VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-011-0040-3
Math. Slovaca **61** (2011), No. 5, 705–716

ON β-EXPANSIONS OF UNITY FOR RATIONAL AND TRANSCENDENTAL NUMBERS β

Artūras Dubickas

(*Communicated by Stanislav Jakubec*)

ABSTRACT. We investigate the sequence of integers $x_1, x_2, x_3, \dots$ lying in $\{0, 1, \dots, [\beta]\}$ in a so-called Rényi β-expansion of unity $1 = \sum_{j=1}^{\infty} x_j \beta^{-j}$ for rational and transcendental numbers $\beta > 1$. In particular, we obtain an upper bound for two strings of consecutive zeros in the β-expansion of unity for rational β. For transcendental numbers β which are badly approximable by algebraic numbers of every large degree and bounded height, we obtain an upper bound for the Diophantine exponent of the sequence $X = (x_j)_{j=1}^{\infty}$ in terms of β.

1. Introduction

Given a real number $\beta > 1$, the β-transformation T_β is defined on $[0, 1]$ by $T_\beta \colon x \mapsto \beta x \pmod 1$. For every integer $j \geqslant 1$ we define iteratively $T_\beta^j(x) = T_\beta(T_\beta^{j-1}(x))$, where $T_\beta^0(x) = x$. Suppose that β is not a rational integer. Then the *Rényi β-expansion of a real number* $\alpha \in [0, 1]$ is given by

$$\alpha = \sum_{k=1}^{\infty} x_k \beta^{-k},$$

where $x_k = [\beta T_\beta^{k-1}(\alpha)]$. (Throughout, $[\cdot]$ denotes the integral part of a number and $\{\cdot\}$ denotes the fractional part of a number.) Clearly, the 'digits' x_k of α all lie in the set $\{0, 1, \dots, [\beta]\}$. This representation of a number in a non-integer base β was introduced by Rényi in [11] and then studied by Parry [10]. The

2010 Mathematics Subject Classification: Primary 11A63; Secondary 11B05, 11J82.
Keywords: β-expansion, Diophantine exponent, transcendental numbers.

investigation of an important link between β-expansions and symbolic dynamics started with the papers of Bertrand-Mathis [2], [3] and Blanchard [4].

The Rényi expansion of a number in base β is obtained by the 'greedy algorithm', i.e., each digit x_k is a unique non-negative integer satisfying

$$\frac{x_k}{\beta^k} \leqslant \alpha - \frac{x_1}{\beta} - \dots - \frac{x_{k-1}}{\beta^{k-1}} < \frac{x_k+1}{\beta^k}.$$

Equivalently,

$$x_k = [\beta^k \alpha - \beta^{k-1} x_1 - \beta^{k-2} x_2 - \dots - \beta x_{k-1}].$$

Putting $y_k = T_\beta^k(\alpha)$, we have $y_0 = \alpha$ and $y_k = \{\beta y_{k-1}\}$, $x_k + y_k = \beta y_{k-1}$ for each $k \geqslant 1$. Hence

$$y_k = \beta^k \alpha - \beta^{k-1} x_1 - \beta^{k-2} x_2 - \dots - \beta x_{k-1} - x_k. \tag{1}$$

In this note, we address the problem of distribution of digits $x_k \in \{0, 1, \dots, [\beta]\}$ of the β-expansion of unity, i.e., $1 = \sum_{j=1}^{\infty} x_j \beta^{-j}$. The importance of the Rényi β-expansion of unity is explained in [10] (see also [13] and [1]). The complexity of the corresponding sequence $X = (x_j)_{j=1}^{\infty}$ for some particular algebraic values of β have been studied by Frougny, Masáková and Pelantová in [7] and [8]. In particular, we are interested in the gaps between non-zero digits of this expansion, the number of non-zero digits, etc. Naturally, as in the case of an integer base, one cannot expect to obtain some strong results in this direction, since even simplest questions are out of reach.

Suppose that the expansion of unity in base β contains infinitely many zeros and infinitely many non-zeros, namely, $x_k = 0$ for infinitely many $k \in \mathbb{N}$ and $x_j \neq 0$ for infinitely many $j \in \mathbb{N}$. Following [13], we shall call such a β-expansion *gappy*. Then there exist two sequences of positive integers $b_1 < b_2 < b_3 < \dots$ and $e_1 < e_2 < e_3 < \dots$ such that $b_k \leqslant e_k < b_{k+1} - 1$ for each $k \geqslant 1$ and $x_j \neq 0$ if and only if $j \in [b_k, e_k]$ for some $k \geqslant 1$. Clearly, $x_j = 0$ if and only if $j < b_1$ or $j \in [e_k + 1, b_{k+1} - 1]$ for some $k \geqslant 1$. How large are the gaps between the end of the nth block of non-zero digits $e_n = e_n(\beta)$ and the beginning of the $n+1$st block $b_{n+1} = b_{n+1}(\beta)$? An old result of Mahler ([9, Theorem 1]) implies that then

$$\limsup_{n\to\infty} \frac{b_{n+1}(\beta)}{e_n(\beta)} < \infty$$

if $\beta > 1$ is an algebraic number. Recently, Verger-Gaugry [13] showed that if the expansion of unity in algebraic base $\beta > 1$ is gappy then

$$\limsup_{n\to\infty} \frac{b_{n+1}(\beta)}{e_n(\beta)} \leqslant \frac{\log M(\beta)}{\log \beta}, \tag{2}$$

where $M(\beta)$ is the Mahler measure of β. Recall that $M(\beta) = b\prod_{j=1}^{d}\max(1,|\beta_j|)$, where $Q(z) = b(z-\beta_1)\dots(z-\beta_d) \in \mathbb{Z}[z]$, $b \in \mathbb{N}$, is the minimal polynomial of β in $\mathbb{Z}[z]$. The main ingredient in the proof of (2) is a Liouville-type argument [13]. A far reaching generalization of (2) with the same upper bound $\log M(\beta)/\log\beta$ for a so-called Diophantine exponent of the sequence $(x_k)_{k=1}^{\infty}$ was obtained by Adamczewski and Bugeaud in [1]. Both in [1] and in the subsequent paper of Bugeaud [5] the main ingredient is the Subspace Theorem.

In the next section, using the theorem of Ridout [12] we shall prove the bound

$$\limsup_{n\to\infty}\frac{e_{n+2}(\beta)}{b_n(\beta)} \leqslant \frac{\log M(\beta)}{\log\beta} = \frac{\log p}{\log(p/q)} \tag{3}$$

for rational numbers $\beta = p/q$ satisfying $1 < \beta < 2$ (Thm. 4). Evidently, (3) is at least as strong as (2), because $b_n(\beta) \leqslant e_n(\beta)$ and $e_{n+2}(\beta) \geqslant e_{n+1}(\beta) \geqslant b_{n+1}(\beta)$. Since, by Lemma 1 below, $\lim_{n\to\infty} e_n(p/q)/b_n(p/q) = 1$ for $1 < p/q < 2$, the inequality of Verger-Gaugry (2) applied to the quotient

$$\frac{e_{n+2}(p/q)}{b_n(p/q)} = \frac{e_{n+2}(p/q)}{b_{n+2}(p/q)}\frac{b_{n+2}(p/q)}{e_{n+1}(p/q)}\frac{e_{n+1}(p/q)}{b_{n+1}(p/q)}\frac{b_{n+1}(p/q)}{e_n(p/q)}$$

at $\beta = p/q$ implies that

$$\limsup_{n\to\infty}\frac{e_{n+2}(p/q)}{b_n(p/q)} \leqslant \left(\frac{\log p}{\log(p/q)}\right)^2,$$

which is weaker than (3).

In Section 3, we shall give an estimate for the Diophantine exponent of the sequence $(x_k)_{k=1}^{\infty}$ for most transcendental numbers β (Thm. 5). Furthermore, we shall estimate the number of nonzero digits x_k (among $x_1,\dots,x_N$) of the Rényi expansion of unity in a transcendental base β for most β as

$$\#\{k:\ x_k \neq 0,\ \ 1 \leqslant k \leqslant n\} > c_0 \log\log N$$

(Thm. 6).

2. Expansion of unity in a rational base β

Lemma 1. *Let β be a real number satisfying $1 < \beta < 2$. Then there is a positive constant $c = c(\beta)$ such that $e_n(\beta) - b_n(\beta) \leqslant c$ for each $n \in \mathbb{N}$.*

Proof. Fix $n \in \mathbb{N}$. Suppose that $b_n = k+1$ and $e_n = k+\ell$ for some $k \geqslant 0$ and $\ell \geqslant 1$. Then $x_{k+1} = \dots = x_{k+\ell} = 1$. Using (1) with $\alpha = 1$, we obtain

$$\begin{aligned} y_k &= \beta^k - \beta^{k-1}x_1 - \dots - \beta x_{k-1} - x_k \\ &= x_{k+1}\beta^{-1} + \dots + x_{k+\ell}\beta^{-\ell} + x_{k+\ell+1}\beta^{-\ell-1} + \dots \end{aligned}$$

$$\geqslant \beta^{-1} + \cdots + \beta^{-\ell} = \frac{1-\beta^{-\ell}}{\beta - 1}.$$

Since $y_k \leqslant 1$, this yields $\beta^{-\ell} \geqslant 2 - \beta$. Thus $\ell \leqslant \log{(2-\beta)^{-1}}/\log\beta$, which implies the bound

$$e_n(\beta) - b_n(\beta) = \ell - 1 \leqslant c(\beta) = [\log{(2-\beta)^{-1}}/\log\beta] - 1$$

for the difference $e_n - b_n$ for each $n \geqslant 1$. □

Lemma 2. *Let p/q be a rational number (with relatively prime integers p, q) satisfying $1 < p/q < 2$. Then the Rényi p/q-expansion of unity is gappy.*

Proof. Observe that the Rényi p/q-expansion of unity is infinite. Indeed, if there is a positive integer L for which $1 = \sum_{k=1}^{L} x_k (q/p)^k$, where $x_k \in \{0, 1\}$ and $x_L = 1$, then multiplying by p^L yields $q^L = p^L - \sum_{k=1}^{L-1} x_k q^k p^{L-k}$. This equality is impossible, because its right hand side is divisible by p whereas its left hand side is not. The assertion of the lemma now follows from Lemma 1. □

Lemma 3. *Let $p > q > 1$ be two fixed relatively prime integers and let b_0 be a fixed positive integer. Then for each $\varepsilon > 0$ there is a positive integer $M = M(\varepsilon, b_0, p, q)$ such that*

$$|p^n a - q^m b| > p^n \mathrm{e}^{-\varepsilon m}$$

for all positive integers n, m, a, b satisfying $(a, q) = 1$, $(b, p) = 1$, $1 \leqslant b \leqslant b_0$ and $m \geqslant M$.

Proof. By the theorem of Ridout [12], for every positive ε there are only finitely many positive rational numbers P/Q, with P and Q of the form $P = p^n a$, $Q = q^m b$, such that $0 < |1 - P/Q| p^{-n} q^{-m} < Q^{-2-\varepsilon}$, or, equivalently,

$$0 < |p^n a - q^m b| = |P - Q| < p^n q^m (q^m b)^{-1-\varepsilon} = p^n b^{-1-\varepsilon} q^{-\varepsilon m}.$$

Since $b \leqslant b_0$, this implies that the inequality $|p^n a - q^m b| > p^n \mathrm{e}^{-\varepsilon m}$ holds for $m \geqslant M$ provided that ε is so small and M is so large that

$$\varepsilon > (1+\varepsilon) M^{-1} \log b_0 + \varepsilon \log q.$$

□

Theorem 4. *Let $p > q > 1$ be two relatively prime integers satisfying $1 < p/q < 2$, and let*

$$1 = \sum_{k=1}^{\infty} x_k (q/p)^k = \sum_{n=1}^{\infty} \sum_{j=b_n}^{e_n} (q/p)^j$$

be the Rényi expansion of unity in base p/q. *Then the sequence* $(x_k)_{k=1}^{\infty}$ *of* $0, 1$ *is gappy and for the sequence* $(b_n)_{n=1}^{\infty}$ *we have*

$$\limsup_{n\to\infty} \frac{b_{n+2}}{b_n} \leqslant \frac{\log p}{\log(p/q)}.$$

P r o o f. The first assertion of the theorem follows from Lemma 2. To prove the second claim we select, in (1), $\alpha = 1$, $\beta = p/q$ and $k = e_{n+1}$. Then $x_{b_{n+1}} = \dots = x_{e_{n+1}} = 1$ and $x_{e_n+1} = \dots = x_{b_{n+1}-1} = 0$, so

$$y_{e_{n+1}} = (p/q)^{e_{n+1}} - \sum_{j=1}^{e_n} x_j (p/q)^{e_{n+1}-j} - \sum_{j=0}^{e_{n+1}-b_{n+1}} (p/q)^j.$$

Multiplying by $q^{e_{n+1}}$, we obtain

$$y_{e_{n+1}} q^{e_{n+1}} = p^{e_{n+1}-e_n} \left(p^{e_n} - \sum_{j=1}^{e_n} x_j p^{e_n-j} q^j \right) - q^{b_{n+1}} \sum_{j=0}^{e_{n+1}-b_{n+1}} p^j q^{e_{n+1}-b_{n+1}-j}. \tag{4}$$

By Lemma 1, the difference $e_{n+1} - b_{n+1}$ is bounded from above by a constant c independent of n. Hence, by Lemma 3 applied to (4), for each fixed positive ε and each $n > n(\varepsilon)$ we have

$$y_{e_{n+1}} q^{e_{n+1}} > p^{e_{n+1}-e_n} \mathrm{e}^{-\varepsilon b_{n+1}}.$$

From (1) and $x_{e_{n+1}+1} = \dots = x_{b_{n+2}-1} = 0$ it follows that

$$y_{b_{n+2}-1} = (p/q)^{b_{n+2}-e_{n+1}-1} y_{e_{n+1}},$$

thus

$$y_{b_{n+2}-1} > (p/q)^{b_{n+2}-e_{n+1}-1} q^{-e_{n+1}} p^{e_{n+1}-e_n} \mathrm{e}^{-\varepsilon b_{n+1}} = p^{b_{n+2}-e_n-1} q^{1-b_{n+2}} \mathrm{e}^{-\varepsilon b_{n+1}}.$$

Since $y_{b_{n+2}-1} < 1$ and $e_n - b_n \leqslant c$, this yields

$$p^{b_{n+2}-b_n} q^{-b_{n+2}} < y_{b_{n+2}-1} p^{e_n-b_n+1} q^{-1} \mathrm{e}^{\varepsilon b_{n+1}} < p^{c+1} \mathrm{e}^{\varepsilon b_{n+1}} < \mathrm{e}^{2\varepsilon b_{n+1}}$$

for each sufficiently large n. Hence

$$b_{n+2} \log(p/q) - b_n \log p < 2\varepsilon b_{n+1} < 2\varepsilon b_{n+2},$$

giving

$$\frac{b_{n+2}}{b_n} < \frac{\log p}{\log(p/q) - 2\varepsilon}.$$

This this proves the inequality $\limsup_{n\to\infty} b_{n+2}/b_n \leqslant \log p / \log(p/q)$, because we can take ε arbitrarily small. $\square$

Note that, by Lemma 1, $e_{n+2} - b_{n+2} \leqslant c$, so

$$\limsup_{n\to\infty} b_{n+2}/b_n = \limsup_{n\to\infty} e_{n+2}/b_n.$$

Hence Theorem 4 implies (3).

3. Expansion of unity in a transcendental base β

Following [1], we say that an infinite sequence $X = (x_k)_{k=1}^{\infty}$ over a finite alphabet $\mathscr{A}$ satisfies Condition $(*)_{\varrho}$ if there exist two sequences of finite words $(U_n)_{n=1}^{\infty}$ and $(V_n)_{n=1}^{\infty}$ over $\mathscr{A}$ and a sequence of positive real numbers $(\tau_n)_{n=1}^{\infty}$ such that

(i) for any $n \geqslant 1$ the word $U_n V_n^{\tau_n}$ is a prefix of X,

(ii) $|U_n V_n^{\tau_n}| \geqslant \varrho |U_n V_n|$ for every $n \geqslant 1$,

(iii) $|V_n^{\tau_n}| \to \infty$ as $n \to \infty$.

Then the *Diophantine exponent* of X, $\mathrm{Dio}(X)$, is defined as the supremum of the real numbers ϱ for which $X = (x_k)_{k=1}^{\infty}$ satisfies Condition $(*)_{\varrho}$.

In [1] it was shown that if $\alpha \in [0,1]$ and $\beta > 1$ are algebraic numbers and the Rényi β-expansion of α is given by

$$\alpha = \sum_{k=1}^{\infty} x_k \beta^{-k}$$

then either $X = (x_k)_{k=1}^{\infty}$ is eventually periodic (and so $\mathrm{Dio}(X) = \infty$) or

$$\mathrm{Dio}(X) \leqslant \frac{\log M(\beta)}{\log \beta}. \tag{5}$$

It was observed in [1] that (5) implies (2), by choosing

$$U_n = x_1 x_2 \dots x_{e_n}, \quad V_n = 0 \quad \text{and} \quad \tau_n = b_{n+1} - e_n - 1.$$

Given a complex number β and two positive integers H, n, let

$$d_n(\beta, H) = \max |P(\beta)|^{-1/n}, \tag{6}$$

where the maximum is taken over every monic polynomial $P(z) \in \mathbb{Z}[z]$ of degree n and height $\leqslant H$ satisfying $P(\beta) \neq 0$. Set

$$D(\beta, H) = \limsup_{n\to\infty} d_n(\beta, H). \tag{7}$$

Assuming that $D(\beta, H) < \infty$ we prove the following:

Theorem 5. *Let $\beta > 1$ be a real transcendental number or an algebraic number which is not an algebraic integer. If the Rényi β-expansion of unity is given by $1 = \sum_{k=1}^{\infty} x_k \beta^{-k}$ then for the sequence $X = (x_k)_{k=1}^{\infty}$ we have*

$$\mathrm{Dio}(X) \leqslant \frac{\log(\beta D(\beta, [\beta]+1))}{\log \beta}.$$

P r o o f. Suppose $U_m V_m^{\tau_m}$ with some integer m is a prefix of X. Put $k = |U_m|$, $t = |V_m|$, $s = |V_m^{\tau_m}|$. Write $V_m = x_{k+1}x_{k+2}\dots x_{k+t}$ and define, by periodicity, $a_{k+j\ell} = x_{k+\ell}$ for every integer $j \geqslant 1$ and every $\ell \in \{1, \dots, t\}$. We shall estimate the difference between

$$y_k = \beta^k - \beta^{k-1}x_1 - \dots - x_k = x_{k+1}\beta^{-1} + x_{k+2}\beta^{-2} + \dots \tag{8}$$

and

$$\begin{aligned} Y_k &= \sum_{j=1}^{\infty} a_{k+j}\beta^{-j} = \sum_{j=1}^{t} a_{k+j}\beta^{-j} \sum_{l=0}^{\infty} \beta^{-lt} \\ &= \frac{x_{k+1}\beta^{t-1} + x_{k+2}\beta^{t-2} + \dots + x_{k+t}}{\beta^t - 1}. \end{aligned} \tag{9}$$

Since $x_{k+j}, a_{k+j} \in \{0, 1, \dots, [\beta]\}$, we have $|x_{k+j} - a_{k+j}| \leqslant [\beta] \leqslant \beta$ for every $j \geqslant 1$. Moreover, the first s terms of the series $y_k = \sum_{j=1}^{\infty} x_{k+j}\beta^{-j}$ and $Y_k = \sum_{j=1}^{\infty} a_{k+j}\beta^{-j}$ are equal. Hence

$$\begin{aligned} |y_k - Y_k| &= \Big| \sum_{j=1}^{\infty} (x_{k+j} - a_{k+j})\beta^{-j} \Big| \\ &= \Big| \sum_{j=s+1}^{\infty} (x_{k+j} - a_{k+j})\beta^{-j} \Big| \leqslant \frac{1}{\beta^{s-1}(\beta - 1)}. \end{aligned} \tag{10}$$

On the other hand, by (8) and (9),

$$(\beta^t - 1)(y_k - Y_k) = (\beta^t - 1)(\beta^k - \beta^{k-1}x_1 - \dots - x_k) - (x_{k+1}\beta^{t-1} + \dots + x_{k+t})$$

is a monic polynomial in β of degree $t + k$ with integer coefficients of moduli at most $[\beta] + 1$. Furthermore, $(\beta^t - 1)(y_k - Y_k)$ is non-zero, because $\beta > 1$ is not an algebraic integer. Using (6) and (10), we obtain

$$d_{t+k}(\beta, [\beta]+1)^{-t-k} \leqslant (\beta^t - 1)|y_k - Y_k| \leqslant \frac{\beta^t - 1}{\beta^{s-1}(\beta - 1)} < \frac{\beta^{t-s}}{1 - 1/\beta}.$$

It follows that $\log(1-1/\beta)+(s-t)\log\beta<(t+k)\log d_{t+k}(\beta,[\beta]+1)$. Adding $(t+k)\log\beta$ to both sides, we deduce that

$$\log(1-1/\beta)+(k+s)\log\beta<(t+k)\log(\beta d_{t+k}(\beta,[\beta]+1)).$$

Dividing by $(t+k)\log\beta$ yields

$$\frac{|U_m V_m^{\tau_m}|}{|U_m V_m|}=\frac{|U_m|+|V_m^{\tau_m}|}{|U_m|+|V_m|}=\frac{k+s}{k+t}<\frac{\log(\beta d_{t+k}(\beta,[\beta]+1))}{\log\beta}-\frac{\log(1-1/\beta)}{(k+t)\log\beta}.$$

Since $s=|V_m^{\tau_m}|\to\infty$ as $m\to\infty$, we must have $k+t=|U_m V_m|\to\infty$ as $m\to\infty$, because otherwise the left hand side $(k+s)/(k+t)$ is unbounded whereas the right hand side is bounded from above by a positive constant. Hence

$$\frac{\log(1-1/\beta)}{(k+t)\log\beta}=\frac{\log(1-1/\beta)}{|U_m V_m|\log\beta}\to 0$$

as $m\to\infty$. We conclude that

$$\begin{aligned}\mathrm{Dio}(X)&=\limsup_{m\to\infty}\frac{|U_m V_m^{\tau_m}|}{|U_m V_m|}\\ &\leqslant\limsup_{k+t\to\infty}\frac{\log(\beta d_{t+k}(\beta,[\beta]+1))}{\log\beta}\\ &=\frac{\log(\beta D(\beta,[\beta]+1))}{\log\beta},\end{aligned}$$

by (7), as claimed. □

Note that if $\beta>1$ is an algebraic number with the leading coefficient of its minimal polynomial b and the conjugates $\beta_1=\beta,\beta_2,\dots,\beta_d$, then for $P(z)=z^n+c_{n-1}z^{n-1}+\dots+c_0\in\mathbb{Z}[z]$ with $H(P)\leqslant[\beta]+1$ we have either $P(\beta)=0$ or $b^n\prod_{j=1}^{d}|P(\beta_j)|\geqslant 1$. By estimating $|P(\beta_j)|\leqslant([\beta]+1)(n+1)\max(1,|\beta_j|)^n$ for $j=2,\dots,d$ and using $b\prod_{j=2}^{d}\max(1,|\beta_j|)=M(\beta)/|\beta|$, we obtain

$$|P(\beta)|\geqslant\big((n+1)([\beta]+1)\big)^{1-d}(M(\beta)/|\beta|)^{-n}.$$

So, by (6) and (7),

$$D(\beta,[\beta]+1)\leqslant M(\beta)/|\beta|$$

for every algebraic number $\beta\neq 0$. By [1, Lemma 2], $y_k\neq Y_k$ for every $k\geqslant 1$. (Indeed, in terms of the notation of [1, Lemma 2] we have $\alpha-\alpha_n=(y_k-Y_k)\beta^{-k}$ with $\alpha=1$, so $\alpha\neq\alpha_n$ is equivalent to $y_k\neq Y_k$.) Hence Theorem 5 (combined with [1, Lemma 2]) implies the upper bound (5) for $\alpha=1$ without any use of the Subspace Theorem.

Suppose that for a given transcendental number β there exists a function $f\colon \mathbb{N} \mapsto \mathbb{R}_{>0}$ and a real number $s \geqslant 1$ such that

$$|P(\beta)| > \mathrm{e}^{-n^s f(H)} \tag{11}$$

for each polynomial $P(z) \in \mathbb{Z}[z]$ of degree n and height $\leqslant H$. The function $\mathrm{e}^{-n^s f(H)}$ (and, more generally, any positive function in two variables $\sigma(n, H)$ on the right hand side of (11)) is called the *transcendence measure of* β. The situation with $s = 1$ is covered by Theorem 5. Unfortunately, for most explicit transcendental numbers β, like

$$\pi,\ \ \log\alpha,\ \ \mathrm{e}^{\alpha},\ \ \log\alpha/\log\gamma,\ \ \alpha^{\gamma},\ \ \mathrm{e}^{\pi},$$

where α and γ are algebraic numbers, the transcendence measure is known to be of the form $\mathrm{e}^{-n^s f(H)}$ with $s > 1$ only. See, for instance, the paper of Waldschmidt [14] which is based on classical methods of Mahler and Gelfond (as is all subsequent work on the transcendence measure of these constants). For those numbers we have the following:

Theorem 6. *Let $\beta > 1$ be a real transcendental number with transcendence measure $\mathrm{e}^{-n^s f(H)}$, where $1 < s < \infty$. If the Rényi β-expansion of unity is given by $1 = \sum_{k=1}^{\infty} x_k \beta^{-k}$ then there is a constant $c_0 = c_0(\beta) > 0$ such that at least $c_0 \log\log N$ numbers of the set $\{x_1, x_2, \dots, x_N\}$ are non-zero for each sufficiently large N.*

Proof. Let $u(N)$ be the number of non-zero integers among $x_1, \dots, x_N$. Clearly, the expansion $1 = \sum_{k=1}^{\infty} x_k \beta^{-k}$ is infinite. If it is not gappy then $u(N)/N \to 1$ as $N \to \infty$, so $u(N) > (1 - \varepsilon)N$ for each $\varepsilon > 0$ and $N > N(\varepsilon)$.

Suppose it is gappy. Let us write this expansion in the form

$$1 = \sum_{k=1}^{\infty} x_k \beta^{-k} = \sum_{n=1}^{\infty} \sum_{j=b_n}^{e_n} x_j \beta^{-j}.$$

Take m such that $e_m \leqslant N < e_{m+1}$. Put $E_0 = b_1 - 1$ and $E_j = b_{j+1} - e_j - 1$, $F_j = e_j - b_j + 1$ for $j \in \mathbb{N}$. Then

$$u(N) = \sum_{j=1}^{m} F_j + \max(N, b_{m+1} - 1) - b_{m+1} + 1. \tag{12}$$

Setting in (1) $\alpha = 1$ and $k = e_n$ we obtain

$$y_{e_n} = \beta^{e_n} - \sum_{k=1}^{e_n} \beta^{e_n - k} x_k.$$

This expression is an integer polynomial in β of degree e_n and height $\leqslant [\beta]$, so $\log y_{e_n} > -e_n^s g(\beta)$ with some positive number $g(\beta)$. On the other hand, $y_{b_{n+1}-1} = \beta^{b_{n+1}-e_n-1} y_{e_n} < 1$, so $\log y_{e_n} < -(b_{n+1} - e_n - 1)\log\beta$. Thus $b_{n+1} - e_n - 1 < e_n^s g_1(\beta)$ with some positive number $g_1(\beta)$. Below, we shall write $u \ll v$ if $u, v > 0$ and the quotient u/v does not exceed a constant depending only on β (and s). With this notation, we have

$$E_n = b_{n+1} - e_n - 1 \ll e_n^s \qquad \text{for every} \quad n \geqslant 1.$$

Hence

$$E_n \ll e_n^s = (F_n + b_n - 1)^s \ll F_n^s + (b_n - 1)^s. \tag{13}$$

Note that $b_{j+1} = E_j + F_j + b_j$, so, by (13), we have

$$\begin{aligned} L &= N - \max(N, b_{m+1} - 1) + b_{m+1} - 1 \\ &\leqslant b_{m+1} - 1 = E_m + F_m + b_m - 1 \\ &\ll F_m^s + (b_m - 1)^s + F_m + b_m - 1 \\ &\ll F_m^s + (b_m - 1)^s. \end{aligned}$$

On applying the bound $b_m - 1 \ll F_{m-1}^s + (b_{m-1} - 1)^s$, we deduce that $L \ll F_m^s + F_{m-1}^{s^2} + (b_{m-1} - 1)^{s^2}$ and so on, until we get

$$L \ll c_1^{s^{m-1}} \big(F_m^s + F_{m-1}^{s^2} + \cdots + F_2^{s^{m-1}} + (b_2 - 1)^{s^{m-1}}\big)$$

with an absolute constant $c_1 > 0$. Thus, using (12), we deduce that for some absolute positive constants $c_1, c_2 > 1$

$$\begin{aligned} N &\leqslant \max(N, b_{m+1} - 1) - b_{m+1} + 1 + c_1^{s^m} \big(F_m^s + F_{m-1}^{s^2} + \cdots + F_1^{s^m} + c_2^{s^m}\big) \\ &< \big(c_1 u(N) + c_2\big)^{s^m}. \end{aligned} \tag{14}$$

The required bound $u(N) > c_0 \log\log N$ is immediate if $u(N) > \log N$. Suppose that $u(N) \leqslant \log N$. Taking logarithms in (14) (twice) we obtain

$$\log\log N < m \log s + \log\log(c_1 \log N + c_2).$$

Since, by (12), $u(N) \geqslant m$, this yields

$$u(N) \geqslant m > \frac{\log\log N - \log\log(c_1 \log N + c_2)}{\log s} > c_0 \log\log N$$

with some constant c_0 depending on β and s (which depends on β). $\square$

We remark that for some transcendental numbers β the function $u(N)$ defined as the number of non-zeros among the first N digits $x_1, \ldots, x_N$ of the β-expansion $1 = \sum_{k=1} x_k \beta^{-k}$ can be arbitrarily small. Indeed, let $\delta(n)$ be a non-decreasing sequence of real numbers satisfying $\delta(n) \to \infty$ as $n \to \infty$. Let us construct a transcendental number β for which $u(N) < \delta(N)$ for each $N \geqslant N_0$.

Take a sequence of positive integers $b_1 = 1 < b_2 < b_3 < \dots$ which is increasing so fast that $b_{n+1} \geqslant 2b_n$ and $\delta(b_n) > n$ for each $n \geqslant 2$. Define $\beta > 1$ by the equality

$$1 = \sum_{j=1}^{\infty} \beta^{-b_j}. \tag{15}$$

Clearly, $1 < \beta < 2$. By [6, Corollary 5], the number β is transcendental. Also, it is easy to see that the β-expansion of unity is given by (15). The number of non-zero elements among $x_1, \dots, x_N$ is equal to n, where n is chosen so that $b_n \leqslant N < b_{n+1}$. So

$$u(N) = u(b_n) = n < \delta(b_n) \leqslant \delta(N)$$

for $N \geqslant b_2$, as claimed.

REFERENCES

[1] ADAMCZEWSKI, B.—BUGEAUD, Y: *Dynamics for β-shifts and Diophantine approximation*, Ergodic Theory Dynam. Systems **27** (2007), 1695–1711.

[2] BERTRAND, A.: *Développements en base de Pisot et répartition modulo* 1, C. R. Math. Acad. Sci. Paris Sér. A **285** (1977), 419–421.

[3] BERTRAND-MATHIS, A.: *Développement en base θ, répartition modulo un de la suite $(x\theta^n)$, $n \geqslant 0$, langages codés et θ-shift*, Bull. Soc. Math. France **114** (1986), 271–323.

[4] BLANCHARD, A.: *β-expansions and symbolic dynamics*, Theoret. Comput. Sci. **65** (1989), 131–141.

[5] BUGEAUD, Y.: *On the β-expansion of an algebraic number in an algebraic base β*, Integers **9** (2009), 215–226.

[6] CORVAJA, P.—ZANNIER, U.: *Some new applications of the subspace theorem*, Compos. Math. **131** (2002), 319–340.

[7] FROUGNY, C.—MASÁKOVÁ, Z.—PELANTOVÁ, E.: *Complexity of infinite words associated with beta-expansions*, Theor. Inform. Appl. **38** (2004), 163–185; Corrigendum 269–271.

[8] FROUGNY, C.—MASÁKOVÁ, Z.—PELANTOVÁ, E.: *Infinite special branches in words associated with beta-expansions*, Discrete Math. Theor. Comput. Sci. **9** (2007), 125–144.

[9] MAHLER, K.: *Arithmetic properties of lacunary power series with integral coefficients*, J. Aust. Math. Soc. **5** (1965), 56–64.

[10] PARRY, W.: *On β-expansions of real numbers*, Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416.

[11] RÉNYI, A.: *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. **8** (1957), 477–493.

[12] RIDOUT, D.: *Rational approximations to algebraic numbers*, Mathematika, **4** (1957), 125–131.

[13] VERGER-GAUGRY, J.-L.: *On gaps in Rényi β-expansions of unity for $\beta > 1$ an algebraic number*, Ann. Inst. Fourier (Grenoble) **56** (2006), 2565–2579.

[14] WALDSCHMIDT, M.: *Transcendence measures for exponentials and logarithms*, J. Aust. Math. Soc. Ser. A **25** (1978), 445–465.

Received 10. 3. 2009
Accepted 19. 4. 2011

Department of Mathematics and Informatics
Vilnius University
Naugarduko 24
LT–03225 Vilnius
LITHUANIA
E-mail: arturas.dubickas@mif.vu.lt