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C^*-ALGEBRAS OF GENERALIZED HECKE PAIRS

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ABSTRACT. We generalize the concept of Hecke pairs and study representations of the corresponding C^*-algebra. We introduce the notions of covariant pairs and matrix unit pairs of representations in this general setting and show that covariant pairs are exactly faithful matrix unit pairs.

1. Introduction

A discrete *Hecke pair* is a pair (G, H) where G is a discrete group and H is a subgroup. It is assumed that every double coset HgH contains just finitely many left cosets [1]. Covariant representations (*covariant pairs*) of Hecke pairs [1] and corresponding C^*-algebras [4] are studied. The idea behind a Hecke pair is that the group G acts on the homogeneous space G/H by translation and this action is used in the defining condition of covariant pairs.

The objective of this paper is to consider a more general situation where a discrete group G acts on a semi convo K ([3]). The notion of (semi) convo or a hypergroup [2] could be considered as a generalization of the quotient space of a group by a subgroup. Another example is the dual space of a compact group [2]. We refer the reader to [2] for detailed definition and more details.

The pair (G, K) endowed with the left and right actions $\gamma_L, \gamma_R \colon G \times K \to K$ is called a *generalized Hecke pair*. We write $t \cdot a := \gamma_L(t, a)$ and $a \cdot t := \gamma_R(t, a)$, for

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$t \in G$ and $a \in K$. We also write $G{\cdot}a := \{s{\cdot}a :\ s \in G\}$ and $t{\cdot}K := \{t{\cdot}b :\ b \in K\}$, for $t \in G, a \in K$. We assume that

$$t \cdot (s \cdot a) = (ts) \cdot a, \quad (a \cdot t) \cdot s = a \cdot (ts), \quad t \cdot (a \cdot s) = (ts) \cdot a \quad (t \cdot a) \cdot s = a \cdot (ts)$$

for $t, s \in G$, $a \in K$. Let e be the identity element of K. We assume that $G \cdot e = K$ and take a minimal set $J \subseteq G$ of representatives such that each $a \in K$ is of the form $t \cdot e$, for a unique $t \in J$.

Let $\tilde{K}$ be the hypergroup of "quotients" of K, constructed as follows: Take a copy $\bar{K} = \{\bar{a} :\ a \in K\}$ of K and put $\tilde{K} = \bar{K} * K = \{\bar{a} * b :\ a, b \in K\}$ with convolution and involution defined in such a way that $\tilde{K}$ is a hypergroup. We identify $a \in K$ and $\bar{a} \in \bar{K}$ with $\bar{e} * a \in \tilde{K}$ and $\bar{a} * e \in \tilde{K}$, respectively. For $a \in K$ and $x \in \tilde{K}$, write $a \leq x$ if $\bar{e} * a = x$. When $K = G/H$ we get $\tilde{K} = G//H$ which is partially ordered by inclusion, and $a = uH \leq x = HgH$ is equivalent to $uH \subseteq HgH$, which is in turn equivalent to $HuH = HgH$. Given $x \in \tilde{K}$, put $K_x = \{u \in K :\ u \leq x\}$. We assume that K_x is finite, for each $x \in \tilde{K}$. This is the analog of the condition in [1] that every double coset HgH contains just finitely many left cosets. We also put $J_x = \{t \in J :\ t \cdot e \leq x\}$, for $x \in \tilde{K}$.

The corresponding Hecke algebra is the $*$-algebra $M(\tilde{K})$ of bounded measures on $\tilde{K}$. Dealing with elements of this algebra, we use the notation $x \in \tilde{K}$ for the Dirac measure (point mass) $\delta_x \in M(\tilde{K})$. In particular, for $x, y \in \tilde{K}$, $x * y$ is a convex combination of point masses (a probability measure on $\tilde{K}$). Also dealing with subsets of $\tilde{K}$ we use x to denote the singleton $\{x\}$. On K, we consider the $*$-algebra $c_0(K)$ of complex function vanishing at infinity (i.e. those functions $f\colon K \to \mathbb{C}$ such that for each $\varepsilon > 0$, there is a finite set $I \subseteq K$ with $|f| \leq \varepsilon$ off I). This is a C^*-algebra in sup-norm with respect to the pointwise multiplication and complex conjugate involution. We use $a \in K$ to denote both the characteristic function $\chi_{\{a\}}$ of the singleton $\{a\}$ as an element in $c_0(K)$, and the the singleton $\{a\}$ itself, as a subset of K. In particular, in the former case we see relations such as $ab = 0$, when $a \neq b$, which means that the pointwise multiplication of $\chi_{\{a\}}$ and $\chi_{\{b\}}$ are identically zero, and in the latter case, for a measure m on K, $m(a)$ means $m(\{a\})$. A similar convention is used for $\tilde{K}$.

Lemma 1.1. *With the above notation, $\tilde{K}$ is a hypergroup with respect to the convolution and involution*

(i) $(\delta_x * \delta_y)(z) = \sum_{a \in K} \delta_x(\bar{e} * a)\delta_y(\bar{a} * z)$, *for* $x, y, z \in \tilde{K}$,

(ii) $(\bar{a} * b)^- = \bar{b} * a$, *for* $a, b \in K$.

In particular, for $f, g \in \ell^1(\tilde{K})$,

(iii) $f * g(\bar{e} * b) = \sum_{a \in K} f(\bar{e} * a) g(\bar{a} * b)$,

(iv) $f^*(\bar{e} * b) = \bar{f}(\bar{b} * e)$,

for $a, b \in K$.

Note that the above hypergroup structure on $\tilde{K}$ is different from the structure presented in [2] for $\tilde{K} = G//H$, when $K = G/H$.

2. Covariant representations

Definition 2.1. Let (G, K) be a generalized discrete Hecke pair, ν be a nondegenerate representation of $c_0(K)$ on a Hilbert space $\mathscr{H}$, and V be a $*$-representation of $M(\tilde{K})$ on the same Hilbert space, we say that (ν, V) is

(i) a *matrix unit pair* if

$$\left\{\nu(a) V(\bar{a} * b) \nu(b) : \; a, b \in K\right\}$$

is a set of matrix units in $B(\mathscr{H})$.

(ii) a *covariant pair* if

$$V(x)\nu(a)V(y) = \sum_{s \in J_{\bar{x}}} \sum_{t \in J_y} \nu(a \cdot s) V((\bar{e} \cdot s^{-1}) * (t \cdot e)) \nu(a \cdot t),$$

for each $a \in K$, $x, y \in \tilde{K}$, where e is the identity of K.

Lemma 2.2. *For a pair* (ν, V)*, the following are equivalent:*

(i) $V(x)\nu(b) = \sum_{t \in J_{\bar{x}}} \nu(b \cdot t) V(x) \nu(a) \quad (b \in K, x \in \tilde{K})$,

(ii) $\nu(a) V(x) = \sum_{s \in J_x} \nu(a) V(x) \nu(a \cdot s) \quad (a \in K, x \in \tilde{K})$,

(iii) $\nu(a) V(x) \nu(b) = 0$*, unless* $\bar{a} * b = x \quad (a, b \in K, x \in \tilde{K})$.

If (ν, V) *is a matrix unit pair, then these are equivalent to:*

(iv) $\|V_a(x)\| \le m(K_x)^{\frac{1}{2}} \quad (a \in K, x \in \tilde{K})$,

where $V_a(x)$ *is the restriction of the operator* $V(x)$ *to the closed subspace* $\nu(a)\mathscr{H}$.

P r o o f. The equivalence of (i) and (ii) is proved by taking adjoints. Suppose (ii), then for $a, b \in K$ and $x \in \tilde{K}$,

$$\nu(a)V(x)\nu(b) = \sum_{s \in J_x} \nu(a)V(x)\nu(a \cdot s)\nu(b) = 0$$

unless $a \cdot s = b$, for some $s \in J_x$. Let $a = t \cdot e$ and $b = u \cdot e$, for some $t, u \in J$, then $u \cdot e = (t \cdot e) \cdot s = (ts) \cdot e$. Hence $(t^{-1}u) \cdot e = s \cdot e \leq x$ which implies that

$$\bar{a} * b = (t \cdot e)^{-} * (u \cdot e) = (\bar{e} \cdot t^{-1}) * (u \cdot e) = \bar{e} * ((t^{-1}u) \cdot e) = \bar{e} * (s \cdot e) = x.$$

Conversely if $\bar{a} * b = x$, and $a = t \cdot e$, $b = u \cdot e$, then $a \cdot (t^{-1}u) = (tt^{-1}u) \cdot e = u \cdot e = b$ and $t^{-1}u \in J_x$. Thus (ii) implies (iii).

Next, assume (iii), then since ν is non-degenerate, to prove (ii) it is enough to show that for each $a, b \in K$, $x \in \tilde{K}$, and $\xi \in \mathscr{H}$,

$$\nu(a)V(x)\nu(b)\xi = \sum_{s \in J_x} \nu(a)V(x)\nu(a \cdot s)\nu(b)\xi.$$

By assumption, the left hand side is zero unless $x = \bar{a} * b$, and clearly the right hand side is zero unless $a \cdot s = b$, for some $s \in J_x$, which again happens exactly when $x = \bar{a} * b$. Thus (iii) implies (ii).

Now suppose that (ν, V) is a matrix unit pair such that (i) holds. Let $x \in \tilde{K}$ and $a = e \cdot s \in K$, where $s \in G$, then for each $t \in J_{\bar{x}}$, the matrix unit

$$\nu(a \cdot t)V((x)\nu(a) = \nu(a \cdot t)V((\bar{e} * (t \cdot e))\nu(a) = \nu(e \cdot st)V((\bar{e} \cdot (st)^{-1}) * (s \cdot e))\nu(e \cdot s)$$

is a partial isometry with initial projection $\nu(a)$ and final projection $\nu(a \cdot t)$, and the final projections are orthogonal for different values of $t \in J_{\bar{x}}$ (since $a \cdot t \neq a \cdot u$, for $t, u \in J_{\bar{x}}$ with $t \neq u$). For each $\xi \in \nu(a)\mathscr{H}$,

$$\begin{aligned} \|V(x)\xi\|^2 &= \|V(x)\nu(a)\xi\|^2 \\ &= \Big\| \sum_{t \in J_{\bar{x}}} \nu(a \cdot t)V(x)\nu(a)\xi \Big\|^2 \\ &= \sum_{t \in J_{\bar{x}}} \|\nu(a \cdot t)V(x)\nu(a)\xi\|^2 = |J_{\bar{x}}| \|\xi\|^2. \end{aligned}$$

Thus (i) implies (iv) in this case.

Conversely, suppose (ν, V) is a matrix unit pair such that (iv) holds. Let $x \in \tilde{K}$ and $a = e \cdot s \in K$, where $s \in G$, and set $P = \sum_{t \in J_{\bar{x}}} \nu(a \cdot t)$, then P is an orthogonal projection on $\mathscr{H}$ and for each $\xi \in \mathscr{H}$, $\nu(a \cdot t)V(x)\nu(a)$ is a partial isometry whose initial space contains $\nu(a)\xi$, hence

$$\begin{aligned}
|J_{\bar{x}}|\|\nu(a)\xi\|^2 &\geq \|V_a(x)\|^2\|\nu(a)\xi\|^2 \\
&\geq \|V(x)\nu(a)\xi\|^2 \\
&= \|PV(x)\nu(a)\xi\|^2 + \|(1-P)V(x)\nu(a)\xi\|^2 \\
&= \sum_{t\in J_{\bar{x}}} \|\nu(a\cdot t)V(x)\nu(a)\xi\|^2 + \|(1-P)V(x)\nu(a)\xi\|^2 \\
&= \sum_{t\in J_{\bar{x}}} \|\nu(a)\xi\|^2 + \|(1-P)V(x)\nu(a)\xi\|^2 \\
&= |J_{\bar{x}}|\|\nu(a)\xi\|^2 + \|(1-P)V(x)\nu(a)\xi\|^2,
\end{aligned}$$

and so $(1-P)V(x)\nu(a)\xi = 0$. Therefore $V(x)\nu(a) = PV(x)\nu(a)$, which is (i). □

We say that a pair (ν, V) is *faithful* if it satisfies in any of the above first three equivalent conditions.

Proposition 2.3. *Covariant pairs are exactly faithful matrix unit pairs.*

Proof. For $a, b \in K$ put $v_{a,b} = \nu(a)V(\bar{a} * b)\nu(b)$. If (ν, V) is a covariant pair, then for $a, b, c \in K$,

$$\begin{aligned}
v_{a,b}v_{b,c} &= \nu(a)V(\bar{a} * b)\nu(b)V(\bar{b} * c)\nu(c) \\
&= \sum_{s\in J_{\bar{b}*a}} \sum_{t\in J_{\bar{b}*c}} \nu(a)\nu(b\cdot s)V((\bar{e}\cdot s^{-1}) * (t\cdot e))\nu(b\cdot t)\nu(c).
\end{aligned}$$

By definition of J, there is exactly one $s \in J_{\bar{b}*a}$ such that $a = b \cdot s$ and exactly one $t \in J_{\bar{b}*c}$ such that $c = b \cdot t$, and if $u \in J$ is the unique element such that $b = u \cdot e$, then

$$\begin{aligned}
\bar{a} * c &= (s^{-1}\cdot \bar{b}) * (b\cdot t) = (s^{-1}\cdot(\bar{e}\cdot u^{-1})) * ((u\cdot e)\cdot t) \\
&= (\bar{e}\cdot(s^{-1}u^{-1}u)) * (e\cdot t) = (\bar{e}\cdot(s^{-1})) * (e\cdot t),
\end{aligned}$$

hence $v_{a,b}v_{b,c} = \nu(a)V(\bar{a} * c)\nu(c) = v_{a,c}$. Similarly $v^*_{a,b} = v_{b,a}$ and $v_{a,b}v_{c,d} = 0$, unless $b = c$, and so (ν, V) is a matrix unit pair. If in the covariant pair condition, we put $y = \bar{e} * e$, we get equality (i) of Lemma 1.3, since $J_y = \{1\}$, where $1 \in G$ is the identity element, which shows that (ν, V) is faithful.

Conversely, if (ν, V) is a faithful matrix unit pair, then for each $a \in K$, $x, y \in \tilde{K}$, and $s \in J_{\bar{x}}$, $t \in J_y$, we have $x = (a \cdot s)^{-} * a$ and $y = \bar{a} * (a \cdot t)$ and so by conditions (i) and (ii) of Lemma 1.3,

$$\begin{aligned} V(x)\nu(a)V(y) &= V(x)\nu(a)\nu(a)V(y) \\ &= \sum_{s \in J_{\bar{x}}} \sum_{t \in J_y} \nu(a \cdot s)V(x)\nu(a)\nu(a)V(y)\nu(a \cdot t) \\ &= \sum_{s \in J_{\bar{x}}} \sum_{t \in J_y} \nu(a \cdot s)V((a \cdot s)^{-} * a)\nu(a)\nu(a)V(\bar{a} * (a \cdot t))\nu(a \cdot t) \\ &= \sum_{s \in J_{\bar{x}}} \sum_{t \in J_y} v_{a \cdot s, a} v_{a, a \cdot t} = \sum_{s \in J_{\bar{x}}} \sum_{t \in J_y} v_{a \cdot s, a \cdot t} \\ &= \sum_{s \in J_{\bar{x}}} \sum_{t \in J_y} \nu(a \cdot s)V((\bar{e} \cdot s^{-1}) * (t \cdot e))\nu(a \cdot t), \end{aligned}$$

hence (ν, V) is a covariant pair. □

Example 2.4. Let M be the representation of $c_0(K)$ on $\ell^2(K)$ defined by

$$M(f)(\delta_a) = f(a)\delta_a \qquad (f \in c_0(K), \;\; a \in K)$$

where δ_a is the characteristic function $\chi_{\{a\}}$ as an element of $\ell^2(K)$, and ρ be the right regular representation of $\tilde{K}$ on $\ell^2(K)$ defined by

$$\rho(x)(\delta_a) = \sum_{t \in J_{\bar{x}}} \delta_{a \cdot t} \qquad (x \in \tilde{K}, \;\; a \in K).$$

Then (M, ρ) is a covariant pair.

To see this, for $\xi, \zeta \in \ell^2(K)$, consider the rank-one operators $(\xi \otimes \zeta)(\eta) = \langle \eta, \zeta \rangle \xi$, $\eta \in \ell^2(K)$, and observe that $M(a) = \delta_{a \cdot t} \otimes \delta_a$. Then

$$\rho(x)M(a) = \sum_{t \in J_{\bar{x}}} \delta_{a \cdot t} \otimes \delta_a \qquad (x \in \tilde{K}, \;\; a \in K).$$

Hence

$$M(a)\rho(\bar{a} * b)M(b) = \sum_{t \in J_{\overline{b * a}}} (\delta_a \otimes \delta_a)(\delta_{b \cdot t} \otimes \delta_b) = \delta_a \otimes \delta_b,$$

for each $x \in \tilde{K}$, $a, b \in K$. Therefore $(M, \tilde{\rho})$ is a faithful matrix unit pair, and so a covariant pair.

Theorem 2.5. *Let (G, K) be a generalized discrete Hecke pair, ν be a non-degenerate representation of $c_0(K)$ on a Hilbert space $\mathscr{H}$, and V be a $*$-representation of $M(\tilde{K})$ on the same Hilbert space, then*

(i) *(ν, V) is a matrix unit pair if and only if there is a Hilbert space $\mathscr{K}$ and a representation $\tilde{V}$ of $M(\tilde{K})$ on $\mathscr{K} \otimes \ell^2(K)$ such that (ν, V) is unitarily equivalent to $(1 \otimes M, \tilde{V})$ and*

$$(1 \otimes M(a))\tilde{V}(\bar{a} * b)(1 \otimes M(b)) = 1 \otimes M(a)\rho(\bar{a} * b)M(b),$$

for each $a, b \in K$.

(ii) *(ν, V) is a covariant pair if and only if there is a Hilbert space $\mathscr{K}$ such that (ν, V) is unitarily equivalent to the covariant representation $(1 \otimes M, 1 \otimes \rho)$ on $\mathscr{K} \otimes \ell^2(K)$.*

In particular, ν is equivalent to a multiple of M if and only if (ν, V) is a covariant pair, for some V.

Proof. Suppose that (ν, V) is a matrix unit pair on $\mathscr{H}$ and consider matrix units $v_{a,b} = \nu(a)V(\bar{a} * b)\nu(b)$, and put $\mathscr{K} = \nu(e)\mathscr{H}$, then by non-degeneracy of ν, $\nu(a)\xi \mapsto v_{e,a}\xi \otimes \delta_a$ $(\xi \in \mathscr{H})$ extends to a unitary isomorphism Ψ of $\mathscr{H}$ onto $\mathscr{K} \otimes \ell^2(K)$ such that $\operatorname{Ad}\Psi(v_{a,b}) = 1 \otimes (\delta_a \otimes \delta_a)$, for $a, b \in K$. Then

$$\operatorname{Ad}\Psi(\nu(a)) = \operatorname{Ad}\Psi(v_{a,b}) = 1 \otimes (\delta_a \otimes \delta_a) = 1 \otimes M(a),$$

for $a \in K$, thus $\operatorname{Ad}\Psi \circ \nu = 1 \otimes M$. Similarly the representation $\tilde{V} = \operatorname{Ad}\Psi \circ V$ of $M(\tilde{K})$ on $\mathscr{K} \otimes \ell^2(K)$ satisfies

$$\begin{aligned}(1 \otimes M(a))\tilde{V}(\bar{a} * b)(1 \otimes M(b)) &= \operatorname{Ad}\Psi(\nu(a)V(\bar{a} * b)\nu(b)) = \operatorname{Ad}\Psi(v_{a,b}) \\ &= 1 \otimes (\delta_a \otimes \delta_a) \\ &= 1 \otimes (M(a))\rho(\bar{a} * b)(1 \otimes M(b)),\end{aligned}$$

for $a, b \in K$. This proves half of (i). The converse follows easily as the relation stated in (i) between M and $\tilde{V}$ implies that $(1 \otimes M, \tilde{V})$ is a matrix unit pair.

Next suppose that (ν, V) is a covariant pair, and let $\mathscr{K}$, Ψ, and $\tilde{V}$ be as in (i). Then $(1 \otimes M, \tilde{V}) = (\operatorname{Ad}\Psi \circ \nu, \operatorname{Ad}\Psi \circ V)$ is covariant and, by (i) and Lemma 2.2(iii),

$$(1 \otimes M(a))\tilde{V}(x)(1 \otimes M(b)) = 1 \otimes (M(a))\rho(x)(1 \otimes M(b)),$$

for $a, b \in K$, $x \in \tilde{K}$. It follows that $\tilde{V} = 1 \otimes \rho$, which proves half of (ii). The other half follows from the fact that $(1 \otimes M, 1 \otimes \rho)$ is a covariant pair. Finally, observe that if a unitary Ψ intertwines ν and $1 \otimes M$, then $V = \operatorname{Ad}\Psi^* \circ (1 \otimes \rho)$ is a representation of $M(\tilde{K})$ such that (ν, V) is a covariant pair. □

Corollary 2.6. *With the above notation, for every matrix unit pair (ν, V), there is a representation W of $M(\tilde{K})$ such that (ν, W) is a covariant pair.*

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