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THE CROSSING NUMBER OF $G \square C_n$ FOR THE GRAPH G ON SIX VERTICES

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ABSTRACT. The crossing numbers of Cartesian products of paths, cycles or stars with all graphs of order at most four are known. The crossing numbers of $G \square C_n$ for some graphs G on five and six vertices and the cycle C_n are also given. In this paper, we extend these results by determining the crossing number of the Cartesian product $G \square C_n$, where G is a specific graph on six vertices.

1. Introduction

Let G be a simple graph with vertex set V and edge set E. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of crossings of edges in a drawing of G in the plane such that no three edges cross in a point. It is easy to verify that a drawing with minimum number of crossings (an *optimal* drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. Computing the crossing number of a given graph is in general a very difficult problem, and crossing numbers of few families of graphs are known. Most of these graphs are Cartesian products of special graphs. (For a definition of Cartesian product, see [4].)

Let C_n be the cycle of length n, P_n be the path of length n, and S_n be the star isomorphic to $K_{1,n}$. Beineke and Ringeisen in [4], Jendrol' and Ščerbová in [9] determined the crossing numbers of the Cartesian products of all graphs on four vertices with cycles. Klešč in [10], [12], [13], [14], Klešč, Richter and Stobert in [15], and Klešč and Kocúrová in [16] gave the crossing numbers of $G \square C_n$

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for several graphs of order five. Harary et al. [8] conjectured that the crossing number of $C_m \square C_n$ is $(m-2)n$, for all m, n satisfying $3 \leq m \leq n$. This has been proved only for m, n satisfying $n \geq m$, $m \leq 7$ ([1], [2], [3], [4], [5], [15], [17], [18]). It was recently proved by Glebsky and Salazar [7] that the crossing number of $C_m \square C_n$ equals its long-conjectured value at least for $n \geq m(m+1)$.

In [12] and [14], all known values of crossing numbers for the Cartesian products of cycles and graphs of order five are presented. We are interested in the crossing numbers of Cartesian products of graphs on six vertices with cycles. Except for the star S_5, in [6] there are given the crossing numbers of $G \square C_n$ for all trees G on six vertices. For the star on six vertices an upper bound is presented. It seems natural to enquire about crossing numbers of Cartesian products for other 6-vertex graphs with cycles. In this paper, we give the crossing number of the Cartesian product $G \square C_n$ for the specific 6-vertex graph shown in Figure 1. This result can help to establish crossing numbers for some other suitable Cartesian products of cycles with graphs of order at least six.

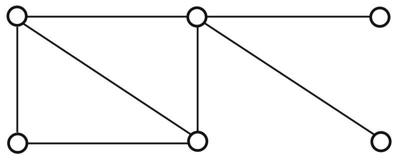

Figure 1. The graph G

Let D be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge disjoint subgraphs of G. We denote by $\mathrm{cr}_D(G_i, G_j)$ the number of crossings between edges of G_i and edges of G_j, and by $\mathrm{cr}_D(G_i)$ the number of crossings between edges of G_i in D.

2. The crossing numbers of $G \square C_n$

Assume $n \geq 3$, and consider the graph $G \square C_n$ in the following way: it has $6n$ vertices and edges that are the edges in the n copies G^i, $i = 0, 1, \ldots, n-1$, and in the six cycles of length n. For $i = 0, 1, \ldots, n-1$, let a_i and b_i be the vertices of G^i of degree one, c_i the vertex of degree four and let d_i and e_i be the vertices of G^i of degree three, f_i the vertex of degree two (see Figure 2). Thus, for $x \in \{a, b, c, d, e, f\}$, the n-cycle C_n^x is induced by the vertices $x_0, x_1, \ldots, x_{n-1}$. For $i = 0, 1, \ldots, n-1$, let P^i denote the subgraph of $G \square C_n$ containing the vertices of G^i and G^{i+1} and six edges joining G^i to G^{i+1}, i taken modulo n. Let T^x, $x \in \{a, b, d, e\}$, be the subgraph of the graph $G \square C_n$ consisting of the cycle C_n^x together with the vertices of C_n^c and of the edges joining C_n^x with C_n^c. For

$i = 0, 1, \ldots, n-1$, let K^i be the subgraph of G^i induced by the vertices c_i, d_i, e_i, and f_i. We denote by P_K^i the subgraph of P^i consisting only of the edges joining K^i to K^{i+1}. Let us denote by I the subgraph of $G \square C_n$ consisting of the vertices in C_n^d and C_n^e and of the edges $\{d_i, e_i\}$ for all $i = 0, 1, \ldots, n-1$, and let X^f be the subgraph of $G \square C_n$ induced by the edges incident with the vertices of C_n^f. It is easy to see that T^a, T^b, T^d, T^e, C_n^c, I, and X^f are edge-disjoint subgraphs and that

$$G \square C_n = T^a \cup T^b \cup C_n^c \cup T^d \cup T^e \cup I \cup X^f.$$

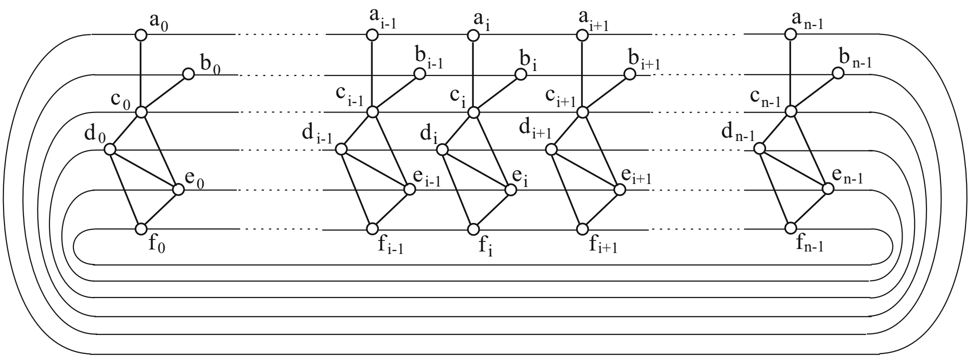

FIGURE 2. The drawing of the graph $G \square C_n$

We say that a good drawing of $G \square C_n$ is *coherent* if for each K^i the vertices of the subgraph $(G \square C_n) - V(G^i)$ lie in the same region in the view of the subdrawing of K^i. In the proofs of the paper, we will often use the term "region" also in nonplanar drawings. In this case, crossings are considered to be vertices of the "map".

Consider the subgraph $T^a \cup T^b \cup C_n^c \cup T^d \cup T^e$ of the graph $G \square C_n$. Clearly, the subgraph $T^a \cup T^b \cup C_n^c \cup T^d \cup T^e$ is isomorphic to the graph $S_4 \square C_n$. The next two lemmas will be very helpful in some proofs of the paper.

Lemma 2.1. *Let D be a good drawing of the graph $T^a \cup T^b \cup C_n^c \cup T^d \cup T^e$, $n \in \{4, 5, 6\}$, in which $\mathrm{cr}_D(T^x \cup T^y) = 1$ for some $x, y \in \{a, b, d, e\}$. Then $\mathrm{cr}_D(T^x \cup T^y, C_n^c) \geq 2$ and $\mathrm{cr}_D(T^x \cup T^y, T^z) \geq 2$ for every $z \in \{a, b, d, e\}$, $z \neq x$, $z \neq y$.*

P r o o f. For the graph $C_n \square P_1$, $n \geq 4$, there is no good drawing with exactly one crossing, because for any two edges which cross each other one can find two vertex-disjoint cycles such that crossed edges are in different cycles. Two vertex-disjoint cycles cannot cross only once. The subgraph $T^x \cup T^y$ is obtained from $C_n \square P_1$ by an elementary subdivision of every edge joining two n-cycles

C_n^x and C_n^y. Consider now the subdrawing D' induced from D by the subgraph $T^x \cup T^y$. The only one crossing in D' appears between an edge incident with a vertex of degree two and an edge of the cycle C_n^x or the cycle C_n^y. In this case, the cycle C_n^x or the cycle C_n^y separates in D a vertex c_i of the cycle C_n^c from the other vertices of C_n^c. Hence, the cycle C_n^c crosses in D the edges of $T^x \cup T^y$ at least twice. The removing of the separated vertex c_i of the cycle C_n^c from D' results in the drawing without crossings. This drawing divides the plane in such a way that there are at most two vertices of C_n^c on the boundary of every region. As the vertex c_i is in D' separated from the other vertices of C_n^c, in the subdrawing D' of $T^x \cup T^y$ with one crossings there are at most two vertices of C_n^c on the boundary of a region. If the cycle C_n^z of T^z crosses the 2-connected subgraph $T^x \cup T^y$, it crosses $T^x \cup T^y$ at least two times. Otherwise C_n^z is in D placed in one region in the view of the subdrawing of $T^x \cup T^y$ and at least two edges of T^z joining C_n^z with the vertices of C_n^c cross the edges of $T^x \cup T^y$. This completes the proof. $\square$

Lemma 2.2. *Let D be a good drawing of the graph $T^a \cup T^b \cup C_n^c \cup T^d \cup T^e$, $n \in \{5, 6\}$, in which $\mathrm{cr}_D(T^x, T^y) = 0$ for some $x, y \in \{a, b, d, e\}$. Then for every $z \in \{a, b, d, e\}$, $z \neq x$, $z \neq y$, $\mathrm{cr}_D(T^x \cup T^y, T^z) \geq 4$ for $n = 5$, and $\mathrm{cr}_D(T^x \cup T^y, T^z) \geq 6$ if $n = 6$.*

P r o o f. As $T^x \cup T^y$ is the subdivision of the 3-connected planar graph $P_1 \square C_n$, the subdrawing D' of $T^x \cup T^y$ induced from D divides the plane into several regions without vertices of C_n^c on their boundaries and into five or six regions, respectively, having exactly two vertices of C_n^c on the boundary of one region. For $n = 6$, Figure 3 shows the drawing D' in which possible crossings among the edges of T^x are inside the left disc bounded by the dotted cycle and possible crossings among the edges of T^y are inside the right disc bounded by the dotted cycle. A similar drawing we consider for the case $n = 5$. We can suppose that if, in D, an edge not incident with a vertex of C_n^x or C_n^y passes through one of these two discs, then it crosses the edges of $T^x \cup T^y$ at least twice. Consider now a subgraph T^z, $z \in \{a, b, d, e\}$, $z \neq x$, $z \neq y$. Both C_n^z and $T^x \cup T^y$ are 2-connected graphs and hence, $\mathrm{cr}_D(C_n^z, T^x \cup T^y) \neq 1$. If, in D, the cycle C_n^z is placed in a region of D' with fewer than two vertices of C_n^c on its boundary, then $\mathrm{cr}_D(T^x \cup T^y, T^z) \geq n$. If C_n^z is placed in a region with two vertices of C_n^c on the boundary, then one vertex of C_n^c is separated from C_n^z by at least two vertex-disjoint cycles for $n = 5$, and in the case $n = 6$ there are at least two such vertices of C_n^c. Hence, $\mathrm{cr}_D(T^x \cup T^y, T^z) \geq 4$ for $n = 5$, and $\mathrm{cr}_D(T^x \cup T^y, T^z) \geq 6$ if $n = 6$. If the cycle C_n^z crosses in D the edges of $T^x \cup T^y$ two or three times, then it is placed in two regions of D' with at most three vertices of C_n^c on their boundaries and the edges joining C_n^z with C_n^c cross in D the edges of $T^x \cup T^y$ at least four times when $n = 5$, and at least six times for $n = 6$. If there are four vertices of C_n^c on the boundaries of the regions in D' in which C_n^z is placed in D, at least four crossings between the edges of C_n^z and the edges of $T^x \cup T^y$ are necessary. For $n = 6$, the edges joining C_6^z to C_6^c cross the subgraph $T^x \cup T^y$ at least twice, and $\mathrm{cr}_D(T^x \cup T^y, T^z) \geq 6$. If C_6^z crosses in D the edges of D' five times, then there is at least one vertex of C_6^c separated in D' from the cycle C_6^z and $\mathrm{cr}_D(T^x \cup T^y, T^z) \geq 6$ again. This completes the proof. $\square$

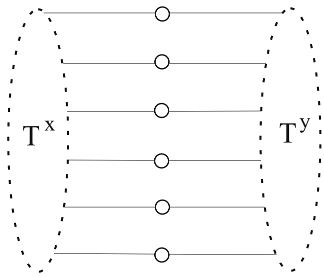

FIGURE 3. The subdrawing of the subgraph $T^x \cup T^y$

THEOREM 2.1. $\mathrm{cr}(G \square C_3) = 7$.

P r o o f. In Figure 4 it is easy to see that the removing the edges of two subgraphs G^i in the left results in the drawing of a subdivision of the graph $G \square C_3$ with seven crossings. Hence, $\mathrm{cr}(G \square C_3) \leq 7$. Assume that there is a good drawing of $G \square C_3$ with at most 6 crossings and let D be such a drawing. The subgraph $T^d \cup T^e \cup I \cup X^f$ of the graph $G \square C_3$ is isomorphic to the graph $(K_4 - e) \square C_3$ and it is proved in [11] that $\mathrm{cr}((K_4 - e) \square C_3) = 6$. Thus, in D there is no crossing on the edges of $T^a \cup T^b$. The planar subdrawing of $T^a \cup T^b$ induced by D is the unique within isomorphism and divides the plane into 5 regions with at most two vertices of C_3^c on the boundary of a region. Then, in D, an edges of T^d cross the edges of $T^a \cup T^b$ at least once. This ontradiction completes the proof. $\square$

THEOREM 2.2. $\mathrm{cr}(G \square C_4) = 10$.

P r o o f. By deleting the edges of the left subgraph G^i in Figure 4, the drawing of a subdivision of the graph $G \square C_4$ with ten crossings is obtained. Hence, $\mathrm{cr}(G \square C_4) \leq 10$. Assume that there is a good drawing of $G \square C_4$ with at most 9 crossings and let D be such a drawing. The graph $G \square C_4$ contains the graph $(K_4 - e) \square C_4$ as a subgraph and $\mathrm{cr}((K_4 - e) \square C_4) = 8$ (see [4]). Thus, in D there is at most one crossing on the edges of $T^a \cup T^b$.

Consider now the subgraph $T^a \cup T^b$ of the graph $G \square C_4$ and let D' be its subdrawing induced by D. Assume first, that $\mathrm{cr}_D(T^a \cup T^b) = 0$. As $T^a \cup T^b$ is

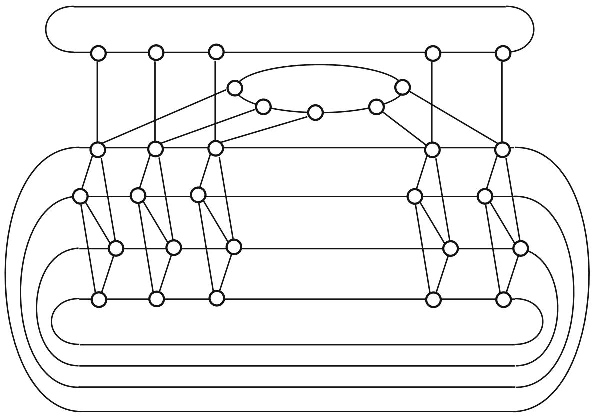

FIGURE 4. The drawing of the graph $G \square C_5$

the subdivision of the 3-connected planar graph $P_1 \square C_4$, the planar subdrawing of $T^a \cup T^b$ induced from D is the unique within isomorphism and it divides the plane into two quadrangular and four hexagonal regions in such a way that there are at most two of the vertices c_0, c_1, c_2, and c_3 on the boundary of a region. Consider now the cycle C_4^d. As in D there is at most one crossing on the edges of $T^a \cup T^b$, the cycle C_4^d is placed in D in one region in the view of the subdrawing D'. So, in D, the edges joining the vertices of C_4^d cross the edges of $T^a \cup T^b$ at least twice, which is a contradiction.

Assume now that there is one crossing among the edges of $T^a \cup T^b$ in D'. By Lemma 2.1, $\mathrm{cr}_D(T^a \cup T^b, C_4^c) \geq 2$ and this contradiction completes the proof. $\square$

THEOREM 2.3. $\mathrm{cr}(G \square C_5) = 14$.

Proof. In the drawing of the graph $G \square C_5$ in Figure 4 one can easily see that $\mathrm{cr}(G \square C_5) \leq 14$. Assume that there is a good drawing of $G \square C_5$ with at most 13 crossings and let D be such a drawing. The graph $G \square C_5$ contains the graph $(K_4 - e) \square C_5$ as a subgraph and $\mathrm{cr}((K_4 - e) \square C_5) = 10$ (see [4]). Thus, in D there are at most three crossings on the edges of $T^a \cup T^b$. Consider now the subgraph $T^a \cup T^b$ of the graph $G \square C_5$ and let D' be its subdrawing induced by D. Clearly, $\mathrm{cr}_D(T^a, T^b) \neq 0$ otherwise, by Lemma 2.2, $\mathrm{cr}_D(T^a \cup T^b, T^d) \geq 4$, a contradiction. If there is only one crossing in the subdrawing D' of $T^a \cup T^b$, by Lemma 2.1, $\mathrm{cr}_D(T^a \cup T^b, C_5^c) \geq 2$ and $\mathrm{cr}_D(T^a \cup T^b, T^d) \neq 0$. In this case there are more than three crossings on the edges of $T^a \cup T^b$ again.

Consider now that there are at least two crossings in D'. Then at least one subgraph T^d or T^e does not cross in D the edges of $T^a \cup T^b$. Without loss of generality, let $\mathrm{cr}_D(T^a \cup T^b, T^d) = 0$. In this case $\mathrm{cr}_D(T^a, T^d) = 0$ and, by Lemma 2.2, $\mathrm{cr}_D(T^a \cup T^d, T^b) \geq 4$. As $\mathrm{cr}_D(T^d, T^b) = 0$, all considered crossings appear between the edges of T^a and the edges of T^b. This contradiction with the assumption that there are at most three crossings on the edges of $T^a \cup T^b$ completes the proof. $\Box$

Theorem 2.4. $\mathrm{cr}(G \Box C_6) = 18$.

Proof. In the drawing in Figure 2 one can see that $\mathrm{cr}(G \Box C_n) \leq 18$. Assume that there is a good drawing of $G \Box C_6$ with fewer than 18 crossings and let D be such a drawing. The graph $G \Box C_6$ contains $(K_4 - e) \Box C_6$ as a subgraph and $\mathrm{cr}((K_4 - e) \Box C_6) = 12$ (see [4]). Thus, in D there are at most five crossings on the edges of the subgraph $T^a \cup T^b$.

Let us consider the subgraph $T^a \cup T^b$. If $\mathrm{cr}_D(T^a \cup T^b) = 0$, then $\mathrm{cr}_D(T^a, T^b) = 0$, too, and Lemma 2.2 implies that $\mathrm{cr}_D(T^a \cup T^b, T^d) \geq 6$. This contradicts the assumption that there are at most five crossings on the edges of $T^a \cup T^b$. Assume now a subdrawing of $T^a \cup T^b$ with one crossing. By Lemma 2.1, $\mathrm{cr}_D(T^a \cup T^b, C_6^c) \geq 2$, $\mathrm{cr}_D(T^a \cup T^b, T^d) \geq 2$ and $\mathrm{cr}_D(T^a \cup T^b, T^e) \geq 2$, and on the edges of $T^a \cup T^b$ there are more than five crossings again. If $\mathrm{cr}_D(T^a \cup T^b) \geq 2$, one of the subgraphs T^d and T^e crosses the edges of $T^a \cup T^b$ at most once. Assume that $\mathrm{cr}_D(T^a \cup T^b, T^d) \leq 1$. This implies that $\mathrm{cr}_D(T^a, T^d) = 0$ or $\mathrm{cr}_D(T^b, T^d) = 0$. Without loss of generality, let $\mathrm{cr}_D(T^a, T^d) = 0$. It follows from Lemma 2.2 that, in this case, $\mathrm{cr}_D(T^a \cup T^d, T^b) \geq 6$ and in D there are more than five crossings on the edges of T^b. This contradiction completes the proof. $\Box$

Lemma 2.3. *If D is a good drawing of $G \Box C_n$, $n \geq 6$, in which every G^i has at most two crossings on its edges, then D has at least $3n$ crossings.*

Proof. Note that in the whole proof the indices are considered modulo n. By hypothesis, the following Claim 2.1 holds.

Claim 2.1. *The drawing D is coherent and two different subgraphs K^i and K^j do not cross each other.*

If some K^i, $i \in \{0, 1, \dots, n-1\}$, separates vertices of the 3-connected subgraph induced by the vertices $V(K^{i+1}) \cup \dots \cup V(K^{i-1})$, then its edges are crossed at least three times. So, all subgraphs K^j, $j \neq i$, lie in D in the same region in the view of the subdrawing of K^i. Moreover, as the drawing D is good, two different K^i and K^j do not cross each other. Assume now that some triangular cycle of K^i, say Δ^i, separates the vertices a_j and c_j. So, the edge $\{a_j, c_j\}$ crosses K^i. If $j \notin \{i-1, i+1\}$, then Δ^i is crossed by both paths $a_j a_{j+1} c_{j+1}$ and $a_j a_{j-1} c_{j-1}$, and there are at least three crossings on the edges of K^i. Assume, without loss of generality, that $a_j = a_{i+1}$. Then the edges of Δ^i are crossed by the edge $\{a_i, a_{i+1}\}$ and by the path $a_{i+1} a_{i+2} c_{i+2}$ if Δ^i separates a_i from a_{i+1}, and K^i has its edges crossed at least three times. The last possibility is that the edge $\{a_i, a_{i+1}\}$ does not cross Δ^i. In this case, the cycle Δ^i is crossed by both paths $a_{i+1}a_{i+2}c_{i+2}$ and $a_ia_{i-1}c_{i-1}$, and K^i is crossed at least three times again. The similar consideration for the case when K^i separates the vertices b_j and c_j confirms that the drawing D is coherent.

For $i = 0, 1, \dots, n-1$, let Q^i denote the subgraph of $G \,\square\, C_n$ induced by $V(G^{i-1}) \cup V(G^i) \cup V(G^{i+1})$ (see Figure 2) and let Q^i_K be its subgraph obtained by deleting the vertices $a_{i-1}, a_i, a_{i+1}, b_{i-1}, b_i$, and b_{i+1}. Thus, $Q^i = G^{i-1} \cup P^{i-1} \cup G^i \cup P^i \cup G^{i+1}$ and $Q^i_K = K^{i-1} \cup P^{i-1}_K \cup K^i \cup P^i_K \cup K^{i+1}$.

Let us consider the following types of crossings on the edges of Q^i in a drawing of the graph $G \,\square\, C_n$:

(1) a crossing of an edge in $P^{i-1} \cup P^i$ with an edge in G^i,

(2) a crossing of an edge in $G^{i-1} \cup P^{i-1}$ with an edge in $G^{i+1} \cup P^i$,

(3) a self-intersection in G^i,

(4) a self-intersection in P^{i-1},

(5) a self-intersection in P^i,

(6) a crossing of an edge in $G^{i-1} \cup G^{i+1}$ with an edge in G^i.

It is readily seen that every crossing of types (1), (2), and (3) appears in a drawing of the graph $G \,\square\, C_n$ only on the edges of the subgraph Q^i. Every crossing of types (4), (5), and (6) appears in Q^i and in one of Q^{i-1} or Q^{i+1}.

In a good drawing of $G \,\square\, C_n$, we define the *force* $f(Q^i)$ of Q^i in the following way: every crossing of type (1), (2) or (3) contributes the value 1 to $f(Q^i)$ and every crossing of type (4), (5) or (6) contributes the value $\frac{1}{2}$ to $f(Q^i)$ (and $\frac{1}{2}$ to $f(Q^{i-1})$ or to $f(Q^{i+1})$). The *total force* of the drawing is the sum of $f(Q^i)$. It is readily seen that the number of crossings in the drawing is not less then the total force of the drawing. So, the aim of our proof is to show that if every G^i has at most two crossings on its edges, then $f(Q^i) \geq 3$ for all $i = 0, 1, \dots, n-1$.

Assume that, in the drawing D, there is some i, $i \in \{0, 1, \dots, n-1\}$, for which the force $f(Q^i) < 3$. Then for the drawing D the following Claim 2.2 and Claim 2.3 hold.

Claim 2.2. *Let $i \in \{0, 1, \dots, n-1\}$. If $f(Q^i) < 3$, then the edges of K^i do not cross each other in D.*

If $f(Q^i) < 3$ and $\mathrm{cr}_D(K^i) \neq 0$, then $\mathrm{cr}_D(K^i, P^{i-1}_K) = 0$ or $\mathrm{cr}_D(K^i, P^i_K) = 0$. Let D' be the subdrawing of D induced by the subgraph $K^i \cup P^i_K \cup K^{i+1}$. If $\mathrm{cr}_D(K^i, P^i_K) = 0$, then the subdrawing D' divides the plane in such a way that K^{i+1} lies in one region, say unbounded, in the view of the subdrawing of K^i and, since the edges of P^i_K do not cross K^i, every region outside K^i has at most two vertices of K^i on its boundary. As $f(Q^i) < 3$ and $\mathrm{cr}_D(K^i) \neq 0$, in D the 2-connected subgraph K^{i-1} does not cross edges of $K^i \cup P^i_K \cup K^{i+1}$. Hence, by Claim 2.1, K^{i-1} lies in D in one region of the subdrawing D' outside K^i with two vertices of K^i on its boundary. But, in this case, the edges of P_K^{i-1} joining K^{i-1} to K^i cross the edges of $K^i \cup P_K^i \cup K^{i+1}$ at least twice. This contradicts the assumption $f(Q^i) < 3$. The consideration $\mathrm{cr}_D(K^i, P_K^{i-1}) = 0$ gives the same contradiction and therefore $\mathrm{cr}_D(K^i) = 0$.

Claim 2.3. *Let $i \in \{0, 1, \dots, n-1\}$. If $f(Q^i) < 3$, then the edges of K^i are crossed by both P_K^{i-1} and P_K^i.*

Assume that $f(Q^i) < 3$ and let $\mathrm{cr}_D(K^i, P_K^i) = \mathrm{cr}_D(K^i, P_K^{i-1}) = 0$. The similar consideration as in the proof of Claim 2.2 shows that the subdrawing D' of the subgraph $K^i \cup P_K^i \cup K^{i+1}$ induced by D divides the plane in such a way that, say outside K^i, there are four regions such that every region has two vertices of K^i on its boundary. By Claim 2.2, $\mathrm{cr}_D(K^i) = 0$ and the unique subdrawing D' is shown in Figure 5(a). The assumption $f(Q^i) < 3$ enforces that $\mathrm{cr}_D(K^{i-1}, P_K^i) < 3$. So, by Claim 2.1, if K^{i-1} crosses $K^i \cup P_K^i \cup K^{i+1}$, it crosses P_K^i exactly twice and the vertices of K^{i-1} are placed in at most two regions of the subdrawing shown in Figure 5(a). But, in this case, at least one edge of P_K^{i-1} crosses in D the edges of $P_K^i \cup K^{i+1}$. This contradiction with $f(Q^i) < 3$ implies that $\mathrm{cr}_D(K^{i-1}, K^i \cup P_K^i \cup K^{i+1}) = 0$ and that the subgraph K^{i-1} lies in D in only one region of D'. Moreover, it lies in one of two regions, say α, having the vertex c_i on its boundary. Otherwise, if the vertex c_i does not lie on the boundary of α, then in D both paths $c_i a_i a_{i-1} c_{i-1}$ and $c_i b_i b_{i-1} c_{i-1}$ and at least two edges of P_K^{i-1} joining K^{i-1} with K^i cross the boundary of the region α. This contradicts the assumption $f(Q^i) < 3$. So, K^{i-1} lies in D in the region α of D' with the vertex c_i on its boundary and at least two edges of P_K^{i-1} joining K^{i-1} to K^i cross the edges of $P_K^i \cup K^{i+1}$. As none of the edges of P_K^{i-1} can cross K^{i+1} only once, the unique possibility is that P_K^{i-1} crosses P_K^i twice. Moreover, in D' the vertices c_i and c_{i+1} are separated by the cycle $d_i e_i e_{i-1} d_{i-1} d_i$ (see Figure 5(b)). In this case, the cycle $d_i e_i e_{i-1} d_{i-1} d_i$ is crossed in D by both paths $c_i a_i a_{i+1} c_{i+1}$ and $c_i b_i b_{i+1} c_{i+1}$. This contradiction with $f(Q^i) < 3$ determines that K^i is crossed in D by at least one of P_K^i and P_K^{i-1}.

Let us assumed that, without loss of generality, P_K^i does not cross K^i. Then K^i is crossed by P_K^{i-1}. It is shown above that if $\mathrm{cr}_D(K^i, P_K^i) = 0$, then K^{i-1} is placed in D in the region of D' with the vertex c_i on its boundary. In Figure 5(c) one can see that the condition $f(Q^i) < 3$ enforces that, in D, the edge $\{f_i, f_{i-1}\}$ does not cross K^i nor K^{i+1}. Hence, the edge $\{f_i, f_{i-1}\}$ crosses an edge of P_K^i and K^i is crossed once by the edge $\{d_i, d_{i-1}\}$ or by the edge $\{e_i, e_{i-1}\}$. If K^i is crossed by the edge $\{d_i, d_{i-1}\}$, then the cycle $c_{i-1} c_i d_i f_i f_{i-1} c_{i-1}$ separates the vertices e_i and e_{i-1} and the edge $\{e_i, e_{i-1}\}$ crosses in D the cycle $c_{i-1} c_i d_i f_i f_{i-1} c_{i-1}$. These three crossings give $f(Q^i) \geq 3$. The same we obtain if K^i is crossed by the edge $\{e_i, e_{i-1}\}$ crosses K^i. This proofs Claim 2.3.

For the drawing D the following Claim 2.4 also holds.

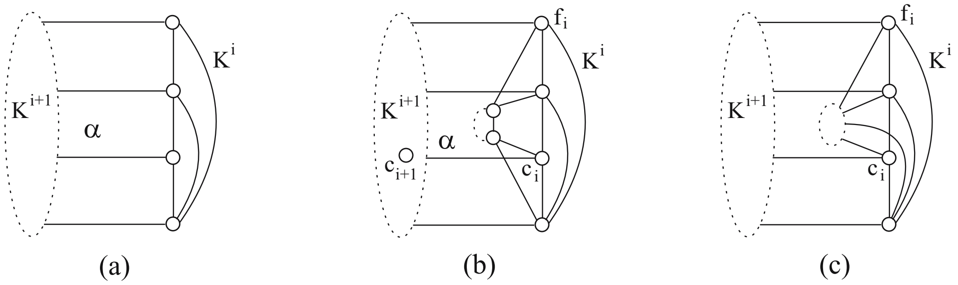

FIGURE 5. The subdrawing of $K^i \cup P_K^i \cup K^{i+1}$ and placements of $K^{i-1} \cup P_K^{i-1}$

CLAIM 2.4. *If both P_K^i and P_K^{i-1} cross K^i, then $f(Q^i) \geq 3$.*

The assumption $\mathrm{cr}_D(K^i, P_K^{i-1}) \neq 0$ and $\mathrm{cr}_D(K^i, P_K^i) \neq 0$ enforces $f(Q^i) \geq 2$. If K^i is crossed by the edges of $P_K^{i-1} \cup P_K^i$ at least three times or there is a crossing on some of the edges $\{c_i, a_i\}$ and $\{c_i, b_i\}$, then $f(Q^i) \geq 3$ and we are done. Assume now that $\mathrm{cr}_D(K^i, P_K^{i-1}) = \mathrm{cr}_D(K^i, P_K^i) = 1$ and the edges $\{c_i, a_i\}$ and $\{c_i, b_i\}$ are not crossed in D. For $x \in \{a, b, d, e\}$, let $X^i, X \in \{A, B, D, E\}$, be the subgraph of $G \square C_n$ induced by the vertices $x_{i-1}, x_i, x_{i+1}, c_{i-1}, c_i$, and c_{i+1}. One can easily verify that, in this case, none of two crossings on K^i which contribute to $f(Q^i) \geq 2$ is an internal crossing in some of the subgraphs A^i, B^i, D^i and E^i. Every internal crossing in X^i contributes at least $\frac{1}{2}$ to the force of Q^i. As the edge $\{c_i, c_{i-1}\}$ does not cross the edge $\{c_i, c_{i+1}\}$, the edges in at least three of the subgraphs X^i, $X \in \{A, B, D, E\}$, do not cross each other. Hence, in both pairs of subgraphs A^i, B^i and C^i, D^i there is at least one subgraph without internal crossings. Without loss of generality assume that A^i and D^i do not have internal crossings.

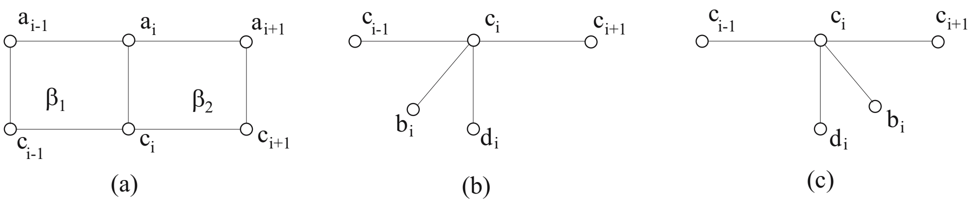

FIGURE 6. The subdrawing of A^i and possible subdrawings of the edges $\{c_i, b_i\}$, $\{c_i, d_i\}$, $\{c_i, c_{i-1}\}$, and $\{c_i, c_{i+1}\}$

The unique subdrawing of A^i induced from D is shown in Figure 6(a). The vertex b_i is not placed in D in the region β_1 of the subdrawing of A^i. Otherwise, as the edge $\{c_i, a_i\}$ is not crossed, the path $b_i b_{i+1} c_{i+1}$ crosses the path $a_i a_{i-1} c_{i-1} c_i$ in D and this crossing enforces $f(Q^i) \geq 3$. Due to symmetry, the vertex b_i cannot be placed in the region β_2. Assume now, that the vertex d_i is placed in D in the region β_1. Then the path $d_i d_{i+1} c_{i+1}$ crosses the path $a_i a_{i-1} c_{i-1} c_i$ in D. As none of two crossings on the edges of K^i is a crossing between the edge $\{d_i, d_{i+1}\}$ and the path $a_i a_{i-1} c_{i-1} c_i$, the assumption that G^i has at most two crossings on its edges implies that the edge $\{d_i, d_{i+1}\}$ does not cross the path $a_i a_{i-1} c_{i-1} c_i$ and that the path $a_i a_{i-1} c_{i-1} c_i$ is crossed by the edge $\{d_{i+1}, c_{i+1}\}$. This enforces $f(Q^i) \geq 3$ again. The similar consideration can be repeated for the region β_2 and hence, d_i is placed in D outside the regions β_1 and β_2. Since the edges $\{c_i, b_i\}$ and $\{c_i, d_i\}$ do not cross the edges of A^i, the only two possible subdrawings of the edges $\{c_i, b_i\}$, $\{c_i, d_i\}$, $\{c_i, c_{i-1}\}$, and $\{c_i, c_{i+1}\}$ are shown in Figure 6(b) and Figure 6(c). As we have assumed above, there is no crossing on the edge $\{c_i, b_i\}$ and the edges of D^i do not cross each other. Thus, for the case shown in Figure 6(b), the path $d_i d_{i-1} c_{i-1}$ separates in D the vertices b_i and c_{i+1} and in D there is a crossing between the paths $b_i b_{i+1} c_{i+1}$ and $c_i c_{i-1} d_{i-1} d_i$ which contributes at least 1 to $f(Q^i)$. A similar consideration for the situation shown in Figure 6(c) gives that the crossing between the paths $b_i b_{i-1} c_{i-1}$ and $c_i c_{i+1} d_{i+1} d_i$ contributes at least 1 to $f(Q^i)$ again. Hence, $f(Q^i) \geq 3$ and Claim 2.4 is true.

The contradiction between Claim 2.3 and Claim 2.4 completes the proof. $\square$

Theorem 2.5. *For $n \geq 6, \mathrm{cr}(G \,\square\, C_n) = 3n$.*

Proof. The drawing in Figure 2 shows that $\mathrm{cr}(G \,\square\, C_n) \leq 3n$ for $n \geq 6$. We prove the reverse inequality by induction on n. By Theorem 2.4, $\mathrm{cr}(G \square C_6) = 18$, so the result is true for $n = 6$. Assume it is true for $n = k$, $k \geq 6$, and suppose that there is a drawing of $G \,\square\, C_{k+1}$ with fewer than $3(k+1)$ crossings. By Lemma 2.3, some G^i must be crossed at least three times. By the removal of all edges of this G^i, we obtain a subdivision of $G \,\square\, C_k$ with fewer than $3k$ crossings. This contradicts the induction hypothesis. $\square$

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REFERENCES

[1] ADAMSON, J.—RICHTER, R. B.: *Arrangements, circular arrangements and the crossing number of $C_7 \times C_n$*, J. Combin. Theory Ser. B **90** (2004), 21–39.

[2] ANDERSON, M.—RICHTER, R. B.—RODNAY, P.: *The crossing number of $C_6 \times C_6$*, Congr. Numer. **118** (1996), 97–107.

[3] ANDERSON, M.—RICHTER, R. B.—RODNAY, P.: *The crossing number of $C_7 \times C_7$*, Congr. Numer. **125** (1997), 97–117.

[4] BEINEKE, L. W.—RINGEISEN, R. D.: *On the crossing numbers of products of cycles and graphs of order four*, J. Graph Theory **4** (1980), 145–155.

[5] DEAN, A. M.—RICHTER, R. B.: *The crossing number of* $C_4 \times C_4$, J. Graph Theory **19** (1995), 125–129.

[6] DRAŽENSKÁ, E.—KLEŠČ, M.: *The crossing numbers of products of cycles with* 6*-vertex trees*, Tatra Mt. Math. Publ. **36** (2007), 109–119.

[7] GLEBSKY, L. Y.—SALAZAR, G.: *The crossing number of* $C_m \times C_n$ *is as conjectured for* $n \geq m(m+1)$, J. Graph Theory **47** (2004), 53–72.

[8] HARARY, F.—KAINEN, P. C.—SCHWENK, A. J.: *Toroidal graphs with arbitrarily high crossing numbers*, Nanta Math. **6** (1973), 58–67.

[9] JENDROĽ, S.—ŠČERBOVÁ, M.: *On the crossing numbers of* $S_m \times P_n$ *and* $S_m \times C_n$, Časopis Pěst. Mat. **107** (1982), 225–230.

[10] KLEŠČ, M.: *On the crossing numbers of Cartesian products of stars and paths or cycles*, Math. Slovaca **41** (1991), 113–120.

[11] KLEŠČ, M.: *The crossing number of* $(K_4 - e) \times C_3$. In: Proc. International Scientific Conference on Mathematics (Herľany, 1999), Univ. Technol. Košice, Košice, 2000, pp. 106–109.

[12] KLEŠČ, M.: *The crossing numbers of Cartesian products of paths with* 5*-vertex graphs*, Discrete Math. **233** (2001), 353–359.

[13] KLEŠČ, M.: *The crossing number of* $K_{2,3} \times C_3$, Discrete Math. **251** (2002), 109–117.

[14] KLEŠČ, M.: *Some crossing numbers of products of cycles*, Discuss. Math. Graph Theory **25** (2005), 197–210.

[15] KLEŠČ, M.—RICHTER, R. B.—STOLBERT, I.: *The crossing number of* $C_5 \times C_n$, J. Graph Theory **22** (1996), 239–243.

[16] KLEŠČ, M.—KOCÚROVÁ, A.: *The crossing number of products of* 5*-vertex graphs with cycles*, Discrete Math. **307** (2007), 1395–1403.

[17] RICHTER, R. B.—SALAZAR, G: *The crossing number of* $C_6 \times C_n$, Australas. J. Combin. **23** (2001), 135–144.

[18] RINGEISEN, R. D.—BEINEKE, L. W.: *The crossing number of* $C_3 \times C_n$, J. Combin. Theory Ser. B **24** (1978), 134–136.

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