $\alpha \in I$, we have that there exists $z_\alpha \in T(x_\alpha)$ satisfying

$$f_\alpha(F(x_\alpha, y, z_\alpha)) \geq 0 \qquad \text{for all} \quad y \in K.$$

By the condition (iii), we have

$$f_\alpha(F(x_\alpha, y, t)) \geq 0 \qquad \text{for all} \quad y \in K, \ \ t \in T(y). \tag{3.2}$$

Note that $W = \{F(x, y, z) : \ x, y \in K, \ \ z \in D\}$ is a bounded subset of Z, define

$$P_W(z^*) := \sup\{|z^*(s)| : \ s \in W\}, \qquad z^* \in Z^*.$$

It is easy to see that P_W is a seminorm of Z^*. For arbitrary $\varepsilon > 0$,

$$U = \{z^* \in Z^* : \ P_W(z^*) < \varepsilon\}$$

is a neighborhood of zero with respect to $\beta(Z^*, Z)$. Since $f_\alpha \to f$, there exists $\alpha_0 \in I$ such that $f_\alpha - f \in U$, for all $\alpha \geq \alpha_0$. It follows that

$$P_W(f_\alpha - f) = \sup\{|(f_\alpha - f)(s)| : \ s \in W\} < \varepsilon \qquad \text{whenever} \quad \alpha \geq \alpha_0.$$

Therefore, for any $y \in K$,

$$|(f_\alpha - f)(F(x_\alpha, y, t))| = |f_\alpha(F(x_\alpha, y, t)) - f(F(x_\alpha, y, t))| < \varepsilon,$$

which implies

$$\lim[f_\alpha(F(x_\alpha, y, t)) - f(F(x_\alpha, y, t))] = 0. \tag{3.3}$$

By the C-upper semicontinuity of F with respect to the first argument,

$$f(F(x, y, t)) \geq \limsup f(F(x_\alpha, y, t)) \qquad \text{for all} \quad y \in K, \ \ t \in T(y). \tag{3.4}$$

From (3.2), (3.3) and (3.4), we have

$$\begin{aligned} 0 &\leq \limsup f_\alpha(F(x_\alpha, y, t)) \\ &= \limsup[f_\alpha(F(x_\alpha, y, t)) - f(F(x_\alpha, y, t)) + f(F(x_\alpha, y, t))] \\ &\leq \limsup[f_\alpha(F(x_\alpha, y, t)) - f(F(x_\alpha, y, t))] + \limsup f(F(x_\alpha, y, t)) \\ &\leq f(F(x, y, t)). \end{aligned}$$

From the proof of Proposition 3.1, we have that there exists $z \in T(x)$ such that

$$f(F(x, y, z)) \geq 0 \qquad \text{for all} \quad y \in K,$$

which implies

$$x \in S_f(K, F) = H(f).$$

It follows that H is a closed mapping and so H is upper semicontinuous on $C^* \setminus \{0\}$. From Lemma 2.4

$$\bigcup_{f \in C^* \setminus \{0\}} S_f(K, F)$$

is connected. Furthermore, for any $x \in K$ and $z \in T(x)$, $F(x, \cdot, z)$ is C-convex on K, then $F(x, K, z) + C$ is a convex set. By Lemma 2.5,

$$S_w(K, F) = \bigcup_{f \in C^* \setminus \{0\}} S_f(K, F)$$

is a connected subset of K. This completes the proof. □

Now we give an example to illustrate Theorem 3.1.

Example 3.1. Let $X = Y = R$, $Z = R^2$, $C = R^2_+$ and $K = D = [0, 1]$. Then $C^* = R^2_+$. Let $T(x) = [0, x]$ and

$$F(x, y, z) = \big(z(y^2 - x^2) + y^2 - x^2, z(y^2 - x^2) + y^2 - x^2\big)$$

for all $x, y \in K$ and $z \in T(x)$. It is easy to see that all assumptions of Theorem 3.1 are satisfied. Thus, by Theorem 3.1, we conclude that $S_w(K, F)$ is a connected subset of K.

From Theorem 3.1, it is easy to have the following corollaries.

Corollary 3.1. *Let $\varphi\colon K \times K \to Z$ and $\psi\colon K \to Z$ be two vector-valued mappings. Let $F(x, y, z) = \varphi(x, y) + \psi(y) - \psi(x)$. Suppose that the following conditions are satisfied:*

(i) *ψ is C-lower semicontinuous and C-convex on K;*

(ii) *for any $x \in K$, $\varphi(x, x) = 0$, and φ is pseudomonotone with respect to ψ, i.e., for any $x, y \in K$, $f \in C^* \setminus \{0\}$, $f(\psi(y)) + f(\varphi(x, y)) \geq f(\psi(x)) \implies f(\psi(y)) - f(\varphi(y, x)) \geq f(\psi(x))$;*

(iii) *for any $y \in K$, $\varphi(\cdot, y)$ are C-upper semicontinuous and C-concave on K;*

(iv) *for any $x \in K$, $\varphi(x, \cdot)$ is C-convex on K;*

(v) *$\psi(K)$ and $W = \big\{\varphi(x, y) : \ x, y \in K\big\}$ are bounded subsets of Z;*

(vi) *there exist a nonempty compact convex subset E of K and $y_0 \in E$ such that for any $x \in K \setminus E$ satisfying $\varphi(x, y) + \psi(y) - \psi(x) \in -\operatorname{int} C$.*

Then, $S_w(K, F)$ is a connected set.

Remark 3.1. The conditions of Corollary 3.1 are different from [11, Theorem 4.5].

Corollary 3.2. *Let $X = Y$, $K = D$ and $L(X, Z)$ be the space of all bounded linear mappings from X into Z. Let $T\colon K \to Z$ and $q\colon K \to Z$ be two vector-valued mappings. Let $F(x, y, T(x)) = \langle T(x), y - x\rangle + q(y) - q(x)$. Suppose that the following conditions are satisfied:*

(i) *T is f-hemicontinuous on K, i.e., for any $x, y \in K$, the mapping*

$$G(t) = f\big(\langle T(ty + (1-t)x),\ y - x\rangle\big), \qquad t \in [0,1]$$

is upper semicontinuous at 0^+;

(ii) *T is pseudomonotone with respect to q, i.e., for any $x, y \in K$, $f \in C^*\backslash\{0\}$, $f(\langle T(x), y-x\rangle) + f(q(y)) - f(q(x)) \geq 0 \implies f(\langle T(y), y-x\rangle) + f(q(y)) - f(q(x)) \geq 0$;*

(iii) *q is C-lower semicontinuous and C-convex on K;*

(iv) *$q(K)$ is a bounded subset of Z;*

(v) *there exist a nonempty compact convex subset E of K and $y_0 \in E$ such that for any $x \in K\backslash E$ satisfying $\langle T(x), y-x\rangle + q(y) - q(x) \in -\operatorname{int} C$.*

Then, $S_w(K,F)$ is a connected set.

Remark 3.2. Corollary 3.2 improves [10, Theorem 4.2] in the following two aspects:

(i) The compactness of K is dropped;

(ii) The monotonicity of T is replaced by the pseudomonotonicity of T.

4. Conclusions

In this paper, we study the connectedness of the set of weak efficient solutions for GVEP in locally convex spaces. By using the scalarization method due to Gong [11], we show the existence of weak efficient solutions and the connectedness of the set of weak efficient solutions for GVEP without the compactness. We also give an example to show that the assumptions of Theorem 3.1 hold true.

It is well known that the scalar equilibrium problem is a special case of the vector equilibrium problem (see, for example, [5, 9] and the references therein). Furthermore, it is easy to see that the generalized vector equilibrium problem considered in this paper includes many scalar and vector equilibrium problems as special cases, such as the scalar equilibrium problem studied by Blum and Oettli [4]. As pointed out by the authors of paper [4], the optimization, saddle points, Nash equilibria in noncooperative games, fixed points, convex differentiable optimization, variational operator inequalities, complementarity problems and variational inequalities with multivalued mappings could be considered as special cases of the scalar equilibrium problem. Therefore, the results presented in this paper can be applied to all the problems mentioned above.

Acknowledgement. The authors are grateful to the referees for their valuable comments and suggestions.

REFERENCES

[1] ANSARI, Q. H.—FLORES-BAZÁN, F.: *Generalized vector quasi-equilibrium problems with applications*, J. Math. Anal. Appl. **277** (2003), 246–256.
[2] AUBIN, J. P.—EKELAND, I.: *Applied Nonlinear Analysis*, John Wiley, New York, 1984.
[3] BIANCHI, M.—HADJISAVVAS, N.—SCHAIBLES, S.: *Vector equilibrium problems with generalized monotone bifunctions*, J. Optim. Theory Appl. **92** (1997), 527–542.
[4] BLUM, B.—OETTLI, W.: *From optimzation and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.
[5] CHEN, G. Y.—HUANG, X. X.—YANG, X. Q.: *Vector Optimization: Set-Valued and Variational Analysis*, Springer, Berlin, 2005.
[6] CHEN, G. Y.—YANG, X. Q.—YU, H.: *A nonlinear scalarization function and generalized quasi-vector equilibrium problem*, J. Global Optim. **32** (2005), 451–466.
[7] CHENG, Y. H.: *On the connectedness of the solution set for the weak vector variational inequality*, J. Math. Anal. Appl. **260** (2001), 1–5.
[8] FAN, K.: *A generalization of Tychonoffs fixed point theorem*, Math. Ann. **142** (1961), 305–310.
[9] *Vector Variational Inequilities and Vector Equilibria: Mathematical Theories* (F. Giannessi, ed.), Kluwer, Dordrechet, 2000.
[10] GONG, X. H.: *Efficiency and Henig efficiency for vector equilibrium problems*, J. Optim. Theory Appl. **108** (2001), 139–154.
[11] GONG, X. H.: *Connectedness of the solution sets and scalarization for vector equilibrium problems*, J. Optim. Theory Appl. **133** (2007), 151–161.
[12] GONG, X. H.—YAO, J. C.: *Connectedness of the set of efficient solutions for generalized systems*, J. Optim. Theory Appl. **138** (2008), 189–196.
[13] HUANG, N. J.—LI, J.—THOMPSON, H. B.: *Implicit vector equilibrium problems with applications*, Math. Comput. Modelling **37** (2003), 1343–1356.
[14] LEE, G. M.—KIM, D. S.—LEE, B. S.—YUN, N. D.: *Vector variational inequalities as a tool for studing vector optimization problems*, Nonlinear Anal. **34** (1998), 745–765.
[15] LONG, X. J.—HUANG, N. J.—TEO, K. L.: *Existence and stability of solutions for generalized strong vector quasi-equilibrium problems*, Math. Comput. Modelling **47** (2008), 445–451.
[16] LUC, D. T.: *Theory of Vector Optimization.* Lecture Notes in Econom. and Math. Systems 319, Springer-Verlag, New York, 1989.
[17] PECK, J. E. L.—DUMAGE, A. L.: *Games on a compact set*, Canad. J. Math. **9** (1957), 450–458.
[18] ROCKAFELLAR, R. T.: *Convex Analysis*, Princeton Univ. Press, Princeton, NJ, 1970.
[19] TAN, N. X.—TINH, P. N.: *On the existence of equilibrium points of vector functions*, Numer. Funct. Anal. Optim. **19** (1998), 141–156.
[20] WARBURTON, A. R.: *Quasiconcave vector maximization: connectedness of the sets of Pareto-optimal and weak Pareto-optimal alternatives*, J. Optim. Theory Appl. **40** (1983), 537–557.

[21] ZHONG, R. Y.—HUANG, N. J.—WONG, M. M.: *Connectedness and path-connectedness of solution sets to symmetric vector equilibrium problems*, Taiwanese J. Math. **13** (2009), 821–836.

Received 27. 6. 2009
Accepted 2. 12. 2009

* *State Key Laboratory of Oil*
and Gas Reservoir Geology and Exploitation
Southwest Petroleum University
Chengdu
Sichuan 610064
P.R. CHINA
E-mail: liuqy66@yahoo.com.cn

** *College of Mathematics and Statistics*
Chongqing Technology and Business University
Chongqing 400067
P.R. CHINA
E-mail: xianjunlong@hotmail.com

*** *Corresponding author:*
Department of Mathematics
Sichuan University
Chengdu
Sichuan 610064
P.R. CHINA
E-mail: nanjinghuang@hotmail.com