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GEOMETRY OF ISOMETRIC REFLECTION VECTORS

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ABSTRACT. In this paper we study the geometry of isometric reflection vectors. In particular, we generalize known results by proving that the minimal face that contains an isometric reflection vector must be an exposed face. We also solve an open question by showing that there are isometric reflection vectors in any two dimensional subspace that are not isometric reflection vectors in the whole space. Finally, we prove that the previous situation does not hold in smooth spaces, and study the orthogonality properties of isometric reflection vectors in those spaces.

1. Introduction and background

The concept of *isometric reflection vector* seems to have been officially introduced by Skorik and Zaidenberg in [6].

Definition 1.1 (Skorik and Zaidenberg, 1997)**.** Let X be a real Banach space. A vector $e \in \mathsf{S}_X$ is said to be an isometric reflection vector if there exists a closed, maximal subspace M of X such that $X = \mathbb{R}e \oplus M$ and $\|\lambda e + m\| = \|\lambda e - m\|$ for every $m \in M$ and every $\lambda \in \mathbb{R}$.

Several results in [6] can be restated as follows.

Theorem 1.1 (Skorik and Zaidenberg, 1997)**.** *Let X be a real Banach space. The following are equivalent:*

(1) *X is a Hilbert space.*

(2) *Every $e \in \mathsf{S}_X$ is an isometric reflection vector.*

In [5] the authors generalize the previous theorem as follows.

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Theorem 1.2 (Becerra-Guerrero and Rodríguez-Palacios, 1997). *Let X be a real Banach space. The following are equivalent:*

(1) *X is a Hilbert space.*

(2) *The set of isometric reflection vectors of norm 1 has non-empty interior relative to the unit sphere.*

In [4] we give a shorter proof of the previous theorem and also prove the following result.

Theorem 1.3 (Aizpuru, García-Pacheco, and Rambla, 2004). *Let X be a real Banach space. Let $e \in \mathsf{S}_X$ be an isometric reflection vector. Then:*

(1) *The functional $e^* \in \mathsf{S}_{X^*}$ such that $e^*(e) = 1$ and $M = \ker(e^*)$ is an isometric reflection vector of X^* and is called the isometric reflection functional associated to e.*

(2) *e is an extreme point of B_X if and only if it is an exposed point of B_X.*

(3) *If $C \subset \mathsf{S}_X$ is a maximal segment whose interior contains e, then e is the midpoint of C.*

In this manuscript we generalize parts (2) and (3) in the previous theorem and analyze other geometrical properties of isometric reflection vectors, solving several open questions on this topic.

2. Extremal structure of isometric reflection vectors

Before going into the main results of this section we would like to review some well-known geometrical concepts that we use.

Definition 2.1. Let X be a real topological vector space. Let H be a convex subset of X. A subset C of H is said to be

(1) a face of H if C is convex and verifies the extremal condition, that is, if $x, y \in H$, $t \in (0,1)$, and $tx + (1-t)y \in C$, then $x, y \in C$;

(2) an exposed face of H if there exists a functional $f \in X^*$ such that $C = \{h \in H : f(h) = \sup f(H)\}$;

(3) a smooth face of H if it is a face of H with non-empty interior relative to $\mathrm{bd}(H)$.

The first result that we want to present in this section is a generalization of part (2) in Theorem 1.3.

Theorem 2.1. *Let X be a real Banach space. Let $e \in \mathsf{S}_X$ be an isometric reflection vector. Then $(e^*)^{-1}(1) \cap \mathsf{B}_X$ is the smallest face of B_X that contains e. In particular, if e is an extreme point of B_X, then it is an exposed point.*

P r o o f. Let C be a face of B_X containing e. Let $x \in (e^*)^{-1}(1) \cap \mathsf{B}_X - e$. Then $x \in \ker(e^*)$ and $\|e + x\| = 1$. Therefore, $\|e - x\| = 1$. Now,

$$e = \frac{e+x}{2} + \frac{e-x}{2},$$

which means that $e + x \in C$, that is, $x \in C - e$. This proves that $(e^*)^{-1}(1) \cap \mathsf{B}_X - e \subseteq C - e$, in other words, $(e^*)^{-1}(1) \cap \mathsf{B}_X \subseteq C$. □

Now, we will approach part (3) of Theorem 1.3 from a more general viewpoint.

Lemma 2.1. *Let X be a real topological vector space. Let H be an absolutely convex subset of X. Then:*

(1) *If C is a face of H containing 0, then $C = H$.*

(2) *If C is a maximal segment of H containing 0, then 0 is the midpoint of C.*

P r o o f.

(1) Let $h \in H \setminus \{0\}$. We have that $0 = \frac{1}{2}h + \frac{1}{2}(-h)$. Since C is a face of H and $0 \in C$, we deduce that $h \in C$.

(2) Notice that C must be balanced because it is a maximal segment of an absolutely convex set. Therefore 0 must be the midpoint of C. □

Theorem 2.2. *Let X be a real Banach space. Let $e \in \mathsf{S}_X$ be an isometric reflection vector. Let C be a face of B_X containing e. Then $C - e$ is balanced if and only if $C = (e^*)^{-1}(1) \cap \mathsf{B}_X$.*

P r o o f. Let us see first that $(e^*)^{-1}(1) \cap \mathsf{B}_X - e$ is balanced. Let $x \in (e^*)^{-1}(1) \cap \mathsf{B}_X - e$. Then $x \in \ker(e^*)$ and $\|e + x\| = 1$. Therefore, $\|e - x\| = 1$ and $e - x \in (e^*)^{-1}(1) \cap \mathsf{B}_X$. Hence, $-x \in (e^*)^{-1}(1) \cap \mathsf{B}_X - e$. Conversely, let C be a face of B_X containing e such that $C - e$ is balanced. By Theorem 2.1, we have that $(e^*)^{-1}(1) \cap \mathsf{B}_X - e$ is a face of $C - e$ containing 0, therefore we have that $C - e = (e^*)^{-1}(1) \cap \mathsf{B}_X - e$ in virtue of the part (1) in Lemma 2.1. □

Theorem 2.3. *Let X be a real Banach space. Let $e \in \mathsf{S}_X$ be an isometric reflection vector. If $C \subset \mathsf{S}_X$ is any segment whose interior contains e, then $C \subseteq (e^*)^{-1}(1) \cap \mathsf{B}_X$. In particular, if C is maximal, then e is the midpoint of C.*

Proof. Let $c \in C \setminus \{e\}$. Consider $d \in C$ such that $e \in (c, d)$. Since $(e^*)^{-1}(1) \cap \mathsf{B}_X$ is a face of B_X, we deduce that $c \in (e^*)^{-1}(1) \cap \mathsf{B}_X$. Finally, if C is maximal, then $C - e$ is a maximal segment of $(e^*)^{-1}(1) \cap \mathsf{B}_X - e$ that contains 0. By part (2) in Lemma 2.1, we deduce that 0 is the midpoint of $C - e$, that is, e is the midpoint of C. □

To finish this section, we present the following question, which has close connections with the Separable Quotient Problem. (It is, in fact, equivalent to the Separable Quotient Problem as shown in [2]).

Question 1. Let X be a real Banach space. Can X be equivalently renormed to have an isometric reflection vector $e \in \mathsf{S}_X$ such that

$$\operatorname{span}\big((e^*)^{-1}(1) \cap \mathsf{B}_X - e\big)$$

is a proper dense subspace of $\ker(e^*)$?

Before presenting our results on the previous question, we state, without a proof, the following useful lemma, which can be found in almost any text about convex geometry (see, for instance, [3]).

Lemma 2.2. *Let X be a Hausdorff, locally convex, real topological vector space. Let H be a closed, convex subset of X with non-empty interior. Let C be a proper exposed face of H and let $f \in X^* \setminus \{0\}$ be the functional that supports H on C. Let $c \in C$. The following conditions are equivalent:*

(1) *c belongs to the interior of C relative to* $\operatorname{bd}(H)$.

(2) *c belongs to the interior of C relative to* $f^{-1}(\sup f(H))$.

Theorem 2.4. *Let X be a real Banach space. Let $e \in \mathsf{S}_X$ be an isometric reflection vector. Then:*

(1) $\overline{\operatorname{span}}\big((e^*)^{-1}(1) \cap \mathsf{B}_X - e\big) \subseteq \ker(e^*)$.

(2) *If $m \in \ker(e^*)$ is so that there exists $\lambda \neq 0$ such that $\|e + \lambda m\| = 1$, then* $m \in \operatorname{span}\big((e^*)^{-1}(1) \cap \mathsf{B}_X - e\big)$.

(3) $\operatorname{span}\big((e^*)^{-1}(1) \cap \mathsf{B}_X - e\big) = \ker(e^*)$ *if and only if $(e^*)^{-1}(1) \cap \mathsf{B}_X$ is a smooth face of B_X.*

P r o o f. First off, note that for all $m \in \ker(e^*)$ we have that

$$1 = \|e\| \leq \frac{1}{2}\|e+m\| + \frac{1}{2}\|e-m\|.$$

Therefore, $\|e+m\| \geq 1$ since $\|e+m\| = \|e-m\|$.

(1) Obviously, $(e^*)^{-1}(1) \cap \mathsf{B}_X - e \subseteq \ker(e^*)$, thus

$$\overline{\operatorname{span}}\big((e^*)^{-1}(1) \cap \mathsf{B}_X - e\big) \subseteq \ker(e^*).$$

(2) If $m \in \ker(e^*)$ and there exists $\lambda \neq 0$ such that $\|\lambda m + e\| = 1$, then $\lambda m = (\lambda m + e) - e \in (e^*)^{-1}(1) \cap \mathsf{B}_X - e$.

(3) Assume first that $\operatorname{span}\big((e^*)^{-1}(1) \cap \mathsf{B}_X - e\big) = \ker(e^*)$. Note that $\ker(e^*)$ is a barrelled space and $(e^*)^{-1}(1) \cap \mathsf{B}_X - e$ is a barrel of $\ker(e^*)$. Therefore, $(e^*)^{-1}(1) \cap \mathsf{B}_X - e$ has non-empty interior in $\ker(e^*)$, in other words, $(e^*)^{-1}(1) \cap \mathsf{B}_X$ has non-empty interior in $(e^*)^{-1}(1)$. In virtue of Lemma 2.2, we deduce that $(e^*)^{-1}(1) \cap \mathsf{B}_X$ is a smooth face of B_X. Conversely, if $(e^*)^{-1}(1) \cap \mathsf{B}_X$ is a smooth face of B_X, then by applying again Lemma 2.5, we have that $(e^*)^{-1}(1) \cap \mathsf{B}_X$ has non-empty interior in $(e^*)^{-1}(1)$, that is, $(e^*)^{-1}(1) \cap \mathsf{B}_X - e$ has non-empty interior in $\ker(e^*)$ and hence $\operatorname{span}\big((e^*)^{-1}(1) \cap \mathsf{B}_X - e\big) = \ker(e^*)$. □

Finally, we present the following result that could be considered as an approach to a positive answer to Question 1.

Theorem 2.5. *Let X be an infinite dimensional, real Banach space. Then X can be equivalently renormed to have an isometric reflection vector $e \in \mathsf{S}_X$ such that* $\operatorname{span}\big((e^*)^{-1}(1) \cap \mathsf{B}_X - e\big)$ *is a closed, maximal subspace of* $\ker(e^*)$.

P r o o f. Observe that X is isomorphic to a space of the form $Y := \ell_2^2 \oplus_\infty N$, where N is some real Banach space. Now, any $e \in \mathsf{S}_{\ell_2^2}$ is an isometric reflection vector of Y such that $(e^*)^{-1}(1) \cap \mathsf{B}_X - e = \mathsf{B}_N$ and hence

$$\operatorname{span}\big((e^*)^{-1}(1) \cap \mathsf{B}_X - e\big) = N.$$

□

3. 2-dimensional properties of isometric reflection vectors

We will begin this section with the following observation.

Remark 1. Let X be a real Banach space and consider $e \in \mathsf{S}_X$ to be an isometric reflection vector. Then:

(1) The operator

$$\begin{aligned} M \oplus \mathbb{R}e &\to M \oplus \mathbb{R}e \\ m + \lambda e &\mapsto m - \lambda e \end{aligned}$$

is a surjective, linear isometry on X that maps e to $-e$.

(2) Assume now that X is 2-dimensional and its unit ball is a polygon. If $f \in \mathsf{S}_X$ is in isometric reflection with e, that is, $\|\lambda e + \gamma f\| = \|\lambda e - \gamma f\|$ for all $\lambda, \gamma \in \mathbb{R}$, then there exists a surjective, linear isometry on X that maps e to f.

The previous remark motivates the following question.

Question 2. If, in a 2-dimensional, real Banach space, two isometric reflection vectors are in isometric reflection, does there always exist a surjective, linear isometry on the space mapping one to the other?

The next example answers the previous question in the negative.

Example 1. Consider $\mathbb{R}^2$ endowed with the norm given by

$$\|(a, b)\| = \int_{-1}^{1} |at + b| \, \mathrm{d}t,$$

for every $(a, b) \in \mathbb{R}^2$. We have that $\|(a, b)\| = \|(a, -b)\|$ for all $(a, b) \in \mathbb{R}^2$, therefore $e := (1, 0)$ and $f := (0, 1)$ are in isometric reflection. Finally, e is a extreme point of the unit ball of $\mathbb{R}^2$ with this norm, but f it is not.

Let us take a look also at the following remark.

Remark 2. Let X be a real Banach space. Let $e \in \mathsf{S}_X$ be an isometric reflection vector. Then for every 2-dimensional subspace Y of X containing e, we have that e is an isometric reflection vector of Y.

Observe that the previous remark rises the following question.

Question 3. Is the converse to Remark 2 true? In other words, is "being an isometric reflection vector" a 2-dimensional property?

We will answer the previous question negatively.

Lemma 3.1. *Let K be a compact, Hausdorff topological space. Let $g \in \mathsf{S}_{\mathscr{C}(K)} \setminus \mathbf{1}$. Then there exist $\alpha, \beta \in \mathbb{R}$ such that:*

(1) $\|\alpha\mathbf{1} + \beta g\| = 1$.

(2) *There exists $t \in K$ such that $(\alpha\mathbf{1} + \beta g)(t) = 1$.*

(3) *There exists $s \in K$ such that $(\alpha\mathbf{1} + \beta g)(s) = -1$.*

P r o o f. Without loss we can assume that there are $t, s \in K$ such that $g(t) = 1$ and $g(s) = \inf g(K)$. Let us take

$$\alpha := \frac{-1 - g(s)}{1 - g(s)} \qquad \text{and} \qquad \beta := \frac{2}{1 - g(s)}.$$

Observe that α and β are well defined because $g \neq \mathbf{1}$. Furthermore, $(\alpha\mathbf{1} + \beta g)(t) = 1$ and $(\alpha\mathbf{1} + \beta g)(s) = -1$. Finally, let $k \in K$. Then

$$|(\alpha\mathbf{1} + \beta g)(k)| = \frac{|2g(k) - 1 - g(s)|}{1 - g(s)}.$$

On the one hand,

$$2g(k) - 1 - g(s) \leq 2 - 1 - g(s) = 1 - g(s).$$

On the other hand,

$$-2g(k) + 1 + g(s) \leq -2g(s) + 1 + g(k) = 1 - g(s).$$

□

Theorem 3.1. *Let K be a compact, Hausdorff topological space. Then the constant function $\mathbf{1}$ is an isometric reflection vector of any 2-dimensional subspace containing it.*

P r o o f. Let $g \in \mathsf{S}_{\mathscr{C}(K)} \setminus \{\mathbf{1}\}$. By Lemma 3.1, there exist $\alpha, \beta \in \mathbb{R}$ such that:

(1) $\|\alpha\mathbf{1} + \beta g\| = 1$.

(2) There exists $t \in K$ such that $(\alpha\mathbf{1} + \beta g)(t) = 1$.

(3) There exists $s \in K$ such that $(\alpha\mathbf{1} + \beta g)(s) = -1$.

Let $h := \alpha\mathbf{1} + \beta g$. Then $\|\lambda\mathbf{1} + \gamma h\| = |\lambda| + |\gamma| = \|\lambda\mathbf{1} - \gamma h\|$. □

Remark 3. Observe that the proof of the previous theorem shows more than what is stated. Indeed, if K is a compact, Hausdorff topological space, then the constant function $\mathbf{1}$ is an L^1-summand vector of any 2-dimensional subspace containing it.

Example 2. In [1] it is shown that if L is an uncountably discrete topological space and $\widehat{L}$ denotes its one-point compactification, then the constant function $\mathbf{1}$ in $\mathscr{C}(\widehat{L})$ is an exposed point of the unit ball of every 2-dimensional subspace containing it, but it is not an exposed point of $\mathsf{B}_{\mathscr{C}(\widehat{L})}$. Therefore, in accordance to part (2) in Theorem 1.3, we deduce that $\mathbf{1}$ is not an isometric reflection vector of $\mathscr{C}(\widehat{L})$.

4. Isometric reflection vectors in smooth Banach spaces

Before presenting the results of this section we want to explain the notation that we will make use of.

Remark 4. If X is a smooth, real Banach space and $x \in X$, then $\mathsf{J}_X(x)$ denotes the unique functional in X^* such that $\mathsf{J}_X(x)(x) = \|x\|^2$ and $\|\mathsf{J}_X(x)\| = \|x\|$. The map $\mathsf{J}_X : X \to X^*$ is known as the dual mapping of X.

The first result that we present in this section shows that "being an isometric reflection vector" is a 2-dimensional property in the class of all smooth Banach spaces.

Theorem 4.1. *Let X be a smooth, real Banach space. Let $e \in \mathsf{S}_X$. The following conditions are equivalent:*

(1) *e is an isometric reflection vector.*
(2) *e is an isometric reflection vector of every 2-dimensional subspace containing it.*

Proof. Let $m \in \ker(\mathsf{J}_X(e))$ and consider $Y = \operatorname{span}\{e, m\}$. By the smoothness, we have that $e^*_Y = \mathsf{J}_X(e)|_Y$. It follows that $\|\lambda e + m\| = \|\lambda e - m\|$ for all $\lambda \in \mathbb{R}$. □

We will conclude this section, and therefore the paper, by showing that isometric reflection vectors enjoy some orthogonality properties in smooth Banach spaces.

Theorem 4.2. *Let X be a smooth Banach space and consider a point $e \in X$. If e is an isometric reflection vector, then*

$$\mathbb{R}e = \bigcap \left\{ \ker\big(\mathsf{J}_X(m)\big) : \ m \in \ker\big(\mathsf{J}_X(e)\big) \right\}.$$

P r o o f. Consider the operator

$$\begin{aligned} T\colon M \oplus \mathbb{R}e &\to M \oplus \mathbb{R}e \\ m + \lambda e &\mapsto m - \lambda e. \end{aligned}$$

According to part (1) in Remark 1, T is a surjective linear isometry on X that maps e to $-e$. Furthermore, T fixes the elements of $M = \ker(\mathsf{J}_X(e))$. Therefore,

$$\mathsf{J}_X(m)(e) = -\left(\mathsf{J}_X(m) \circ T\right)(e) = -\mathsf{J}_X(m)(e)$$

for all $m \in M$. Then

$$\mathbb{R}e \subseteq \bigcap \left\{\ker(\mathsf{J}_X(m)) : \ m \in \ker(\mathsf{J}_X(e))\right\}.$$

Finally, let $x \in \ker(\mathsf{J}_X(m))$ for all $m \in \ker(\mathsf{J}_X(e))$. We can write $x = m + \lambda e$ with $m \in M$ and $\lambda \in \mathbb{R}$. Then

$$0 = \mathsf{J}_X(m)(x) = \|m\|^2 + \lambda \mathsf{J}_X(m)(e) = \|m\|^2 .$$

This proves the result. □

The previous theorem motivates the following question.

Question 4. Does the converse to Theorem 4.2 remain true?

The next example shows a negative answer to the previous question.

Example 3. Consider in $\mathbb{R}^2$ a smooth norm whose unit sphere contains the maximal segments $[(-1,3),(2,3)]$ and $[(3,2),(3,-1)]$. Notice that $e := (0,3)$ is not an isometric reflection vector, since e is not the midpoint of the segment $[(-1,3),(2,3)]$ (see part (3) in Theorem 1.3). However,

$$\mathbb{R}e = \bigcap \left\{\ker\big(\mathsf{J}_X(m)\big) : \ m \in \ker\big(\mathsf{J}_X(e)\big)\right\}.$$

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