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Mathematica Slovaca

DOI: 10.2478/s12175-011-0057-7
Math. Slovaca **61** (2011), No. 6, 907–920

ARGUMENT ESTIMATES OF CERTAIN MEROMORPHICALLY P-VALENT FUNCTIONS ASSOCIATED WITH A FAMILY OF LINEAR OPERATOR

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(*Communicated by Lajos Molnar*)

ABSTRACT. Making use of linear operator we investigate some inclusion relationships and some argument properties of certain meromorphically p-valent functions.

1. Introduction

Let Σ_p denote the class of functions $f(z)$ of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \qquad (p \in \mathbb{N} = \{1, 2, 3, \dots\}), \tag{1.1}$$

which are analytic and p-valent in the punctured open unit disc $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. If $f(z)$ and $g(z)$ are analytic in $U = U^* \cup \{0\}$, we say that $f(z)$ is subordinate to $g(z)$, written $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function $w(z)$ in U, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$, $(z \in U)$. Furthermore, if $g(z)$ is univalent in U, then the following equivalence relationship holds true (see [3] and [9]):

$$(\forall z \in U)\big(f(z) \prec g(z)\big) \iff \big(f(0) = g(0) \ \ \& \ \ f(U) \subset g(U)\big).$$

2010 Mathematics Subject Classification: Primary 30C45.
Keywords: meromorphically multivalent functions, Hadamard product, linear operator, argument estimates.

For functions $f(z) \in \Sigma_p$, given by (1.1) and $g(z) \in \Sigma_p$ defined by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p} \qquad (p \in \mathbb{N}), \tag{1.2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p} = (g * f)(z). \tag{1.3}$$

Now, we define the function $\varphi_p(a, c; z)$ by

$$\begin{gathered} \varphi_p(a, c; z) = z^{-p} + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k-p} \\ (a \in \mathbb{R}; \quad c \in \mathbb{R} \backslash \mathbb{Z}_0^-; \quad \mathbb{Z}_0^- = \{0, -1, -2, \dots\}; \quad z \in U^*), \end{gathered} \tag{1.4}$$

where $(x)_k$ is the Pochhammer symbol defined by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & \text{if } k = 0, \\ x(x+1)\dots(x+k-1) & \text{if } k \in \mathbb{N}. \end{cases}$$

Corresponding to the function $\varphi_p(a, c; z)$, we define a linear operator $L_p(a, c)$ on Σ_p by the following Hadamard product (see [8]):

$$L_p(a, c) f(z) = \varphi_p(a, c; z) * f(z) \qquad (f \in \Sigma_p). \tag{1.5}$$

Corresponding to the function $\varphi_p(a, c; z)$, defined by (1.4), we introduce a function $\varphi_p^{\lambda}(a, c; z)$ by

$$\begin{gathered} \varphi_p(a, c; z) * \varphi_p^{\lambda}(a, c; z) = \frac{1}{z^p (1-z)^{p+\lambda}} \\ (a, c \in \mathbb{R} \backslash \mathbb{Z}_0^-; \quad \lambda > -p; \quad p \in \mathbb{N}; \quad z \in U^*), \end{gathered} \tag{1.6}$$

which leads us to the following family of linear operators $L_p^{\lambda}(a, c)$ analogous to $L_p(a, c)$:

$$\begin{gathered} L_p^{\lambda}(a, c) f(z) = \varphi_p^{\lambda}(a, c; z) * f(z) = z^{-p} + \sum_{k=1}^{\infty} \frac{(c)_k (p+\lambda)_k}{(a)_k k!} a_{k-p} z^{k-p} \\ (a, c \in \mathbb{R} \backslash \mathbb{Z}_0^-; \quad \lambda > -p; \quad p \in \mathbb{N}; \quad f \in \Sigma_p; \quad z \in U^*). \end{gathered} \tag{1.7}$$

It is easily verified from (1.6) and (1.7) that

$$z \left(L_p^{\lambda}(a, c) f(z) \right)' = (\lambda + p) L_p^{\lambda+1}(a, c) f(z) - (\lambda + 2p) L_p^{\lambda}(a, c) f(z) \tag{1.8}$$

and

$$z \left(L_p^{\lambda}(a+1, c) f(z) \right)' = a L_p^{\lambda}(a, c) f(z) - (a+p) L_p^{\lambda}(a+1, c) f(z). \tag{1.9}$$

For a function $f(z) \in \Sigma_p$ and $\mu > 0$, the integral operator $F_{\mu,p}\colon \Sigma_p \to \Sigma_p$ is defined by

$$F_{\mu,p}(f)(z) = \frac{\mu}{z^{\mu+p}} \int_0^z t^{\mu+p-1} f(t)\,\mathrm{d}t = \left(z^{-p} + \sum_{k=1}^{\infty} \frac{\mu}{\mu+k} z^{k-p} \right) * f(z) \qquad (\mu > 0; \quad z \in U). \tag{1.10}$$

It follows from (1.10) that

$$z(L_p^\lambda(a,c)F_{\mu,p}(f)(z))' = \mu L_p^\lambda(a,c)f(z) - (\mu+p)L_p^\lambda(a,c)F_{\mu,p}(f)(z). \tag{1.11}$$

The operator $F_{\mu,p}(f)(z)$ was investigated by many authors (see for example [6] and [13]).

We note that

(i) $L_p^0(p,1)f(z) = L_p^1(p+1,1)f(z) = f(z)$ and $L_p^1(p,1)f(z) = \frac{2pf(z)+zf'(z)}{p}$;

(ii) $L_1^{\lambda-1}(n+1,1)f(z) = I_{n,\lambda}f(z)$ $(\lambda > 0;\ n > -1)$ (Yuan et al. [14]);

(iii) $L_1^\lambda(a,c)f(z) = Ł^\lambda(a,c)f(z)$ (Aghalary [1]);

(iv) $L_1^{\lambda-1}(a,c)f(z) = I_\lambda(a,c)f(z)$ $(\lambda > 0)$ (Cho and Noor [4]);

(v) $L_p^n(a,a)f(z) = D^{n+p-1}f(z) = \frac{1}{z^p(1-z)^{n+p}} * f(z)$ $(n > -p)$ (Uralegaddi and Somanatha [12] and Aouf [2]);

(vi) $L_p^{\mu-p}(\mu+1,1)f(z) = F_{\mu,p}(f)(z)$ $(\mu > 0)$.

Let M be the class of functions $h(z)$ which are analytic and univalent in U and for which $h(U)$ is convex with $h(0) = 1$ and $\mathrm{Re}\{h(z)\} > 0$, $z \in U$.

Now, by using the linear operator $L_p^\lambda(a,c)$, we define a subclass of Σ_p by

$$\Sigma_p^\lambda(a,c;h) = \left\{ f:\ f \in \Sigma_p \ \text{ and } \ -\frac{zL_p^\lambda(a,c)f(z))'}{pL_p^\lambda(a,c)f(z)} \prec h(z)\ \ (h \in M;\ \ z \in U) \right\}. \tag{1.12}$$

We also set

$$\Sigma_p^\lambda\left(a,c;\frac{1+Az}{1+Bz}\right) = \Sigma_p^\lambda(a,c;A,B) \qquad (-1 < B < A \le 1;\ \ z \in U). \tag{1.13}$$

From (1.12) and (1.13) and by using the result of Silverman and Silvia [11], we observe that a function $f(z)$ is in $\Sigma_p^\lambda(a,c;A,B)$ if and only if

$$\left| \frac{z\left(L_p^\lambda(a,c)f(z)\right)'}{L_p^\lambda(a,c)f(z)} + \frac{p(1-AB)}{1-B^2} \right| < \frac{p(A-B)}{1-B^2} \qquad (-1 < B < A \le 1;\ \ z \in U). \tag{1.14}$$

In the present paper, we investigate some inclusion relationships and argument properties of certain meromorphically p-valent functions in the punctured open unit disk in connection with the linear operator $L_p^\lambda(a,c)$.

2. The main inclusion relationships

In order to show our main results, we need the following lemmas.

Lemma 2.1. ([5]) *Let β and υ be complex constants and let $h(z)$ be convex (univalent) in U with $h(0) = 1$ and $\operatorname{Re}\{\beta h(z) + \upsilon\} > 0$. If $q(z) = 1 + q_1 z + \dots$ is analytic in U, then*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \upsilon} \prec h(z) \qquad (z \in U),$$

implies

$$q(z) \prec h(z) \qquad (z \in U).$$

Lemma 2.2. ([9]) *Let $h(z)$ be convex (univalent) in U and $\psi(z)$ be analytic in U with $\operatorname{Re}\{\psi(z)\} \geq 0$. If q is analytic in U and $q(0) = h(0)$, then*

$$q(z) + \psi(z)zq'(z) \prec h(z) \qquad (z \in U)$$

implies

$$q(z) \prec h(z) \qquad (z \in U).$$

Lemma 2.3. ([10]) *Let $q(z)$ be analytic in U, with $q(0) = 1$ and $q(z) \neq 0$ $(z \in U)$. If there exists a point $z_0 \in U$, such that*

$$|\arg q(z)| < \frac{\pi}{2}\alpha \qquad \text{for} \quad |z| < |z_0| \tag{2.1}$$

and

$$|\arg q(z_0)| = \frac{\pi}{2}\alpha \qquad (0 < \alpha \leq 1). \tag{2.2}$$

Then we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = \mathrm{i}m\alpha, \tag{2.3}$$

where

$$m \geq \frac{1}{2}\left(b + \frac{1}{b}\right) \qquad \text{when} \quad \arg q(z_0) = \frac{\pi}{2}\alpha, \tag{2.4}$$

$$m \geq -\frac{1}{2}\left(b + \frac{1}{b}\right) \qquad \text{when} \quad \arg q(z_0) = -\frac{\pi}{2}\alpha, \tag{2.5}$$

and

$$q(z_0)^{\frac{1}{\alpha}} = \pm \mathrm{i}b \qquad (b > 0). \tag{2.6}$$

With the help of Lemma 2.1, we obtain the following results:

Theorem 2.1. *Let $h(z) \in M$ with*

$$\max_{z \in U} \operatorname{Re}\{h(z)\} < \min\left\{\frac{\lambda + 2p}{p}, \frac{a + p}{p}\right\}$$
$$(a, c \in \mathbb{R}\backslash\mathbb{Z}_0^-; \quad \lambda > -p; \quad p \in \mathbb{N}).$$

Then

$$\Sigma_p^{\lambda+1}(a,c;h) \subset \Sigma_p^{\lambda}(a,c;h) \subset \Sigma_p^{\lambda}(a+1,c;h).$$

P r o o f. First of all, we show that $\Sigma_p^{\lambda+1}(a,c;h) \subset \Sigma_p^{\lambda}(a,c;h)$.

Let $f \in \Sigma_p^{\lambda+1}(a,c;h)$ and set

$$q(z) = -\frac{z\left(L_p^{\lambda}(a,c)f(z)\right)'}{pL_p^{\lambda}(a,c)f(z)} \qquad (z \in U). \tag{2.7}$$

Applying (1.8) in (2.7), we obtain

$$pq(z) - (\lambda+2p) = -(\lambda+p)\frac{L_p^{\lambda+1}(a,c)f(z)}{L_p^{\lambda}(a,c)f(z)}. \tag{2.8}$$

Differentiating (2.8) logarithmically with respect to z and multiplying by z, we have

$$q(z) + \frac{zq'(z)}{-pq(z)+\lambda+2p} = -\frac{z\left(L_p^{\lambda+1}(a,c)f(z)\right)'}{pL_p^{\lambda+1}(a,c)f(z)} \prec h(z) \qquad (z \in U), \tag{2.9}$$

from Lemma 2.1, it follows that $q(z) \prec h(z)$ in U, that is, that $f \in \Sigma_p^{\lambda}(a,c;h)$.

To prove the second part of Theorem 2.1, let $f \in \Sigma_p^{\lambda}(a,c;h)$ and put

$$s(z) = -\frac{z\left(L_p^{\lambda}(a+1,c)f(z)\right)'}{pL_p^{\lambda}(a+1,c)f(z)}$$

then, by using the arguments similar to those detailed above with (1.9), it follows that $s(z) \prec h(z)$ in U, which implies $f \in \Sigma_p^{\lambda}(a+1,c;h)$. The proof of Theorem 2.1 is thus completed. □

Taking $h(z) = \frac{1+Az}{1+Bz}$ $(-1 < B < A \leq 1)$ in Theorem 2.1, we have:

Corollary 2.1. *Let* $(1+A)/(1+B) < \min\left\{\frac{\lambda+2p}{p}, \frac{a+p}{p}\right\}$ *and* $-1 < B < A \leq 1$. *Then*

$$\Sigma_p^{\lambda+1}(a,c;A,B) \subset \Sigma_p^{\lambda}(a,c;A,B) \subset \Sigma_p^{\lambda}(a+1,c;A,B).$$

Theorem 2.2. *Let* $h(z) \in M$ *with* $\operatorname{Re}\{h(z)\} < \frac{\mu+p}{p}$, *if* $f \in \Sigma_p^{\lambda}(a,c;h)$, *then* $F_{\mu,p}(f) \in \Sigma_p^{\lambda}(a,c;h)$.

P r o o f. Let $f \in \Sigma_p^{\lambda}(a,c;h)$ and set

$$q(z) = -\frac{z\left(L_p^{\lambda}(a,c)F_{\mu,p}(f)(z)\right)'}{pL_p^{\lambda}(a,c)F_{\mu,p}(f)(z)}. \tag{2.10}$$

By applying (1.11) to (2.10), we get

$$pq(z) - (\mu+p) = -\mu\frac{L_p^{\lambda}(a,c)f(z)}{L_p^{\lambda}(a,c)F_{\mu,p}(f)(z)}. \tag{2.11}$$

Differentiating (2.11) logarithmically with respect to z and multiplying by z, we have

$$q(z) + \frac{zq'(z)}{-pq(z) + \mu + p} = -\frac{z\left(L_p^{\lambda+1}(a,c)f(z)\right)'}{pL_p^{\lambda+1}(a,c)f(z)} \prec h(z) \qquad (z \in U).$$

Hence, by virtue of Lemma 2.1, we conclude that $q(z) \prec h(z)$ in U, which implies that $F_{\mu,p}(f) \in \Sigma_p^\lambda(a,c;h)$. □

Taking $h(z) = \frac{1+Az}{1+Bz}$ $(-1 < B < A \leq 1)$ in Theorem 2.2, we have

Corollary 2.2. *Let $(1 + A)/(1 + B) < \frac{\mu+p}{p}$ and $-1 < B < A \leq 1$, if $f \in \Sigma_p^\lambda(a,c;A,B)$, then $F_{\mu,p}(f) \in \Sigma_p^\lambda(a,c;A,B)$.*

3. Argument properties

Theorem 3.1. *Let $f(z) \in \Sigma_p$, $0 < \delta \leq 1$, $0 \leq \gamma < p$ and*

$$\lambda \geq \frac{p(A-B)}{1+B} - p \qquad (-1 < B < A \leq 1;\ \ p \in \mathbb{N}).$$

If

$$\left|\arg\left(-\frac{z\left(L_p^{\lambda+1}(a,c)f(z)\right)'}{L_p^{\lambda+1}(a,c)g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta$$

for some $g(z) \in \Sigma_p^{\lambda+1}(a,c;A,B)$, then

$$\left|\arg\left(-\frac{z\left(L_p^{\lambda}(a,c)f(z)\right)'}{L_p^{\lambda}(a,c)g(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha \leq 1)$ is the solution of the equation

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\cos\frac{\pi}{2}t(A,B)}{\frac{(\lambda+p)(1-B)+p(A-B)}{1-B} + \alpha\sin\frac{\pi}{2}t(A,B)}\right) \tag{3.1}$$

and

$$t(A,B) = \frac{2}{\pi}\sin^{-1}\left(\frac{p(A-B)}{(\lambda+2p)(1-B^2) - p(1-AB)}\right) \tag{3.2}$$

Proof. Let

$$q(z) = \frac{1}{p-\gamma}\left(-\frac{z\left(L_p^{\lambda}(a,c)f(z)\right)'}{L_p^{\lambda}(a,c)g(z)} - \gamma\right). \tag{3.3}$$

Using the identity (1.8), we obtain

$$\begin{aligned}
&\frac{1}{p-\gamma}\left(-\frac{z\left(L_p^{\lambda+1}(a,c)f(z)\right)'}{L_p^{\lambda+1}(a,c)g(z)}-\gamma\right)\\
&=\frac{1}{p-\gamma}\left(-\frac{\frac{z\left[z\left(L_p^{\lambda}(a,c)f(z)\right)'\right]'}{L_p^{\lambda}(a,c)g(z)}+(\lambda+2p)\frac{z\left(L_p^{\lambda}(a,c)f(z)\right)'}{L_p^{\lambda}(a,c)g(z)}}{\frac{z(L_p^{\lambda}(a,c)g(z))'}{L_p^{\lambda}(a,c)g(z)}+\lambda+2p}-\gamma\right).
\end{aligned}\tag{3.4}$$

By virtue of (3.3) and (3.4), we observe that

$$\frac{1}{p-\gamma}\left(-\frac{z\left(L_p^{\lambda+1}(a,c)f(z)\right)'}{L_p^{\lambda+1}(a,c)g(z)}-\gamma\right)=q(z)+\frac{zq'(z)}{-r(z)+\lambda+2p},$$

where

$$r(z)=-\frac{z\left(L_p^{\lambda}(a,c)g(z)\right)'}{L_p^{\lambda}(a,c)g(z)}.$$

Since $g(z)\in\Sigma_p^{\lambda+1}(a,c;A,B)\subset\Sigma_p^{\lambda}(a,c;A,B)$, by Corollary 2.1, then from (1.14) we have

$$r(z)\prec p\frac{1+Az}{1+Bz}\qquad(z\in U).$$

we let

$$-r(z)+\lambda+2p=\rho e^{i\frac{\pi}{2}\phi}\quad(z\in U),$$

then it follows from (1.14) that

$$\frac{(\lambda+p)(1+B)-p(A-B)}{1+B}<\rho<\frac{(\lambda+p)(1-B)+p(A-B)}{1-B}$$

and

$$-t(A,B)<\phi<t(A,B),$$

where t is defined by (3.2).

Let h be a function which maps U onto the angular domain $\left\{\omega:\ |\arg\omega|<\frac{\pi}{2}\delta\right\}$ with $h(0)=1$. Applying Lemma 2.2, for this h with $\psi(z)=\frac{1}{-r(z)+\lambda+2p}$, we see that $\mathrm{Re}\{q(z)\}>0$ in U and hence $q(z)\neq 0$ in U. If there exists a point $z_0\in U$ such that the conditions (2.1) and (2.2) are satisfied, then by Lemma 2.3, we have (2.3) under the restrictions (2.4)–(2.5).

At first, we suppose that $q(z_0)^{\frac{1}{\alpha}} = \mathrm{i}b$ $(b > 0)$. Then we obtain

$$\begin{aligned}
&\arg\left[-\frac{1}{p-\gamma}\left(\frac{z_0\left(L_p^{\lambda+1}(a,c)f(z_0)\right)'}{L_p^{\lambda+1}(a,c)g(z_0)}+\gamma\right)\right] \\
= \quad &\arg\left(q(z_0)+\frac{z_0 q'(z_0)}{-r(z_0)+\lambda+2p}\right) \\
= \quad &\frac{\pi}{2}\alpha+\arg\left(1+\mathrm{i}m\alpha\left(\rho e^{\mathrm{i}\frac{\pi}{2}\phi}\right)^{-1}\right) \\
= \quad &\frac{\pi}{2}\alpha+\tan^{-1}\left(\frac{m\alpha\sin\frac{\pi}{2}(1-\phi)}{\rho+m\alpha\cos\frac{\pi}{2}(1-\phi)}\right) \\
\geq \quad &\frac{\pi}{2}\alpha+\tan^{-1}\left(\frac{\alpha\cos\frac{\pi}{2}t(A,B)}{\frac{(\lambda+p)(1-B)+p(A-B)}{1-B}+\alpha\sin\frac{\pi}{2}t(A,B)}\right)=\frac{\pi}{2}\delta,
\end{aligned}$$

where δ and $t(A,B)$ are given by (3.1) and (3.2), respectively. This contradicts to the assumption of our theorem.

Next, we suppose that $q(z_0)^{\frac{1}{\alpha}} = -\mathrm{i}b$ $(b > 0)$. Applying the same method as the above, we have

$$\begin{aligned}
&\arg\left[-\frac{1}{p-\gamma}\left(\frac{z_0\left(L_p^{\lambda+1}(a,c)f(z_0)\right)'}{L_p^{\lambda+1}(a,c)g(z_0)}+\gamma\right)\right] \\
\leq \quad &-\frac{\pi}{2}\alpha-\tan^{-1}\left(\frac{\alpha\cos\frac{\pi}{2}t(A,B)}{\frac{(\lambda+p)(1-B)+p(A-B)}{1-B}+\alpha\sin\frac{\pi}{2}t(A,B)}\right)=-\frac{\pi}{2}\delta,
\end{aligned}$$

where δ and $t(A,B)$ are given by (3.1) and (3.2),respectively, which contradicts the assumption. Therefore we complete the proof of our theorem. $\square$

Taking $\lambda = \mu - p$, $p \in \mathbb{N}$, $a = \mu$ and $c = 1$ in Theorem 3.1, we have

Corollary 3.1. *Let $f(z) \in \Sigma_p$, $0 < \delta \leq 1$, $0 \leq \gamma < p$ and*

$$\mu \geq \frac{p(A-B)}{1+B} \qquad (-1 < B < A \leq 1; \quad p \in \mathbb{N}).$$

If

$$\left|\arg\left(-\frac{z(zf''(z)+(\mu+p+1)f'(z))}{zg'(z)+(\mu+p)g(z)}-\gamma\right)\right| < \frac{\pi}{2}\delta$$

for some $g(z) \in \Sigma_p$ satisfying the condition

$$\left|\frac{z(zg''(z)+(\mu+p+1)g'(z))}{zg'(z)+(\mu+p)g(z)}+p\right| < p$$

then

$$\left|\arg\left(-\frac{zf'(z)}{g(z)}-\gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha \leq 1)$ is the solution of equation (3.1) with $\lambda = \mu - p$.

Taking $\lambda = n$ $(n > -p)$ and $a = c$ in Theorem 3.1, we have:

Corollary 3.2. *Let* $f(z) \in \Sigma_p$, $0 < \delta \leq 1$, $0 \leq \gamma < p$ *and*

$$n \geq \frac{p(A-B)}{1+B} - p \qquad (-1 < B < A \leq 1; \quad p \in \mathbb{N}).$$

If

$$\left| \arg\left(-\frac{z\left(D^{n+p} f(z)\right)'}{D^{n+p} g(z)} - \gamma \right) \right| < \frac{\pi}{2}\delta$$

for some $g(z) \in \Sigma_p$ *satisfying the condition*

$$\left| \frac{z\left(D^{n+p} g(z)\right)'}{D^{n+p} g(z)} + p \right| < p \qquad (z \in U),$$

then

$$\left| \arg\left(-\frac{z\left(D^{n+p-1} f(z)\right)'}{D^{n+p-1} g(z)} - \gamma \right) \right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha \leq 1)$ *is the solution of equation (3.1) with* $\lambda = n$ $(n > -p)$.

Taking $\delta = 1$, $\lambda = 0$, $a = p$, $A = c = 1$ and $B = 0$ in Theorem 3.1, we have:

Corollary 3.3. *Let* $f(z) \in \Sigma_p$. *If*

$$-\operatorname{Re}\left\{ \frac{z(zf''(z) + (1+2p)f'(z))}{zg'(z) + 2pg(z)} \right\} > \gamma \qquad (0 \leq \gamma < p)$$

for some $g(z) \in \Sigma_p$ *satisfying the condition*

$$\left| \frac{z(zg''(z) + (1+2p)g'(z))}{zg'(z) + 2pg(z)} + p \right| < p$$

then

$$-\operatorname{Re}\left\{ \frac{zf'(z)}{g(z)} \right\} > \gamma.$$

Remark 3.1. The results of Corollaries 3.2 and 3.3 were also obtained by Lashin [7].

Theorem 3.2. *Let* $f(z) \in \Sigma_p$, $0 < \delta \leq 1$, $\gamma > p$ *and*

$$\lambda \geq \frac{p(A-B)}{1+B} - p \qquad (-1 < B < A \leq 1; \quad p \in \mathbb{N}).$$

If

$$\left| \arg\left(\frac{z\left(L_p^{\lambda+1}(a,c) f(z)\right)'}{L_p^{\lambda+1}(a,c) g(z)} + \gamma \right) \right| < \frac{\pi}{2}\delta$$

for some $g(z) \in \Sigma_p^{\lambda+1}(a, c; A, B)$, *then*

$$\left|\arg\left(\frac{z\left(L_p^{\lambda}(a,c)f(z)\right)'}{L_p^{\lambda}(a,c)g(z)} + \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha \leq 1)$ *is the solution of equation* (3.1).

The proof of the above theorem is much akin to that of Theorem 3.1. The details may be omitted.

Theorem 3.3. *Let* $f(z) \in \Sigma_p$, $0 < \delta \leq 1$, $0 \leq \gamma < p$ *and*

$$a \geq \frac{p(A-B)}{1+B} \qquad (-1 < B < A \leq 1; \quad p \in \mathbb{N}).$$

If

$$\left|\arg\left(-\frac{z\left(L_p^{\lambda}(a,c)f(z)\right)'}{L_p^{\lambda}(a,c)g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta$$

for some $g(z) \in \Sigma_p^{\lambda}(a, c; A, B)$, *then*

$$\left|\arg\left(-\frac{z\left(L_p^{\lambda}(a+1,c)f(z)\right)'}{L_p^{\lambda}(a+1,c)g(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha \leq 1)$ *is the solution of equation* (3.1) *with* $\lambda = a - p$.

The proof of the Theorem 3.3 is much akin to that of Theorem 3.1 and so the details involved may be omitted.

Taking $\lambda = \mu - p$, $p \in \mathbb{N}$, $a = \mu$ and $c = 1$ in Theorem 3.3, we have:

Corollary 3.4. *Let* $f(z) \in \Sigma_p$, $0 < \delta \leq 1$, $0 \leq \gamma < p$ *and*

$$\mu \geq \frac{p(A-B)}{1+B} \qquad (-1 < B < A \leq 1; \quad p \in \mathbb{N}).$$

If

$$\left|\arg\left(-\frac{zf'(z)}{g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta$$

for some $g(z) \in \Sigma_p$ *satisfying the condition*

$$\left|\frac{zg'(z)}{g(z)} + p\right| < p$$

then

$$\left|\arg\left(-\frac{z(F_{\mu,p}(f)(z))'}{F_{\mu,p}(g)(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha \leq 1)$ *is the solution of equation* (3.1) *with* $\lambda = \mu - p$.

Remark 3.2. The above result was also obtained by Lashin [7].

From Corollaries 2.1, 3.1 and 3.4, we have the following remark:

Remark 3.3. Let $f(z) \in \Sigma_p$, $0 < \delta \leq 1$, $0 \leq \gamma < p$ and

$$\mu \geq \frac{p(A-B)}{1+B} \qquad (-1 < B < A \leq 1; \quad p \in \mathbb{N}).$$

If

$$\left|\arg\left(-\frac{z(zf''(z)+(\mu+p+1)f'(z))}{zg'(z)+(\mu+p)g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta$$

for some $g(z) \in \Sigma_p$ satisfying the condition

$$\left|\frac{z(zg''(z)+(\mu+p+1)g'(z))}{zg'(z)+(\mu+p)g(z)} + p\right| < p$$

then

$$\left|\arg\left(-\frac{z(F_{\mu,p}(f)(z))'}{F_{\mu,p}(g)(z)} - \gamma\right)\right| < \frac{\pi}{2}\sigma,$$

where $\sigma (0 < \sigma \leq 1)$ is the solution of the equation

$$\alpha = \sigma + \frac{2}{\pi}\tan^{-1}\left(\frac{\sigma\cos\frac{\pi}{2}t(A,B)}{\frac{\mu(1-B)+p(A-B)}{1-B} + \sigma\sin\frac{\pi}{2}t(A,B)}\right)$$

and α $(0 < \alpha \leq 1)$ is the solution of the equation

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\cos\frac{\pi}{2}t(A,B)}{\frac{\mu(1-B)+p(A-B)}{1-B} + \alpha\sin\frac{\pi}{2}t(A,B)}\right),$$

where

$$t(A,B) = \frac{2}{\pi}\sin^{-1}\left(\frac{p(A-B)}{(\mu+p)(1-B^2)-p(1-AB)}\right).$$

Taking $\delta = 1$, $\lambda = 0$, $a = p$, $A = c = 1$ and $B = 0$ in Theorem 3.3, we have:

Corollary 3.5. *Let $f(z) \in \Sigma_p$. If*

$$-\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \gamma \qquad (0 \leq \gamma < p)$$

for some $g(z) \in \Sigma_p$ satisfying the condition

$$\left|\frac{zg'(z)}{g(z)} + p\right| < p$$

then

$$-\operatorname{Re}\left\{\frac{z(F_{p,p}(f)(z))'}{F_{p,p}(g)(z)}\right\} > \gamma.$$

From Corollaries 2.1, 3.3 and 3.5, we have the following remark:

Remark 3.4. Let $f(z) \in \Sigma_p$. If

$$-\operatorname{Re}\left\{\frac{z(zf''(z)+(1+2p)f'(z))}{zg'(z)+2pg(z)}\right\} > \gamma \qquad (0 \le \gamma < p)$$

for some $g(z) \in \Sigma_p$ satisfying the condition

$$\left|\frac{z(zg''(z)+(1+2p)g'(z))}{zg'(z)+2pg(z)}+p\right| < p$$

then

$$-\operatorname{Re}\left\{\frac{z(F_{p,p}(f)(z))'}{F_{p,p}(g)(z)}\right\} > \gamma.$$

Letting $g(z) = \frac{1}{z^p}$ in Remark 3.4, we have:

Remark 3.5. Let $f(z) \in \Sigma_p$. If

$$-\operatorname{Re}\left\{z^{p+1}(zf''(z)+(1+2p)f'(z))\right\} > p\gamma \qquad (0 \le \gamma < p),$$

then

$$-\operatorname{Re}\left\{z^{p+1}(F_{p,p}(f)(z))'\right\} > \gamma.$$

Remark 3.6. By taking $\delta = A = 1$, $\mu = p$, $p \in \mathbb{N}$ and $B = 0$ in Remark 3.3, we obtain the result of Remark 3.4.

Theorem 3.4. *Let $f(z) \in \Sigma_p$, $0 < \delta \le 1$, $\gamma > p$ and*

$$a \ge \frac{p(A-B)}{1+B} \qquad (-1 < B < A \le 1; \quad p \in \mathbb{N}).$$

If

$$\left|\arg\left(\frac{z\left(L_p^\lambda(a,c)f(z)\right)'}{L_p^\lambda(a,c)g(z)}+\gamma\right)\right| < \frac{\pi}{2}\delta$$

for some $g(z) \in \Sigma_p^\lambda(a,c;A,B)$, then

$$\left|\arg\left(\frac{z\left(L_p^\lambda(a+1,c)f(z)\right)'}{L_p^\lambda(a+1,c)g(z)}+\gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha \le 1)$ is the solution of equation (3.1) with $\lambda = a - p$.

Theorem 3.5. *Let $f(z) \in \Sigma_p$, $0 < \delta \le 1$, $0 \le \gamma < p$ and*

$$\mu \ge \frac{p(A-B)}{1+B} \qquad (-1 < B < A \le 1; \quad p \in \mathbb{N}).$$

If

$$\left|\arg\left(-\frac{z\left(L_p^\lambda(a,c)f(z)\right)'}{L_p^\lambda(a,c)g(z)}-\gamma\right)\right| < \frac{\pi}{2}\delta$$

for some $g(z) \in \Sigma_p^\lambda(a,c;A,B)$, *then*

$$\left|\arg\left(-\frac{z\left(L_p^\lambda(a,c)F_{\mu,p}(f)(z)\right)'}{L_p^\lambda(a,c)F_{\mu,p}(g)(z)}-\gamma\right)\right|<\frac{\pi}{2}\alpha,$$

where α $(0<\alpha\le 1)$ *is the solution of equation* (3.1) *with* $\lambda=\mu-p$.

P r o o f. Let

$$q(z)=\frac{1}{p-\gamma}\left(-\frac{z\left(L_p^\lambda(a,c)F_{\mu,p}(f)(z)\right)'}{L_p^\lambda(a,c)F_{\mu,p}(g)(z)}-\gamma\right). \tag{3.5}$$

Using the identity (1.11), we obtain

$$\begin{aligned}&\frac{1}{p-\gamma}\left(-\frac{z\left(L_p^\lambda(a,c)f(z)\right)'}{L_p^\lambda(a,c)g(z)}-\gamma\right)\\&=\frac{1}{p-\gamma}\left(-\frac{\frac{z\left[z\left(L_p^\lambda(a,c)F_{\mu,p}(f)(z)\right)'\right]'}{L_p^\lambda(a,c)F_{\mu,p}(g)(z)}+(\mu+p)\frac{z\left(L_p^\lambda(a,c)F_{\mu,p}(f)(z)\right)'}{L_p^\lambda(a,c)F_{\mu,p}(g)(z)}}{\frac{z(L_p^\lambda(a,c)F_{\mu,p}(g)(z))'}{L_p^\lambda(a,c)F_{\mu,p}(g)(z)}+\mu+p}-\gamma\right).\end{aligned} \tag{3.6}$$

By virtue of (3.5) and (3.6), we observe that

$$\frac{1}{p-\gamma}\left(-\frac{z\left(L_p^\lambda(a,c)f(z)\right)'}{L_p^\lambda(a,c)g(z)}-\gamma\right)=q(z)+\frac{zq'(z)}{-\rho(z)+\mu+p},$$

where

$$\rho(z)=-\frac{z\left(L_p^\lambda(a,c)F_{\mu,p}(g)(z)\right)'}{L_p^\lambda(a,c)F_{\mu,p}(g)(z)}.$$

The remaining part of the proof of Theorem 3.5 is similar to that of Theorem 3.1 and so we omit the details involved. □

Theorem 3.6. *Let* $f(z)\in\Sigma_p$, $0<\delta\le 1$, $\gamma>p$ *and*

$$\mu\ge\frac{p(A-B)}{1+B}\qquad(-1<B<A\le 1;\quad p\in\mathbb{N}).$$

If

$$\left|\arg\left(\frac{z\left(L_p^\lambda(a,c)f(z)\right)'}{L_p^\lambda(a,c)g(z)}+\gamma\right)\right|<\frac{\pi}{2}\delta$$

for some $g(z) \in \Sigma_p^\lambda(a,c;A,B)$, *then*

$$\left|\arg\left(\frac{z\left(L_p^\lambda(a,c)F_{\mu,p}(f)(z)\right)'}{L_p^\lambda(a,c)F_{\mu,p}(g)(z)}+\gamma\right)\right|<\frac{\pi}{2}\alpha,$$

where α $(0<\alpha\le 1)$ *is the solution of equation* (3.1) *with* $\lambda=\mu-p$.

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Received 27. 5. 2009
Accepted 21. 8. 2009

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