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ON CONTINUITY PROPERTIES OF SOME CLASSES OF VECTOR-VALUED FUNCTIONS

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ABSTRACT. We extend the VBG^* property to the context of vector-valued functions and give some characterizations of this property. Necessary and sufficient conditions for vector-valued VBG^* functions to be continuous or weakly continuous, except at most on a countable set, are obtained.

1. Introduction

This paper discusses the continuity properties of some classes of functions that occur in the theory of vector-valued integration. The familiar result states that in the scalar case a function of bounded variation on an interval has all the unilateral limits at each point of the interval. In the vector case similar properties of functions *of weakly bounded variation* were studied in connection with some problems in the theory of stochastic differential equations (see [9] and the references therein). Other contributions to this subject were made in [2] and [3]. Among the results, established in those papers, the following two come close to ours:

- a separably-valued function of weakly bounded variation is weakly continuous, except at most on a countable set [2];
- a Banach space X does not contain an isomorphic copy of c_0 if and only if each X-valued function of weakly bounded variation is *regulated* [3].

The class of functions *of generalized weakly bounded variation in the restricted sense* (the VBG^* class, in short), introduced in [12], is associated with the

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Henstock-Stieltjes integration process and has applications, for example, in proving integration-by-parts type theorems for this integral. In the present paper we give additional characterizations of VBG^* functions and demonstrate that both the afore mentioned results can be extended in a natural way to the VBG^* class. More recent papers to which our ideas may be relevant are [4] and [13].

For the most part our notation and terminology are standard, or can be found in [7].

Throughout this paper $[a, b]$ will denote a fixed non-degenerate interval of the real line and I its closed non-degenerate subinterval. X denotes a real Banach space and X^* its dual. The closed unit ball of X is denoted by B_X. Given $f : [a, b] \to X$ and $A \subset X$, $\Delta f(I)$ and $\operatorname{absco}(A)$ denote the *increment* of f on I and the *absolutely convex hull* of A. If E is a subset of the real line, then $\operatorname{Int} E$, $\overline{E}$, and ∂E will denote the *interior* of E, the *closure* of E, and the *boundary* of E, respectively. Finally, $\mathscr{C}$ and $\mathscr{L}$ will refer to the classes of at most countable and Lebesgue negligible subsets of the real line, respectively.

2. VB^* functions

We begin with the notion of weakly bounded variation on a set.

Definition 2.1. Let $f : [a, b] \to X$ and let E be a non-empty subset of $[a, b]$. f is said to be *of weakly bounded variation in the restricted sense* (VB^*) on E if there exists a positive number M such that

$$\left\| \sum_{k=1}^{K} \lambda_k \Delta f(I_k) \right\| \le M$$

for each finite collection of pairwise non-overlapping intervals $\{I_k\}_{k=1}^{K}$ with $\partial I_k \cap E \neq \emptyset$ and for each finite collection of scalars $\{\lambda_k\}_{k=1}^{K}$ with $\max_k |\lambda_k| \le 1$. We denote by $\mathbf{W}_*(f, E)$ the lower bound of those M.

It follows from Definition 2.1 that a VB^* function f on E is necessarily bounded on $[a, b]$. We should observe at this point that in the case where $X = \mathbb{R}$ this notion is equivalent to the classical notion of a VB^* function on a set under the hypothesis that the function involved is bounded on $[a, b]$ (see [10, Lemma 5.3.8]).

Definition 2.2. Let $\Lambda \subset B_{X^*}$.

(a) Λ is said to be *w^*-λ-norming* for some $\lambda \geq 1$ (or *w^*-norming*, in short) if

$$\inf_{\|x\|=1} \sup_{x^* \in \Lambda} |x^*(x)| \geq \lambda^{-1};$$

(b) Λ is said to be *w^*-thin* if $\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$ so that $\Lambda_1 \subset \Lambda_2 \subset \ldots$ and Λ_n is *w^*-non-norming*, that is

$$\inf_{\|x\|=1} \sup_{x^* \in \Lambda_n} |x^*(x)| = 0 \qquad \text{for all} \quad n;$$

(c) Λ is *w^*-thick* if Λ is not w^*-thin.

Remark 1. By the Hahn-Banach Separation Theorem [7, Theorem 3.18], Λ is w^*-norming if and only if $\overline{\text{absco}}^{w^*}(\Lambda) \supset rB_{X^*}$ for some positive r.

As an illustration, if X contains no isomorphic copy of c_0, then any *James boundary of* X is w^*-thick [8]. The reader should refer to Nygaard's survey [15] for an extensive study of thick sets. Now we make the following definition.

Definition 2.3. Let $\Lambda \subset B_{X^*}$, $f\colon [a,b] \to X$ and let E be a non-empty subset of $[a,b]$. f is VB^*_Λ on E if x^*f is VB^* function on E for each $x^* \in \Lambda$.

Remark 2. A standard argument shows that f is VB^*_Λ on E if and only if for each $x^* \in \Lambda$ there exists a positive number M such that

$$\sum_{k=1}^{K} |\Delta(x^*f)(I_k)| \leq M$$

for each finite collection of pairwise non-overlapping intervals $\{I_k\}_{k=1}^K$ with $\partial I_k \cap E \neq \emptyset$.

Lemma 2.1 below states that in the case where Λ is w^*-thick all the above sums are uniformly bounded. Note that Alexiewicz actually proved this fact, cf. [2, Theorem 3], by using a different characterization of the w^*-thickness [14, Theorem 3.5]. Our proof based on Definition 2.2 is included for completeness.

Lemma 2.1. *Let $f\colon [a,b] \to X$ and let $\Lambda \subset B_{X^*}$ be w^*-thick. If f is VB^*_Λ on $E \subset [a,b]$, then*

$$\sup_{x^* \in B_{X^*}} \mathbf{W}_*(x^*f, E) < \infty.$$

P r o o f. For each positive integer m let $\Lambda_m = \{x^* \in \Lambda : \ \mathbf{W}_*(x^*f, E) \leq m\}$. Then $\Lambda = \bigcup_m \Lambda_m$ and $\Lambda_1 \subset \Lambda_2 \subset \ldots$. As Λ is w^*-thick, there exist M and $r > 0$ such that $\overline{\text{absco}}^{w^*}(\Lambda_M) \supset rB_{X^*}$. It is evident that $\mathbf{W}_*(x^*f, E) \leq M$ for each

$x^* \in \operatorname{absco}(\Lambda_M)$. We next show that the same inequality is fulfilled for each $x^* \in \overline{\operatorname{absco}}^{w^*}(\Lambda_M)$.

Let $\{x^*_\alpha\}$ be a net in $\operatorname{absco}(\Lambda_M)$ w^*-convergent to x^*. Fix a finite collection of non-overlapping intervals $\{I_k\}_{k=1}^K$ with $\partial I_k \cap E \neq \emptyset$ and compute

$$\sum_{k=1}^{K} |\Delta(x^* f)(I_k)| = \lim_{\alpha} \left\{ \sum_{k=1}^{K} |\Delta(x^*_\alpha f)(I_k)| \right\} \leq M.$$

Finally, for each $x^* \in B_{X^*}$ we have $\mathbf{W}_*(x^* f, E) \leq M r^{-1}$. □

The converse of Lemma 2.1 reads:

Theorem 2.1. *Let $\Lambda \subset B_{X^*}$. If f is $VB^*_{B_{X^*}}$ on $[a, b]$ whenever $f\colon [a, b] \to X$ is VB^*_Λ on $[a, b]$, then Λ is w^*-thick.*

Proof. On the contrary, assume Λ is w^*-thin. By [1, Corollary 2.4], there exists a series $\sum_n x_n$ in X such that $\sum_n |x^*(x_n)| < \infty$ for each $x^* \in \Lambda$ and $\sum_n |x^*_0(x_n)| = \infty$ for some $x^*_0 \in B_{X^*}$.

Let $\{b_n\}$ be a fixed sequence such that $a = b_1 < b_2 < \dots$ and $\lim_n b_n = b$. Define a function $f\colon [a, b] \to X$ by $f = \sum_{N=1}^{\infty} \Big(\sum_{n=1}^{N} x_n \Big) \chi_{[b_N, b_{N+1})}$. Then for each $x^* \in \Lambda$ we have $\mathbf{W}_*\big(x^* f, [a, b]\big) \leq \sum_n |x^*(x_n)| < \infty$ which means that f is VB^*_Λ on $[a, b]$. On the other hand, $\mathbf{W}_*\big(x^*_0 f, [a, b]\big) \geq \sum_n |x^*_0(x_n)| = \infty$. This is the desired contradiction. □

Definition 2.4. Let $f\colon [a, b] \to X$ and let E be a non-empty subset of $[a, b]$. f is said to be *of outside bounded variation in the restricted sense* (*outside* VB^*) on E if there exists a positive number M such that

$$\left\| \sum_{k=1}^{K} \Delta f(I_k) \right\| \leq M$$

for each finite collection of pairwise non-overlapping intervals $\{I_k\}_{k=1}^K$ with $\partial I_k \cap E \neq \emptyset$. We denote by $\mathbf{V}_*(f, E)$ the lower bound of those M.

In fact three function classes, namely VB^*_Λ in the case where Λ is w^*-thick, VB^* and outside VB^*, coincide. We present complete proof of this important fact for the reader's convenience.

Theorem 2.2. *Let $f\colon [a,b] \to X$, $E \subset [a,b]$ and let $\Lambda \subset B_{X^*}$ be w^*-thick. The following statements are equivalent.*

(i) *f is VB^*_Λ on E;*

(ii) *f is VB^* on E;*

(iii) *f is outside VB^* on E.*

Proof.

(i) $\implies$ (ii). Fix a finite collection of pairwise non-overlapping intervals $\{I_k\}_{k=1}^K$ with $\partial I_k \cap E \neq \emptyset$ and a finite collection of scalars $\{\lambda_k\}_{k=1}^K$ with $\max\limits_k |\lambda_k| \le 1$. Choose $x_0^* \in X^*$ so that $\|x_0^*\| = 1$ and

$$x_0^*\Big(\sum_{k=1}^K \lambda_k \Delta f(I_k)\Big) = \Big\|\sum_{k=1}^K \lambda_k \Delta f(I_k)\Big\|.$$

Then we have

$$\begin{aligned}\Big\|\sum_{k=1}^K \lambda_k \Delta f(I_k)\Big\| &= \Big|\sum_{k=1}^K \lambda_k \Delta(x_0^* f)(I_k)\Big|\\ &\le \sum_{k=1}^K |\Delta(x_0^* f)(I_k)| \le \mathbf{W}_*(x_0^* f, E) \le \sup_{x^*\in B_{X^*}} \mathbf{W}_*(x^* f, E).\end{aligned}$$

Now it follows from Lemma 2.1 that $\mathbf{W}_*(f,E) \le \sup\limits_{x^*\in B_{X^*}} \mathbf{W}_*(x^* f, E) < \infty$.

(ii) $\implies$ (iii). This implication is obvious.

(ii) $\implies$ (i). Fix a finite collection of pairwise non-overlapping intervals $\{I_k\}_{k=1}^K$ with $\partial I_k \cap E \neq \emptyset$ and $x^* \in B_{X^*}$. We let λ_k denote $\operatorname{sgn}\{\Delta(x^* f)(I_k)\}$ and have

$$\begin{aligned}\sum_{k=1}^K |\Delta(x^* f)(I_k)| &= \Big|\sum_{k=1}^K \lambda_k \Delta(x^* f)(I_k)\Big|\\ &\le \|x^*\| \cdot \Big\|\sum_{k=1}^K \lambda_k \Delta f(I_k)\Big\| \le \mathbf{W}_*(f,E).\end{aligned}$$

(iii) $\Longrightarrow$ (i). Fix a finite collection of pairwise non-overlapping intervals $\{I_k\}_{k=1}^K$ with $\partial I_k \cap E \neq \emptyset$ and $x^* \in B_{X^*}$. Then we have

$$\sum_{k=1}^{K} |\Delta(x^* f)(I_k)|$$
$$= \Big| \sum_{k:\Delta(x^* f)(I_k)>0} \Delta(x^* f)(I_k) \Big| + \Big| \sum_{k:\Delta(x^* f)(I_k)<0} \Delta(x^* f)(I_k) \Big|$$
$$\leq \|x^*\| \cdot \left(\Big\| \sum_{k:\Delta(x^* f)(I_k)>0} \Delta f(I_k) \Big\| + \Big\| \sum_{k:\Delta(x^* f)(I_k)<0} \Delta f(I_k) \Big\| \right) \leq 2\,\mathbf{V}_*(f, E).$$

□

Remark 3. It is useful to note that

$$\mathbf{V}_*(f, E) \leq \mathbf{W}_*(f, E) = \sup_{x^* \in B_{X^*}} \mathbf{W}_*(x^* f, E) \leq 2\,\mathbf{V}_*(f, E),$$

where f is a VB^* function on a set E.

Corollary 2.2.1. *Let $f\colon [a, b] \to X$ and $E \subset [a, b]$. Then f is VB^* on E if and only if f is VB^* on $\overline{E}$.*

Proof. The corollary follows easily from Theorem 2.2 and [10, Lemma 5.3.9].

□

Definition 2.5. Let $f\colon [a, b] \to X$ and let E be a non-empty subset of $[a, b]$. f is said to be *of generalized weakly bounded variation in the restricted sense* (VBG^*) on E if E can be written as a countable union of sets on each of which f is VB^*.

Remark 4. It follows from Corollary 2.2.1 that if f is VBG^* on an $\mathscr{F}_\sigma$-set E, then E can be written as a countable union of *closed* sets on each of which f is VB^*.

3. Strong continuity of VBG^* functions

In order to study the continuity properties of VBG^* functions, we first introduce some standard notation. Let $f\colon [a, b] \to X$. The oscillation of f at a point $t \in [a, b]$ is defined by

$$\omega f(t) = \lim_{\delta \to 0+} \omega f\big([t - \delta, t + \delta] \cap [a, b]\big),$$

where $\omega f(I)$ represents $\sup\big\{\|f(u) - f(v)\| : \ u, v \in I\big\}$. It is easy to verify that the set $D_\alpha(f) = \big\{t \in [a, b] : \ \omega f(t) \geq \alpha\big\}$ is closed for each real number α.

Further denote by $D(f)$ the set of discontinuities of f on $[a,b]$. We make note of the fact that $D(f) = \bigcup_{n=1}^{\infty} D_{\frac{1}{n}}(f)$.

Recall that a vector-valued function defined on I is said to be *regulated* on I, if it has discontinuities of the first kind only. In other words, such a function has all the unilateral limits at each point of I. That a real-valued function of bounded variation on I is regulated on I is well-known. However, in the vector case the situation changes.

We begin with a simple example, showing that there exists a VB^* function f on $[0,1]$ which is discontinuous *everywhere* on $[0,1]$.

Example 1. Let $\{r_n\}$ be a listing of the rational numbers in $[0,1]$ and define $f\colon [0,1] \to c_0$ by $f(r_n) = e_n$ and $f(t) = 0$ if t is irrational. It is clear that f is discontinuous everywhere on $[0,1]$ while $\mathbf{W}_*(f,[0,1]) = 1$.

In [3], the following theorem was established.

Theorem A. (O. Blasco et al., 2000) *X does not contain an isomorphic copy of c_0 if and only if each VB^* function $f\colon [a,b] \to X$ is regulated on $[a,b]$.*

The proof of Theorem A is based on the Bessaga-Pełczyński Theorem (see e.g. [11, Proposition 2.e.4]). As to the continuity properties of real-valued VBG^* functions, a classical result states that the set of discontinuities of a real-valued VBG^* function on $[a,b]$ is *at most countable* (see, e.g., [6, Theorem 2.10.1]). In the vector case we present Theorem 3.1 which extends the necessity part of Theorem A to the class VBG^*.

Theorem 3.1. *Suppose that X does not contain an isomorphic copy of c_0 and let $f\colon [a,b] \to X$ be VBG^* on $E \subset [a,b]$. Then the set $D(f) \cap E$ is at most countable.*

Proof. Only the case where E is uncountable is interesting. It follows from Corollary 2.2.1 that $E \subset \bigcup_{n=1}^{\infty} E_n$ so that f is VB^* on each E_n and $E_n = \overline{E}_n$. On the contrary, assume the set $D(f) \cap E$ is uncountable, then so is the set $D_\alpha(f) \cap E_{n_0}$ for some positive number α and $n_0 \in \mathbb{N}$. This set is closed. Hence, by the Cantor-Bendixson Theorem, we have $D_\alpha(f) \cap E_{n_0} = P \cup Q$ where P is a perfect set and Q is at most countable. Fix a point $c \in P$ and a positive number δ. Choose an interval $I_1 \subset (c-\delta, c+\delta)$ so that $c \in \partial I_1$, $((c-\delta, c+\delta) \setminus I_1)$ $\cap P \neq \emptyset$, and $\|\Delta f(I_1)\| > \alpha/4$. We continue this process for infinitely many steps and arrive at an infinite sequence of mutually disjoint intervals $\{I_k\}$ for which $\partial I_k \cap P \neq \emptyset$ and $\|\Delta f(I_k)\| > \alpha/4$. We have

$$\sum_{k=1}^{K} |x^*(\Delta f(I_k))| = \sum_{k=1}^{K} |\Delta(x^* f)(I_k)| \leq \mathbf{W}_*(f, P) < \infty$$

for all $x^* \in B_{X^*}$ and for all $K \in \mathbb{N}$. It follows that $\sum\limits_{k=1}^{\infty} |x^*(\Delta f(I_k))| < \infty$ for all $x^* \in B_{X^*}$ and, by the Bessaga-Pełczyński Theorem, the series $\sum\limits_{k=1}^{\infty} \Delta f(I_k)$ converges. Thus, we obtain a contradiction with $\|\Delta f(I_k)\| > \alpha/4$ for all k. □

Corollary 3.1.1. *Suppose that X is weakly sequentially complete (in particular, reflexive) and let $f\colon [a,b] \to X$ be VBG^* on $E \subset [a,b]$. Then the set $D(f) \cap E$ is at most countable.*

Corollary 3.1.2. *Suppose that X does not contain an isomorphic copy of c_0 and let $f\colon [a,b] \to X$ be VBG^* on $E \subset [a,b]$. Then $f\,|_E$ has a separable range.*

4. Weak continuity of vector-valued functions

In this section we study the weak continuity properties of vector-valued functions. Given $f\colon [a,b] \to X$, f is said to be *weakly continuous* at a point $t \in [a,b]$ provided that $x^* f$ is continuous at t for each $x^* \in X^*$. In this case $D_w(f) = \bigcup\limits_{x^* \in X^*} D(x^* f)$ is the set of weak discontinuities of f on $[a,b]$.

Once again, we begin with a simple example, showing that there exists a VB^* function f on the unit interval $[0,1]$ which is weakly discontinuous *everywhere* on $[0,1]$. Here R will denote the Banach space of regulated real-valued functions defined on $[0,1]$ that are continuous on the right with the norm of a function $x(\cdot) \in R$ defined by $\|x\| = \sup\limits_{t \in [0,1]} |x(t)|$.

Example 2. Define $f\colon [0,1] \to R$ by $f(t) = \chi_{[t,1]}$ for each $t \in [0,1]$, $x_s^* \in R^*$ by $x_s^*(x) = x(s)$ for each $s \in [0,1]$, and $x_{1-0}^* \in R^*$ by $x_{1-0}^*(x) = x(1) - x(1-0)$. It is clear that $\mathbf{W}_*(f, [0,1]) = 1$. However, $x_s^* f(t) = \chi_{[0,s]}(t)$ for each $s \in [0,1]$ and $x_{1-0}^* f(t) = \chi_{\{1\}}(t)$. This, in turn, means that $D_w(f) = [0,1]$. Note that R is non-separable. As the set $\left\{x_s^* :\ s \in \mathbb{Q} \cap [0,1]\right\}$ is countable and w^*-1-norming, R^* is w^*-separable though.

Nevertheless a separably-valued VB^* function is weakly discontinuous on at most a countable set of points:

Theorem B. (A. Alexiewicz, 1951) *Let X be separable and let $f\colon [a,b] \to X$ be VB^* on $[a,b]$. Then $D_w(f)$ is at most countable.*

The proof of this theorem presented in [2] depends in an essential way on the fact that a VB^* function on $[a,b]$ is *scalarly regulated*. It is unclear whether a result similar to Theorem B could be valid for VBG^* functions. However, by introducing some new definitions, we have been able in one or two respects to prove more.

Let $\mathscr{N}$ denote a fixed class of subsets of the real line such that

(i) $\emptyset \in \mathscr{N}$;

(ii) $N \in \mathscr{N}$ whenever $N \subset N_1$ and $N_1 \in \mathscr{N}$;

(iii) $\bigcup_i N_i \in \mathscr{N}$ whenever $N_1, N_2, \ldots \in \mathscr{N}$.

The elements of the class $\mathscr{N}$ will be named *$\mathscr{N}$-sets.* $\mathscr{C}$ and $\mathscr{L}$ provide important examples of such classes.

Definition 4.1. Let $f\colon [a,b] \to X$ and let $E \subset [a,b]$.

(a) f is said to be *$\mathscr{N}$-scalarly continuous* on E provided that $D(x^*f) \cap E$ is an $\mathscr{N}$-set for each $x^* \in X^*$;

(b) f is said to be *$\mathscr{N}$-weakly continuous* on E provided that $D_w(f) \cap E$ is an $\mathscr{N}$-set.

Note that an $\mathscr{N}$-weakly continuous function on E is necessarily $\mathscr{N}$-scalarly continuous on E. On the other hand, Example 2 shows that a $\mathscr{C}$-scalarly continuous function may not be $\mathscr{C}$-weakly continuous. The next theorem will give a simple sufficient condition for the $\mathscr{N}$-weak continuity of bounded separably-valued functions.

Theorem 4.1. (cf. [17, Lemma 1]) *Suppose that X^* is separable and $f\colon [a,b] \to X$ is $\mathscr{N}$-scalarly continuous on $E \subset [a,b]$ and bounded on $[a,b]$. Then f is $\mathscr{N}$-weakly continuous on E.*

P r o o f. Let $\{x_n^* : n \in \mathbb{N}\}$ be a countable set dense in X^*. Write N for

$$\bigcup_n \bigl(D(x_n^* f) \cap E\bigr)$$

and M for $\sup_{t\in[a,b]} \|f(t)\|$. Clearly, N is an $\mathscr{N}$-set. Fix $x^* \in X^*$, a point $t_0 \in E \setminus N$, and a positive number ε. Now choose x_m^* so that $\|x^* - x_m^*\| < \varepsilon/4M$.

Next, choose a positive number δ so that $|x_m^* f(t) - x_m^* f(t_0)| < \varepsilon/2$ for all $t \in [a,b] \cap (t_0 - \delta, t_0 + \delta)$. We have

$$\begin{aligned} &|x^* f(t) - x^* f(t_0)| \\ &\le |x^* f(t) - x_m^* f(t)| + |x_m^* f(t) - x_m^* f(t_0)| + |x_m^* f(t_0) - x^* f(t_0)| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon \end{aligned}$$

for all $t \in [a,b] \cap (t_0 - \delta, t_0 + \delta)$. Thus, $D_w(f) \cap E = N$ and the theorem is proved. □

Corollary 4.1.1. *Suppose that X^* is separable and $f\colon [a,b] \to X$ is VBG^* on $E \subset [a,b]$. Then f is $\mathscr{C}$-weakly continuous on E.*

In the situation in which X^* is w^*-separable we establish a necessary and sufficient condition for the $\mathscr{N}$-weak continuity.

Theorem 4.2. (cf. [4, Lemma 2.1]) *Suppose that X^* is w^*-separable and let $f\colon [a,b] \to X$. Then the following statements are equivalent.*

(i) *f is $\mathscr{N}$-weakly continuous on $E \subset [a,b]$;*

(ii) *f is $\mathscr{N}$-scalarly continuous on E and there exists an $\mathscr{N}$-set N such that for each sequence $\{t_n\}$ in $[a,b]$ that converges to a point $t \in E \setminus N$ the sequence $\{f(t_n)\}$ contains a weakly convergent subsequence.*

Proof.

(i) $\implies$ (ii). It is clear that the set $N = D_w(f) \cap E$ has the desired properties.

(ii) $\implies$ (i). Let $\{x_k^* : k \in \mathbb{N}\}$ be a countable set w^*-dense in X^*. Note that the set $\{x_k^* : k \in \mathbb{N}\}$ *separates* points of X. Write N_1 for $\bigcup_k (D(x_k^* f) \cap E)$. Clearly, N_1 is an $\mathscr{N}$-set. Fix $t \in E \setminus (N \cup N_1)$. We claim that f is weakly continuous at t. Choose $\{t_n\}$ convergent to t arbitrarily. It is evident that $x_k^* f(t_n) \to x_k^* f(t)$ as $n \to \infty$ for all k. As the set $\{x_k^* : k \in \mathbb{N}\}$ separates points of X, each subsequence of $\{f(t_n)\}$ contains a further subsequence weakly convergent to $f(t)$. It follows that $\{f(t_n)\}$ is weakly convergent to $f(t)$. Thus we have $D_w(f) \cap E \subset N \cup N_1$. This in turn means that f is $\mathscr{N}$-weakly continuous on E. □

Corollary 4.2.1. *Suppose that X^* is w^*-separable and let $f\colon [a,b] \to X$ be $\mathscr{N}$-scalarly continuous on $E \subset [a,b]$. If there exists a closed $\mathscr{N}$-set N such that $f\,|_{[a,b]\setminus N}$ has a relatively weakly compact range, then f is $\mathscr{N}$-weakly continuous on E.*

P r o o f. Fix a point $t \in E \setminus N$ and a sequence $\{t_n\}$ in $[a, b]$ that converges to t. As the set N is closed, with no loss of generality we may assume $t_n \notin N$ for all n. It therefore follows that the set $\{f(t_k) : \ k \in \mathbb{N}\}$ is relatively weakly compact. Thus the sequence $\{f(t_n)\}$ contains a weakly convergent subsequence and (ii) of the previous theorem holds. □

Corollary 4.2.2. (cf. [5, Theorem 3]) *Suppose that X^* is w^*-separable and let $f : [a, b] \to X$ be VBG^* on $E \subset [a, b]$. If there exists a closed countable set C such that $f\,|_{[a,b]\setminus C}$ has a relatively weakly compact range, then f is $\mathscr{C}$-weakly continuous on E.*

In conclusion it is worth remarking that a $\mathscr{C}$-weakly continuous function necessarily has a *separable* range.

Remark 5. Suppose that $f : [a, b] \to X$ is $\mathscr{C}$-weakly continuous on $E \subset [a, b]$. Then $f\,|_E$ has a separable range. Indeed, as the set $D_w(f) \cap E$ is at most countable, it suffices to show that $f(E \setminus D_w(f))$ is separable. It is clear that f is weakly continuous on the separable set $E \setminus D_w(f)$. Hence, the set $f(E \setminus D_w(f))$ is w-separable. Let S be at most countable and w-dense in $f(E \setminus D_w(f))$. By the Mazur Theorem, we have $\overline{\operatorname{span}}(S) = \overline{\operatorname{span}}^w(S) \supset f(E \setminus D_w(f))$ which is what we desired.

The above discussion reveals the following open question.

Problem. Suppose that X is separable and let $f : [a, b] \to X$ be VBG^* on $E \subset [a, b]$. Is f necessarily $\mathscr{C}$-weakly continuous on E?

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