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SOME ARGUMENT INEQUALITIES FOR CERTAIN ANALYTIC FUNCTIONS

Jin-Lin Liu

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ABSTRACT. For analytic functions $f(z)$ in the open unit disk U and convex functions $g(z)$ in U, Nunokawa et al. [NUNOKAWA, M.—OWA, S.—NISHIWAKI, J.—KUROKI, K.—HAYAMI, T: *Differential subordination and argumental property*, Comput. Math. Appl. **56** (2008), 2733–2736] have proved one theorem which is a generalization of the result [POMMERENKE, CH.: *On close-to-convex analytic functions*, Trans. Amer. Math. Soc. **114** (1965), 176–186]. The object of the present paper is to generalize the theorem due to Nunokawa et al..

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. A function $f(z) \in A$ is said to be convex in U if and only if it satisfies the condition

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z \in U). \tag{1.2}$$

We denote by K the subclass of A consisting of all such functions. A function $f(z) \in A$ is said to be starlike of order α in U if and only if it satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in U) \tag{1.3}$$

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for some α $(0 \le \alpha < 1)$. We denote by $S^*(\alpha)$ the subclass of A consisting of all such functions. When $\alpha = 0$, we write $S^*(0) = S^*$. It is well known that if $f(z) \in K$, then

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2} \qquad (z \in U), \tag{1.4}$$

so that $f(z) \in S^*\left(\frac{1}{2}\right)$.

If $f(z) \in S^*$ satisfies the condition

$$\mathrm{Re}\left\{\frac{f(z)}{zf'(z)}\right\} > \beta \quad (z \in U) \tag{1.5}$$

where $0 \le \beta < 1$, then $f(z)$ is said to be starlike of reciprocal order β. This concept is defined by Nunokawa, Owa, Nishiwaki, Kuroki and Hayami in a recent paper [3].

In [4], Pommerenke proved the following theorem. If $f(z)$ is analytic in $U = \{z : |z| < 1\}$ and $g(z)$ is convex in U and

$$\left|\arg\left\{\frac{f'(z)}{g'(z)}\right\}\right| \le \frac{\pi}{2}\alpha \qquad (0 \le \alpha \le 1), \tag{1.6}$$

then

$$\left|\arg\left\{\frac{f(z_2) - f(z_1)}{g(z_2) - g(z_1)}\right\}\right| \le \frac{\pi}{2}\alpha \qquad (|z_1| < 1 \text{ and } |z_2| < 1). \tag{1.7}$$

Recently, Nunokawa, Owa, Nishiwaki, Kuroki and Hayami [3] generalized Pommerenke's result and obtained the following theorem. Let $f(z) \in A$, $g(z) \in K$ and $g(z)$ is starlike of reciprocal order β and suppose that

$$\left|\arg\left\{\frac{f'(z)}{g'(z)}\right\}\right| < \frac{\pi}{2}\alpha + \tan^{-1}\frac{\alpha\beta}{1+\alpha} \qquad (z \in U), \tag{1.8}$$

where $0 < \alpha \le 1$ and $0 \le \beta < 1$. Then

$$\left|\arg\left\{\frac{f(z)}{g(z)}\right\}\right| < \frac{\pi}{2}\alpha \qquad (z \in U). \tag{1.9}$$

In this note, we will generalize the above theorems in the next section.

2. Main result

Theorem 2.1. *Let $f(z) \in A$, $g(z) \in K$ and $g(z)$ is starlike of reciprocal order β and suppose that*

$$\left|\arg\left\{\frac{f'(z)}{g'(z)} - \gamma\right\}\right| < \frac{\pi}{2}\alpha + \tan^{-1}\frac{\alpha\beta}{1+\alpha} \qquad (z \in U), \tag{2.1}$$

where $0 < \alpha \le 1$, $0 \le \beta < 1$ and $0 \le \gamma < 1$. Then

$$\left|\arg\left\{\frac{f(z)}{g(z)} - \gamma\right\}\right| < \frac{\pi}{2}\alpha \qquad (z \in U). \tag{2.2}$$

Proof. Put

$$p(z) = \frac{1}{1-\gamma}\left(\frac{f(z)}{g(z)} - \gamma\right). \tag{2.3}$$

Then $p(z)$ is analytic in U and $p(0) = 1$. It follows from (2.3) that

$$\begin{aligned} \frac{f'(z)}{g'(z)} - \gamma &= (1-\gamma)\left(p(z) + p'(z)\frac{g(z)}{g'(z)}\right) \\ &= (1-\gamma)p(z)\left(1 + \frac{zp'(z)}{p(z)} \cdot \frac{g(z)}{zg'(z)}\right). \end{aligned} \tag{2.4}$$

If there exists a point $z_0 \in U$ such that

$$|\arg p(z)| < \frac{\pi}{2}\alpha \qquad \text{for} \quad |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha,$$

then from Nunokawa's result [2], we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = \mathrm{i}\alpha m, \tag{2.5}$$

where

$$m \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1 \qquad \text{when} \quad \arg p(z_0) = \frac{\pi}{2}\alpha,$$
$$m \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \leq -1 \qquad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2}\alpha$$

and

$$p(z_0)^{\frac{1}{\alpha}} = \pm \mathrm{i}a \qquad (a > 0).$$

Since $g(z) \in K$, we have

$$\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > \frac{1}{2} \qquad (z \in U).$$

Put $\frac{zg'(z)}{g(z)} = u + \mathrm{i}v$, where $u > \frac{1}{2}$. Then

$$\begin{aligned} \left|\frac{g(z)}{zg'(z)} - 1\right|^2 &= \left|\frac{1-u-\mathrm{i}v}{u+\mathrm{i}v}\right|^2 \\ &= \frac{1-2u+u^2+v^2}{u^2+v^2} < 1. \end{aligned}$$

Therefore, we get

$$\left|\frac{g(z)}{zg'(z)} - 1\right| < 1 \qquad (z \in U), \tag{2.6}$$

which implies that

$$\left|\mathrm{Im}\left\{\frac{g(z)}{zg'(z)}\right\}\right| < 1 \qquad (z \in U).$$

From the assumption of the theorem, we have

$$\operatorname{Re}\left\{\frac{g(z)}{zg'(z)}\right\} > \beta \qquad (z \in U). \tag{2.7}$$

Now for the case $\arg p(z_0) = \frac{\pi}{2}\alpha$, from (2.4) to (2.7), it follows that

$$\begin{aligned}
\arg\left\{\frac{f'(z_0)}{g'(z_0)} - \gamma\right\} &= \arg\left\{(1-\gamma)p(z_0)\left(1 + \frac{z_0 p'(z_0)}{p(z_0)} \cdot \frac{g(z_0)}{z_0 g'(z_0)}\right)\right\} \\
&= \arg p(z_0) + \arg\left\{1 + \frac{z_0 p'(z_0)}{p(z_0)} \cdot \frac{g(z_0)}{z_0 g'(z_0)}\right\} \\
&= \frac{\pi}{2}\alpha + \arg\left\{1 + \mathrm{i}\alpha m\left(\operatorname{Re}\frac{g(z_0)}{z_0 g'(z_0)} + \mathrm{i}\operatorname{Im}\frac{g(z_0)}{z_0 g'(z_0)}\right)\right\} \\
&= \frac{\pi}{2}\alpha + \arg\left\{1 - \alpha m\left(\operatorname{Im}\frac{g(z_0)}{z_0 g'(z_0)}\right) + \mathrm{i}\alpha m\left(\operatorname{Re}\frac{g(z_0)}{z_0 g'(z_0)}\right)\right\} \\
&\geq \frac{\pi}{2}\alpha + \tan^{-1}\left\{\frac{\alpha m \operatorname{Re}\frac{g(z_0)}{z_0 g'(z_0)}}{1 + \alpha m\left|\operatorname{Im}\frac{g(z_0)}{z_0 g'(z_0)}\right|}\right\} \\
&\geq \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{\alpha\beta m}{1+\alpha m}\right) \\
&\geq \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{\alpha\beta}{1+\alpha}\right).
\end{aligned}$$

This contradicts the assumption of the theorem. For the case $\arg p(z_0) = -\frac{\pi}{2}\alpha$, by using the same method as the above, we have a contradiction. This completes the proof of Theorem. □

Remark 2.1. For $\gamma = 0$ in Theorem, we deduce the main result obtained by Nunokawa et al. [3].

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Department of Mathematics
Yangzhou University
Jiangsu Yangzhou
225002
P.R. CHINA
E-mail: jlliu@yzu.edu.cn