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BOUNDS ON THE k-TUPLE DOMATIC NUMBER OF A GRAPH

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ABSTRACT. Let k be a positive integer, and let G be a simple graph with vertex set $V(G)$. A vertex of a graph G dominates itself and all vertices adjacent to it. A subset $S \subseteq V(G)$ is a *k-tuple dominating set* of G if each vertex of $V(G)$ is dominated by at least k vertices in S. The *k-tuple domatic number* of G is the largest number of sets in a partition of $V(G)$ into k-tuple dominating sets.

In this paper, we present a lower bound on the k-tuple domatic number, and we establish Nordhaus-Gaddum inequalities. Some of our results extends those for the classical domatic number.

1. Terminology and introduction

We consider finite, undirected and simple graphs G with vertex set $V(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n = n(G)$.

The *open neighborhood* $N(v) = N_G(v)$ of a vertex v consists of the vertices adjacent to v and $d(v) = d_G(v) = |N(v)|$ is the *degree* of v. The *closed neighborhood* of a vertex v is defined by $N[v] = N_G[v] = N(v) \cup \{v\}$. The *maximum degree* and *minimum degree* of a graph G are denoted by $\Delta(G) = \Delta$ and $\delta(G) = \delta$, respectively. A graph G with $\delta(G) = \Delta(G)$ is called *regular* or *δ-regular*. The complement of a graph G is denoted by $\overline{G}$. For each generic invariant μ of a graph G, let $\mu = \mu(G)$ and $\overline{\mu} = \mu(\overline{G})$. We write K_n for the *complete graph* of order n.

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Let k be a positive integer. Following Harary and Haynes [4], we call a subset $S \subseteq V(G)$ a *k-tuple dominating set* of G, if $|N_G[v] \cap S| \geq k$ for every $v \in V(G)$. The *k-tuple domination number* $\gamma_{\times k}(G)$ is the minimum cardinality among the k-tuple dominating sets of G. Note that the 1-tuple domination number $\gamma_{\times 1}(G)$ is the classical *domination number* $\gamma(G)$. A *k-tuple domatic partition* of G is a partition of $V(G)$ into k-tuple dominating sets. The *k-tuple domatic number* $d_{\times k}(G)$ is the maximum number of sets in a partition of $V(G)$ into k-tuple dominating sets. For $k = 1$, $d_{\times 1}(G)$ is simply the usual *domatic number* $d(G)$ studied first by Cockayne and Hedetniemi [2]. Not every nontrivial connected graph has a k-tuple domination number for $k \geq 2$. A graph G has a k-tuple dominating set if and only if $\delta(G) \geq k-1$. For example, no tree has a 3-tuple domination number, and no cycle C_n has a 4-tuple domination number. In 2003, Liao and Chang [8] proved that the k-tuple domination problem is **NP**-complete even for bipartite and split graphs. The papers by Chang [1], Klasing and Laforest [7] and others show that the area of k-tuple domination is still intensively studied.

For more information on the domatic number and their variants, we refer the reader to the survey article by Zelinka [10]. Harary and Haynes [3] introduced the concept of the k-tuple domatic number, and they provide the following bounds.

Theorem 1. (Harary and Haynes [3] 1998) *If G is a graph with $\delta \geq k-1$, then*

$$d_{\times k} \leq \frac{\delta + 1}{k}.$$

The special case $k = 1$ of this bound can be found in the article by Cockayne and Hedetniemi [2]. Because $\delta + \overline{\delta} \leq n - 1$, Theorem 1 leads immediately to the following Nordhaus-Gaddum inequality.

Corollary 2. (Harary and Haynes [3] 1998) *If G is a graph of order n with $\delta, \overline{\delta} \geq k-1$, then*

$$d_{\times k} + \overline{d}_{\times k} \leq \frac{\delta + 1}{k} + \frac{\overline{\delta} + 1}{k} \leq \frac{n+1}{k}.$$

In this paper, we investigate the family of graphs with $d_{\times k} + \overline{d}_{\times k} = (n+1)/k$. In particular, we show for $k \geq 2$ that there is only a finite number of graphs with $d_{\times k} + \overline{d}_{\times k} = (n+1)/k$, and if $k \geq 2$ is even, then the better bound $d_{\times k} + \overline{d}_{\times k} \leq n/k$ is valid.

For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi and Slater [5, 6].

2. A lower bound on the k-tuple domatic number

In this section we present a lower bound on the k-tuple domatic number, which generalizes the bound due to Zelinka [9] in 1983.

Theorem 3. *If G is a graph of order n and minimum degree $\delta \geq k-1$, then*

$$d_{\times k}(G) \geq \left\lfloor \frac{n}{k(n-\delta)} \right\rfloor .$$

P r o o f. If $k = \delta + 1$, then

$$k(n-\delta) = (\delta+1)(n-\delta) = n + \delta(n-\delta-1) \geq n,$$

and the statement of the theorem is obvious.

Assume next that $k \leq \delta$. Since the desired bound is trivial in the case that $k(n-\delta) > n$, we assume in the following that $k(n-\delta) \leq n$. Let $S \subseteq V(G)$ be any subset with $|S| \geq k(n-\delta)$. It follows that

$$|S| \geq k(n-\delta) = n - \delta + (k-1)(n-\delta) \geq n - \delta + (k-1)$$

and therefore $|V(G) - S| \leq \delta - k + 1$. This inequality implies that

$$|N_G(u) \cap S| \geq \delta - (\delta - k) = k$$

for $u \in V(G) - S$ and

$$|N_G[v] \cap S| \geq \delta + 1 - (\delta - k + 1) = k$$

for $v \in S$. Hence S is a k-tuple dominating set of G. If $n = t \cdot k(n-\delta) + r$ with integers $t \geq 1$ and $0 \leq r < k(n-\delta)$, then one can take any t disjoint subsets, $t-1$ of cardinality $k(n-\delta)$ and one of cardinality $k(n-\delta)+r$, and all these subsets are k-tuple dominating sets of G. This yields a k-tuple domatic partition of cardinality $t = \lfloor n/(k(n-\delta)) \rfloor$, and thus Theorem 3 is proved. □

If H is the complete graph of order $n(H) = kp$, then $d_{\times k}(H) = p$ and $\delta(H) = n(H) - 1$ and thus

$$d_{\times k}(H) = p = \frac{kp}{k} = \frac{n(H)}{k(n(H) - \delta(H))}.$$

This shows that for a complete graph of order kp the upper bound from Theorem 1 and the lower bound from Theorem 3 coincide. Therefore both these bounds are best possible.

The case $k = 1$ in Theorem 3 leads immediately to a classical result on the domatic number due to Zelinka [9].

Corollary 4. (Zelinka [9] 1983) *If G is a graph of order n and minimum degree δ, then*

$$d(G) \geq \left\lfloor \frac{n}{n-\delta} \right\rfloor .$$

3. Nordhaus-Gaddum-type results

First we derive some structural properties on graphs with equality in the inequality chain of Corollary 2.

Theorem 5. *Let G be a graph of order n with δ, $\overline{\delta} \geq k-1$ such that*

$$d_{\times k} + \overline{d}_{\times k} = \frac{n+1}{k} \tag{1}$$

and, without loss of generality, $d_{\times k} \geq \overline{d}_{\times k}$. Then

1. *G is δ-regular.*
2. *For an integer $r \in \{k, k+1, \ldots, 2k-1\}$,*
$$d_{\times k} = \frac{n}{r}. \tag{2}$$
3. *If $k = 1$, then G is isomorphic to the complete graph K_n.*
4. *If $k \geq 2$, then $n < (kr^2)/(r-k)$ for an integer $r \in \{k+1, k+2, \ldots \ldots, 2k-1\}$.*

P r o o f. According to Corollary 2, we have

$$d_{\times k} + \overline{d}_{\times k} \leq \frac{\delta+1}{k} + \frac{\overline{\delta}+1}{k}.$$

If G is not regular, then $\delta + \overline{\delta} \leq n-2$, and we obtain the upper bound $d_{\times k} + \overline{d}_{\times k} \leq \frac{n}{k}$. Thus the identity (1) implies that G is δ-regular.

The hypothesis $d_{\times k} \geq \overline{d}_{\times k}$ and (1) lead to

$$d_{\times k} \geq \frac{n+1}{2k}. \tag{3}$$

Let $S_1, S_2, \ldots, S_t$ be a k-tuple domatic partition of G such that $t = d_{\times k}(G)$ and $r = |S_1| \leq |S_2| \leq \cdots \leq |S_t|$. Clearly, $r \geq k$ and

$$n \geq r \cdot d_{\times k}. \tag{4}$$

If $r \geq 2k$, then (3) and (4) yield the contradiction

$$n \geq r \cdot t \geq 2k \cdot d_{\times k} \geq 2k \cdot \frac{n+1}{2k} = n+1.$$

Therefore we have shown that $k \le r \le 2k-1$. Since S_1 is a k-tuple dominating set, we deduce that

$$\sum_{v \in S_1} d_G(v) \ge k(n-r) + r(k-1) = kn - r$$

and thus $\Delta \ge (kn-r)/r$ and so

$$\overline{\delta} = n - 1 - \Delta \le n - 1 - \frac{kn-r}{r} = \frac{n(r-k)}{r}. \tag{5}$$

Applying Theorem 1, we thus obtain

$$\overline{d}_{\times k} \le \frac{\overline{\delta}+1}{k} \le \frac{rn - kn + r}{rk}.$$

Now (1) leads to

$$d_{\times k} = \frac{n+1}{k} - \overline{d}_{\times k} \ge \frac{rn + r - rn + kn - r}{rk} = \frac{n}{r}.$$

Using this inequality and (4), we arrive at the identity (2), and 2. is proved.

If $k = 1$, then it follows from $k \le r \le 2k-1$ that $r = 1$, and therefore (2) implies $d(G) = n$. However, this is only possible if G is isomorphic to the complete graph K_n, and so 3. is valid.

Assume next that $k \ge 2$.

The hypothesis $\overline{\delta} \ge k-1$ and inequality (5) show that $r = k$ is impossible, and so $k+1 \le r \le 2k-1$.

The identity (2) implies $|S_i| = r$ for every $i \in \{1, 2, \ldots, t\}$. Since $S_1, S_2, \ldots, S_t$ are k-tuple dominating sets of G, each vertex $v \in S_i$ is adjacent to at most $r-k$ vertices in S_j in the graph $\overline{G}$ for $i \ne j$.

Next let F be any minimum k-tuple dominating set in $\overline{G}$. If $S_i \cap F = \emptyset$ for any $i \in \{1, 2, \ldots, t\}$, then the last observation shows that $|F| \ge (kr)/(r-k)$. In the other case that $S_i \cap F \ne \emptyset$ for every $i \in \{1, 2, \ldots, t\}$, we obviously have $|F| \ge t = d_{\times k}$. This leads to

$$\overline{\gamma}_{\times k} \ge \min\left\{d_{\times k}, \frac{kr}{r-k}\right\}. \tag{6}$$

If we suppose to the contrary that $n \ge (kr^2)/(r-k)$, then (2) implies

$$d_{\times k} = \frac{n}{r} \ge \frac{kr}{r-k},$$

and thus it follows from (6) that $\overline{\gamma}_{\times k} \geq (kr)/(r-k)$. Combining this with (1), (2) and the inequality $\overline{\gamma}_{\times k} \cdot \overline{d}_{\times k} \leq n$, we arrive at the contradiction

$$\begin{aligned} \frac{n+1}{k} &= d_{\times k} + \overline{d}_{\times k} \\ &\leq \frac{n}{r} + \frac{n}{\overline{\gamma}_{\times k}} \\ &\leq \frac{n}{r} + \frac{n(r-k)}{kr} = \frac{n}{k}. \end{aligned}$$

Altogether we have shown that $k+1 \leq r \leq 2k-1$ and $n < kr^2/(r-k)$ in the case $k \geq 2$, and this completes the proof of Theorem 5. □

Since $d(K_n) + d(\overline{K}_n) = n+1$, the next well-known result is a special case of Theorem 5.

Corollary 6. (Cockayne and Hedetniemi [2] 1977) *If G is a graph of order n, then $d(G) + d(\overline{G}) = n+1$ if and only if $G = K_n$ or $\overline{K}_n$.*

Corollary 7. *Let $k \geq 2$ be an integer. Then there is only a finite number of graphs G with δ, $\overline{\delta} \geq k-1$ such that*

$$d_{\times k} + \overline{d}_{\times k} = \frac{n(G)+1}{k}. \tag{7}$$

P r o o f. If $k \geq 2$ is a fixed integer, then the hypothesis and Theorem 5 lead to $n(G) < kr^2/(r-k)$ with $k+1 \leq r \leq 2k-1$. This implies that

$$n(G) < \frac{kr^2}{r-k} \leq kr^2 \leq k(2k-1)^2,$$

and the proof is complete. □

Corollary 8. *If $k \geq 2$ is an even integer, and G is a graph of order n with δ, $\overline{\delta} \geq k-1$, then*

$$d_{\times k} + \overline{d}_{\times k} \leq \frac{n}{k}. \tag{8}$$

P r o o f. Suppose on the contrary that

$$d_{\times k} + \overline{d}_{\times k} = \frac{\delta+1}{k} + \frac{\overline{\delta}+1}{k} = \frac{n+1}{k}.$$

Since k is even, this identity shows that n is odd. According to Theorem 5, the graph G is δ-regular, and thus δ is even. Now Theorem 1 leads to $d_{\times k} \leq \delta/k$, and we obtain the contradiction

$$d_{\times k} + \overline{d}_{\times k} \leq \frac{\delta}{k} + \frac{\overline{\delta}+1}{k} \leq \frac{n}{k}.$$

□

For $k = 2$ we present an infinite family of graphs, which show that the bound (8) is tight.

Let $p \geq 1$ an integer, and let $\overline{H}_p = 3K_{2p}$. Then $d_{\times 2}(H_p) = 2p$, $d_{\times 2}(\overline{H}_p) = p$ and thus

$$d_{\times 2}(H_p) + d_{\times 2}(\overline{H}_p) = 3p = \frac{n(H_p)}{2}.$$

Next we give some examples fulfilling the identity (1) for odd k.

Let $k = 2p+1$ with an integer $p \geq 1$, and let F be a $2p$-regular graph of order $n = 4p + 1$. Then $\overline{F}$ is also $2p$-regular and $d_{\times k}(F) = d_{\times k}(\overline{F}) = 1$ and therefore

$$d_{\times k}(F) + d_{\times k}(\overline{F}) = 2 = \frac{4p+2}{2p+1} = \frac{n(F)+1}{k}.$$

We conclude with two open problems:

If $k \geq 3$ is an odd integer, then characterize the graphs of order n with δ, $\overline{\delta} \geq k - 1$ such that

$$d_{\times k} + \overline{d}_{\times k} = \frac{n+1}{k}.$$

If $k \geq 2$ is an even integer, then characterize the graphs of order n with δ, $\overline{\delta} \geq k - 1$ such that

$$d_{\times k} + \overline{d}_{\times k} = \frac{n}{k}.$$

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