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ON SOME NEW GENERALIZED DIFFERENCE SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULI

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ABSTRACT. In this paper, we define the new generalized difference sequence spaces $[V, \lambda, F, p, q]_0(\Delta_v^m)$, $[V, \lambda, F, p, q]_1(\Delta_v^m)$ and $[V, \lambda, F, p, q]_\infty(\Delta_v^m)$. We also study some inclusion relations between these spaces.

1. Introduction

Let ω be the set of all sequences of real or complex numbers and ℓ_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|$$

where $k \in \mathbb{N} = \{1, 2, \dots\}$, the set of positive integers.

Throughout this paper, let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \le \lambda_n + 1$, $\lambda_1 = 1$. The generalized de la Vallée-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, \dots$.

The difference sequence spaces was first introduced by Kızmaz [9] and then the concept was generalized by Et and Çolak [7].

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A function $f\colon [0,\infty) \to [0,\infty)$ is called a modulus function if

(i) $f(t) = 0$ iff $t = 0$,

(ii) $f(t+u) \le f(t) + f(u)$, for all $t, u \ge 0$,

(iii) f is increasing,

(iv) f is continuous from the right at 0.

Since $|f(x) - f(y)| \le f(|x-y|)$, it follows from condition (iv) that f is continuous on $[0,\infty)$. A modulus may be unbounded or bounded.

Ruckle [13] used the idea of a modulus function to construct some spaces of complex sequences. Maddox [11] investigated and discussed some properties of the three sequence spaces defined using a modulus function f. After, Malkowsky and Savaş [12] defined some λ-sequence spaces by using a modulus function. Later on Et [6] defined some sequence spaces by using a modulus function. Recently, Bektaş and Çolak [1] introduced some new sequence spaces by using a sequence of moduli $F = (f_k)$.

Let $X, Y \subset \omega$. Then we shall write ([10])

$$M(X,Y) = \bigcap_{x \in X} x^{-1} * Y = \{a \in \omega : \ ax \in Y \ \text{ for all } \ x \in X\}.$$

The set $X^\alpha = M(X, \ell_1)$ is called Köthe-Toeplitz dual or the α-dual of X. If $X \subset Y$, then $Y^\alpha \subset X^\alpha$. It is clear that $X \subset (X^\alpha)^\alpha = X^{\alpha\alpha}$. If $X = X^{\alpha\alpha}$, then X is called an α-space. In particular, an α-space is called a Köthe space or a perfect sequence space.

Definition 1.1. Let X be a sequence space. Then X is called:

(i) Solid (or normal), if $(\alpha_k x_k) \in X$ whenever $(x_k) \in X$ for all sequences (α_k) of scalar with $|\alpha_k| \le 1$.

(ii) Monotone, provided X contains the canonical preimages of all its step-spaces.

(iii) Perfect, if $X = X^{\alpha\alpha}$.

(iv) Symmetric, if $(x_k) \in X$ implies $\left(x_{\pi(k)}\right) \in X$, where $\pi(k)$ is a permutation of $\mathbb{N}$.

(v) A sequence algebra, if $(x_k), (y_k) \in X$ implies $(x_k y_k) \in X$.

It is well known that if X is perfect, then X is normal [8].

We use the following inequality throughout this paper ([10])

$$|a_k + b_k|^{p_k} \le D\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\} \tag{1}$$

where a_k and b_k are complex numbers, $D = \max\left(1, 2^{G-1}\right)$ and $G = \sup\limits_k p_k < \infty$.

2. Main results

In this section we introduce some new sequence spaces defined by a sequence of modulus functions.

Definition 2.1. Let $F = (f_k)$ be a sequence of moduli, X be a locally convex Haussdorff topological linear spaces whose topology is determined by a set Q of continuous seminorms q and $p = (p_k)$ be a sequence of strictly positive real numbers. By $\omega(X)$ we shall denote the space of all sequences defined over X. Let $v = (v_k)$ be any fixed sequence of nonzero complex numbers. Now we define the following sequence spaces. Let $m \in \mathbb{N}$ be fixed, then

$$[V, \lambda, F, p, q]_1(\Delta_v^m) = \Big\{x \in \omega(X) : \lim_n \tfrac{1}{\lambda_n} \sum_{k \in I_n} [f_k(q(\Delta_v^m x_k - L))]^{p_k} = 0, \text{ for some } L\Big\},$$

$$[V, \lambda, F, p, q]_0(\Delta_v^m) = \Big\{x \in \omega(X) : \lim_n \tfrac{1}{\lambda_n} \sum_{k \in I_n} \big[f_k\big(q(\Delta_v^m x_k)\big)\big]^{p_k} = 0\Big\},$$

$$[V, \lambda, F, p, q]_\infty(\Delta_v^m) = \Big\{x \in \omega(X) : \sup_n \tfrac{1}{\lambda_n} \sum_{k \in I_n} \big[f_k\big(q(\Delta_v^m x_k)\big)\big]^{p_k} < \infty\Big\},$$

where

$$\Delta_v^0 x = (v_k x_k),$$
$$(\Delta_v x_k) = (v_k x_k - v_{k+1} x_{k+1}),$$
$$(\Delta_v^m x_k) = \left(\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1}\right),$$

and so that $\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$.

The above sequence spaces contain some unbounded sequences for $m \geq 1$. For example, let $X = \mathbb{C}$, $f_k(x) = x$ for all $k \in \mathbb{N}$, $q(x) = |x|$, $\lambda_n = n$ for all $n \in \mathbb{N}$, $v = (1, 1, \dots)$ and $p_k = 1$ for all $k \in \mathbb{N}$, then $(k^m) \in [V, \lambda, F, p, q]_\infty(\Delta_v^m)$ but $(k^m) \notin \ell_\infty$.

In the case $f_k(x) = x$ for every k and $p_k = 1$ for all $k \in \mathbb{N}$, we shall write $[V, \lambda, p, q]_Z(\Delta_v^m)$ and $[V, \lambda, F, q]_Z(\Delta_v^m)$ instead of $[V, \lambda, F, p, q]_Z(\Delta_v^m)$, respectively. If we choose $f_k(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$, we have $[V, \lambda, F, p, q]_Z(\Delta_v^m) = [V, \lambda, q]_Z(\Delta_v^m)$.

Throughout the paper Z will denote any one of the notation 0, 1 or ∞.

Theorem 2.2. *Let the sequence (p_k) be bounded. Then $[V, \lambda, F, p, q]_Z(\Delta_v^m)$ are linear spaces over the complex field $\mathbb{C}$.*

The proof is easy and thus omitted.

Theorem 2.3. $[V, \lambda, F, p, q]_0 \left(\Delta_v^m\right)$ *is a paranormed (need not to be totally paranormed) space with*

$$G_\Delta(x) = \sup_n \left(\frac{1}{\lambda_n} \sum_{k \in I_n} \left[f_k\big(q(\Delta_v^m x_k)\big) \right]^{p_k} \right)^{\frac{1}{M}}$$

where $M = \max\Big(1, \sup_k p_k\Big)$.

P r o o f. From Theorem 2.2, for each $x \in [V, \lambda, F, p, q]_0 \left(\Delta_v^m\right)$, $G_\Delta(x)$ exists. Clearly $G_\Delta(x) = G_\Delta(-x)$. It is trivial that $\Delta_v^m x_k = 0$ for $x = \theta$. Hence, we get $G_\Delta(\theta) = 0$. By Minkowski's inequality, we have $G_\Delta(x + y) \leq G_\Delta(x) + G_\Delta(y)$. Now we show that the scalar multiplication is continuous. Let η be any fixed complex numbers. By definition of f_k for all k, we have $x \to \theta$ implies $G_\Delta(\eta x) \to 0$. Similarly we have x fixed and $\eta \to 0$ implies $G_\Delta(\eta x) \to 0$. Finally $x \to \theta$ and $\eta \to 0$ implies $G_\Delta(\eta x) \to 0$.

This completes the proof. $\square$

Theorem 2.4. *Let* $F = (f_k)$ *and* $G = (g_k)$ *be two sequences of moduli. For any two sequences* $p = (p_k)$ *and* $t = (t_k)$ *of strictly positive real numbers and any two seminorms* q_1, q_2 *we have*

(i) $[V, \lambda, F, p, q]_Z \left(\Delta_v^m\right) \cap [V, \lambda, G, p, q]_Z \left(\Delta_v^m\right) \subset [V, \lambda, F + G, p, q]_Z \left(\Delta_v^m\right)$,

(ii) $[V, \lambda, F, p, q_1]_Z \left(\Delta_v^m\right) \cap [V, \lambda, F, p, q_2]_Z \left(\Delta_v^m\right) \subset [V, \lambda, F, p, q_1 + q_2]_Z \left(\Delta_v^m\right)$,

(iii) *if* q_1 *is stronger than* q_2*, then* $[V, \lambda, F, p, q_1]_Z \left(\Delta_v^m\right) \subset [V, \lambda, F, p, q_2]_Z \left(\Delta_v^m\right)$,

(iv) *if* q_1 *is equivalent to* q_2*, then* $[V, \lambda, F, p, q_1]_Z \left(\Delta_v^m\right) = [V, \lambda, F, p, q_2]_Z \left(\Delta_v^m\right)$,

(v) $[V, \lambda, F, p, q_1]_Z \left(\Delta_v^m\right) \cap [V, \lambda, F, t, q_2]_Z \left(\Delta_v^m\right) \neq \emptyset$.

Proof is omitted.

Theorem 2.5. *Let* X *stands for* $[V, \lambda, F, q]_Z$ *and* $m \geq 1$*. Then* $X\left(\Delta_v^{m-1}\right) \subset X\left(\Delta_v^m\right)$ *and inclusions are strict. In general* $X\left(\Delta_v^i\right) \subset X\left(\Delta_v^m\right)$ *for all* $i = 1, 2, \ldots, m - 1$ *and the inclusions are strict.*

P r o o f. We give the proof for $[V, \lambda, F, q]_\infty$ only. In a similar way we proceed for $[V, \lambda, F, q]_1$ and $[V, \lambda, F, q]_0$. Let $x \in [V, \lambda, F, q]_\infty \left(\Delta_v^{m-1}\right)$. Then we have

$$\sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[f_k\left(q\left(\Delta_v^{m-1} x_k\right)\right) \right] < \infty.$$

Since f_k is a modulus for each k and so non-decreasing, we have

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f_k\left(q\left(\Delta_v^m x_k\right)\right)\right]\\
&=\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f_k\left(q\left(\Delta_v^{m-1}x_k-\Delta_v^{m-1}x_{k+1}\right)\right)\right]\\
&\le\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f_k\left(q\left(\Delta_v^{m-1}x_k\right)\right)\right]+\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f_k\left(q\left(\Delta_v^{m-1}x_{k+1}\right)\right)\right]<\infty.
\end{aligned}$$

Thus $[V,\lambda,F,q]_\infty\left(\Delta_v^{m-1}\right)\subset[V,\lambda,F,q]_\infty\left(\Delta_v^m\right)$. Proceeding in this way one will have $[V,\lambda,F,q]_\infty\left(\Delta_v^i\right)\subset[V,\lambda,F,q]_\infty\left(\Delta_v^m\right)$ for $i=1,2,\ldots,m-1$. The sequence $x=(k^m)$, for example, belongs to $[V,\lambda,F,q]_\infty\left(\Delta_v^m\right)$, but does not belong to $[V,\lambda,F,q]_\infty\left(\Delta_v^{m-1}\right)$ for $f_k(u)=u$ (for all $k\in\mathbb{N}$). Therefore the inclusions are strict. □

Theorem 2.6. *Let $0<p_k\le t_k$ and (t_k/p_k) be bounded. Then*

(i) $[V,\lambda,F,t,q]_0\left(\Delta_v^m\right)\subset[V,\lambda,F,p,q]_0\left(\Delta_v^m\right)$,

(ii) $[V,\lambda,F,t,q]_1\left(\Delta_v^m\right)\subset[V,\lambda,F,p,q]_1\left(\Delta_v^m\right)$,

(iii) $[V,\lambda,F,t,q]_\infty\left(\Delta_v^m\right)\subset[V,\lambda,F,p,q]_\infty\left(\Delta_v^m\right)$.

Proof. We shall prove only (i). The other inclusions can be proved similarly. Let $x\in[V,\lambda,F,t,q]_0\left(\Delta_v^m\right)$. Write $w_k=\left[f_k\left(q\left(\Delta_v^m x_k\right)\right)\right]^{t_k}$ and $\mu_k=\frac{p_k}{t_k}$, so that $0<\mu<\mu_k\le 1$ for each k.

We define the sequences (u_k) and (s_k) as follows:

Let $u_k=w_k$ and $s_k=0$ if $w_k\ge 1$, and let $u_k=0$ and $s_k=w_k$ if $w_k<1$. Then it is clear that for all $k\in\mathbb{N}$, we have $w_k=u_k+s_k$, $w_k^{\mu_k}=u_k^{\mu_k}+s_k^{\mu_k}$. Now it follows that $u_k^{\mu_k}\le u_k\le w_k$ and $s_k^{\mu_k}\le s_k^{\mu}$. Therefore

$$\lambda_n^{-1}\sum_{k\in I_n}w_k^{\mu_k}\le\lambda_n^{-1}\sum_{k\in I_n}w_k+\left(\lambda_n^{-1}\sum_{k\in I_n}s_k\right)^{\mu}.$$

Hence $x\in[V,\lambda,F,p,q]_0\left(\Delta_v^m\right)$. □

Theorem 2.7. *For a sequence of moduli $F=(f_k)$,*

$$[V,\lambda,F,p,q]_1\left(\Delta_v^m\right)\subset[V,\lambda,F,p,q]_\infty\left(\Delta_v^m\right)$$

assuming

$$\sup_k\left[f_k(t)\right]^{p_k}<\infty,\qquad \text{for all}\quad t>0. \tag{2}$$

P r o o f. Let $x \in [V, \lambda, F, p, q]_1 (\Delta_v^m)$. By using the definition of modulus function, we have

$$\begin{aligned}&\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f_k\left(q\left(\Delta_v^m x_k\right)\right)\right]^{p_k}\\ &\le D\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f_k\left(q\left(\Delta_v^m x_k - L\right)\right)\right]^{p_k} + D\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f_k\left(q\left(L\right)\right)\right]^{p_k},\end{aligned}$$

where $D = \max\left(1, 2^{G-1}\right)$. Hence

$$\begin{aligned}&\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f_k\left(q\left(\Delta_v^m x_k\right)\right)\right]^{p_k}\\ &\le D\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f_k\left(q\left(\Delta_v^m x_k - L\right)\right)\right]^{p_k} + D\sup_k\left[f_k\left(q\left(L\right)\right)\right]^{p_k}.\end{aligned}$$

Since $x \in [V, \lambda, F, p, q]_1 (\Delta_v^m)$, we get the result by (2). □

Theorem 2.8. *Let $0 < \inf p_k \le \sup p_k < \infty$. Then for a sequence of moduli $F = (f_k)$ the following statements are equivalent:*

(i) $[V, \lambda, p, q]_\infty (\Delta_v^m) \subseteq [V, \lambda, F, p, q]_\infty (\Delta_v^m)$,

(ii) $[V, \lambda, p, q]_0 (\Delta_v^m) \subseteq [V, \lambda, F, p, q]_\infty (\Delta_v^m)$,

(iii) $\sup_n \frac{1}{\lambda_n}\sum_{k\in I_n}\left[f_k(t)\right]^{p_k} < \infty$ *for all* $t > 0$.

P r o o f. It is trivial that (i) implies (ii). Let (ii) hold and suppose that (iii) does not hold. Then for some $t > 0$

$$\sup_n \frac{1}{\lambda_n}\sum_{k\in I_n}\left[f_k(t)\right]^{p_k} = \infty$$

and therefore there exists an increasing sequence (n_i) of positive integers such that

$$\frac{1}{\lambda_{n_i}}\sum_{k\in I_{n_i}}\left[f_k\left(i^{-1}\right)\right]^{p_k} > i, \qquad i = 1, 2, \dots. \tag{3}$$

Define $x = (x_k)$ such that

$$\Delta_v^m x_k = \begin{cases} i^{-1}, & k \in I_{n_i},\ i = 1, 2, \dots \\ 0, & \text{otherwise}. \end{cases}$$

Then $x \in [V, \lambda, p, q]_0 (\Delta_v^m)$, but by (3), $x \notin [V, \lambda, F, p, q]_\infty (\Delta_v^m)$ which contradicts (ii). Hence (iii) must hold. Therefore (ii) implies (iii).

Let (iii) hold and $x \in [V, \lambda, p, q]_\infty(\Delta_v^m)$. Suppose that $x \notin [V, \lambda, F, p, q]_\infty(\Delta_v^m)$. Then we have

$$\sup_n \frac{1}{\lambda_n}\sum_{k\in I_n}\left[f_k\left(q\left(\Delta_v^m x_k\right)\right)\right]^{p_k} = \infty. \tag{4}$$

Let $q(\Delta_v^m x_k) = t$ for each k. Then by (4)

$$\sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} [f_k(t)]^{p_k} = \infty,$$

which contradicts (iii). Hence (i) must hold. Thus (iii) implies (i). This completes the proof. □

Theorem 2.9. *Let $1 < p_k \le \sup p_k < \infty$. Then for a sequence $F = (f_k)$ of moduli, the following statements are equivalent:*

(i) $[V, \lambda, F, p, q]_0 (\Delta_v^m) \subseteq [V, \lambda, p, q]_0 (\Delta_v^m)$,

(ii) $[V, \lambda, F, p, q]_0 (\Delta_v^m) \subseteq [V, \lambda, p, q]_\infty (\Delta_v^m)$,

(iii) $\inf_n \frac{1}{\lambda_n} \sum_{k \in I_n} [f_k(t)]^{p_k} > 0$ *for all* $t > 0$.

Proof.

(i) implies (ii) is obvious.

(ii) implies (iii). Let (ii) hold and suppose that (iii) does not hold. Then for some $t > 0$

$$\inf_n \frac{1}{\lambda_n} \sum_{k \in I_n} [f_k(t)]^{p_k} = 0$$

and therefore there exists an increasing sequence (n_i) of positive integers such that

$$\frac{1}{\lambda_{n_i}} \sum_{k \in I_{n_i}} [f_k(i)]^{p_k} < \frac{1}{i}, \qquad i = 1, 2, \dots. \tag{5}$$

Define the sequence $x = (x_k)$ such that

$$\Delta_v^m x_k = \begin{cases} i, & k \in I_{n_i}, \ \ i = 1, 2, \dots \\ 0, & \text{otherwise.} \end{cases}$$

Thus by (5), $x \in [V, \lambda, F, p, q]_0 (\Delta_v^m)$ but $x \notin [V, \lambda, p, q]_\infty (\Delta_v^m)$ which contradicts (ii). Hence (iii) must hold.

Let (iii) hold and suppose that $x \in [V, \lambda, F, p, q]_0(\Delta_v^m)$ but $x \notin [V, \lambda, p, q]_0(\Delta_v^m)$. Then

$$\frac{1}{\lambda_n} \sum_{k \in I_n} [f_k(q(\Delta_v^m x_k))]^{p_k} \to 0, \qquad \text{as} \quad n \to \infty. \tag{6}$$

For given $\varepsilon > 0$ there exist n^* such that $q(\Delta_v^m x_k) \ge \varepsilon$ and $k \in I_{n^*}$. Therefore

$$[f_k(\varepsilon)]^{p_k} \le [f_k(q(\Delta_v^m x_k))]^{p_k}$$

and consequently by (6), we have

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} [f_k(\varepsilon)]^{p_k} = 0,$$

which contradicts (iii). Hence (i) must hold. Whence (iii) implies (i). □

Theorem 2.10. *Let $1 \leq p_k \leq \sup p_k < \infty$. Then the inclusion*

$$[V, \lambda, F, p, q]_\infty (\Delta_v^m) \subseteq [V, \lambda, p, q]_0 (\Delta_v^m)$$

holds iff

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} [f_k(t)]^{p_k} = \infty \qquad \text{for all} \quad t > 0. \tag{7}$$

P r o o f. For the necessity let $[V, \lambda, F, p, q]_\infty (\Delta_v^m) \subseteq [V, \lambda, p, q]_0 (\Delta_v^m)$ and suppose that (7) does not hold. Then there exist an increasing sequence (n_i) of positive integers and a number $t_0 > 0$ such that

$$\frac{1}{\lambda_{n_i}} \sum_{k \in I_{n_i}} [f_k(t_0)]^{p_k} \leq K < \infty, \qquad i = 1, 2, \dots. \tag{8}$$

Define $x = (x_k)$ such that

$$\Delta_v^m x_k = \begin{cases} t_0, & k \in I_{n_i}, \ i = 1, 2, \dots \\ 0, & \text{otherwise.} \end{cases}$$

Thus by (8), $x \in [V, \lambda, F, p, q]_\infty (\Delta_v^m)$ but $x \notin [V, \lambda, p, q]_0 (\Delta_v^m)$. Hence (7) must hold.

For the sufficiency suppose that (7) hold and let $x \in [V, \lambda, F, p, q]_\infty (\Delta_v^m)$. Then for each n

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \left[f_k(q(\Delta_v^m x_k))\right]^{p_k} \leq K < \infty \tag{9}$$

for some $K > 0$. Suppose that $x \notin [V, \lambda, p, q]_0 (\Delta_v^m)$. Then for given $\varepsilon_0 > 0$ there exist an integer n^* such that $q(\Delta_v^m x_k) \geq \varepsilon_0$ for $k \in I_{n^*}$. Therefore

$$\left[f_k(\varepsilon_0)\right]^{p_k} \leq \left[f_k(q(\Delta_v^m x_k))\right]^{p_k}$$

and hence for each k we get, by (9), that

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \left[f_k(\varepsilon_0)\right]^{p_k} \leq K < \infty$$

for some $K > 0$. This contradicts (7) and hence

$$[V, \lambda, F, p, q]_\infty (\Delta_v^m) \subseteq [V, \lambda, p, q]_0 (\Delta_v^m).$$

□

Theorem 2.11. *Let $1 \leq p_k \leq \sup p_k < \infty$. Then the inclusion*

$$[V, \lambda, p, q]_\infty (\Delta_v^m) \subseteq [V, \lambda, F, p, q]_0 (\Delta_v^m)$$

holds iff

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} [f_k(t)]^{p_k} = 0 \qquad \text{for all} \quad t > 0. \tag{10}$$

Proof. For the necessity let $[V,\lambda,p,q]_\infty(\Delta_v^m) \subseteq [V,\lambda,F,p,q]_0(\Delta_v^m)$ and suppose that (10) does not hold. Then for some $t_0 > 0$, we have

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} [f_k(t_0)]^{p_k} = L \neq 0. \tag{11}$$

Define $x = (x_k)$ by $x_k = t_0 \sum\limits_{u=0}^{k-m} \binom{m+k-u-1}{k-u}$ for $k = 1, 2, \dots$, so $\Delta_v^m x_k = t_0$. Thus $x \notin [V,\lambda,F,p,q]_0(\Delta_v^m)$ by (11), but $x \in [V,\lambda,p,q]_\infty(\Delta_v^m)$ which contradicts the inclusion. Hence (10) must hold.

For the sufficiency suppose that (10) hold and $x \in [V,\lambda,p,q]_\infty(\Delta_v^m)$. Then for every k

$$q(\Delta_v^m x_k) \le K < \infty$$

for some $K > 0$. Therefore

$$[f_k(q(\Delta_v^m x_k))]^{p_k} \le [f_k(K)]^{p_k}$$

and hence, by (10),

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[f_k(q(\Delta_v^m x_k))\right]^{p_k} \le \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} [f_k(K)]^{p_k} = 0.$$

Thus $x \in [V,\lambda,F,p,q]_0(\Delta_v^m)$, i.e., $[V,\lambda,p,q]_\infty(\Delta_v^m) \subseteq [V,\lambda,F,p,q]_0(\Delta_v^m)$. □

Theorem 2.12. *The sequence spaces $[V,\lambda,F,p,q]_Z(\Delta_v^m)$ are not solid for $m \ge 1$.*

Proof. Let $X = \mathbb{C}$, $f_k(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$, $q(x) = |x|$, $v = (1,1,\dots)$ and $\lambda_n = n$ for all $n \in \mathbb{N}$. Then $(x_k) = (k^m) \in [V,\lambda,F,p,q]_\infty(\Delta_v^m)$ but $(\alpha_k x_k) \notin [V,\lambda,F,p,q]_\infty(\Delta_v^m)$ when $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$. Hence $[V,\lambda,F,p,q]_\infty(\Delta_v^m)$ is not solid. The other cases can be proved on considering similar examples. □

From the above theorem we may give the following corollary.

Corollary 2.13. *The sequence spaces $[V,\lambda,F,p,q]_Z(\Delta_v^m)$ are not perfect for $m \ge 1$.*

Theorem 2.14. *The sequence spaces $[V,\lambda,F,p,q]_1(\Delta_v^m)$ and $[V,\lambda,F,p,q]_\infty(\Delta_v^m)$ are not symmetric for $m \ge 1$.*

Proof. Under the restrictions on X, p, f_k, q, v and λ as given in the proof of Theorem 2.12, consider the sequence $x = (k^m)$, then $x \in [V,\lambda,F,p,q]_\infty(\Delta_v^m)$. Let (y_k) be a rearrangement of (x_k), which is defined as follows:

$$(y_k) = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, \dots\}.$$

Then $(y_k) \notin [V,\lambda,F,p,q]_\infty(\Delta_v^m)$. □

Remark 1. The space $[V,\lambda,F,p,q]_0(\Delta_v^m)$ is not symmetric for $m \ge 2$.

THEOREM 2.15. *The sequence spaces* $[V, \lambda, F, p, q]_Z (\Delta_v^m)$ *are not sequence algebras.*

P r o o f. Under the restrictions on X, p, f_k, q, v and λ as given in the proof of Theorem 2.12, consider the sequence $x = \left(k^{m-2}\right)$ and $y = \left(k^{m-2}\right)$, then $x, y \in [V, \lambda, F, p, q]_Z (\Delta_v^m)$ but $x.y \notin [V, \lambda, F, p, q]_Z (\Delta_v^m)$. □

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