VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-011-0029-y
Math. Slovaca **61** (2011), No. 4, 571–578

ON THE ARONSZAJN PROPERTY FOR A DIFFERENTIAL EQUATION OF FRACTIONAL ORDER IN BANACH SPACES

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(*Communicated by Michal Fečkan*)

ABSTRACT. In this paper we investigate some topological properties of solution sets of a differential equation of fractional order in Banach spaces. Our assumptions and proofs are expressed in terms of measures of noncompactness.

Assume that $I = [0, a]$, E is a Banach space,

$$B = \big\{x \in E : \ \|x - x_0\| \le b\big\}$$

and $f \colon I \times B \to E$ is a bounded continuous function.

In this paper we prove the existence of a solution and we study the structure of the solutions set of the nonlinear differential equation of fractional order

$$\begin{aligned} D^\beta x &= f(t, x) \\ x^{(k)}(0) &= x_0^{(k)}, \qquad k = 0, 1, \dots, \lceil\beta\rceil - 1, \end{aligned} \tag{1}$$

where $\beta \in (0, \infty)$ and D^β denotes the fractional derivative of order β in the Caputo sense (cf. [3]). $\lceil\beta\rceil$ is the first integer not less than β.

Fractional differential equations have been of great interest recently. It is connected with their many applications in various fields of science and engineering. In particular, these equations have proved to be valuable tools for modeling many physical, chemical and biological phenomena. Moreover, some problems

2010 Mathematics Subject Classification: Primary 34G20; Secondary 45N05.
Keywords: nonlinear differential equation, derivative of fractional order, measure of noncompactness, solution sets.

in economics can be described with the help of differential equations of fractional order. For details, see [8]–[10], [12]–[15] and references therein. There have appeared several fundamental monographs on the fractional derivative and fractional differential equations, written by Oldham and Spanier [12], Miller and Ross [10], Podlubny [13], Samko, Kilbas, Marichev [14], Kilbas, Srivastava and Trujillo [9] and others. These works provide a comprehensive account of the theory and applications of fractional calculus.

In what follows we shall need the following result of W. Mydlarczyk given in [11].

Theorem 1. *Let $\alpha > 0$ and let $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that $g(0) = 0$, $g(t) > 0$ for $t > 0$. Then the equation*

$$u(t) = \int_0^t (t-s)^{\alpha-1} g(u(s))\,\mathrm{d}s \qquad (t \geq 0)$$

has a nontrivial continuous solution if and only if

$$\int_0^\delta \frac{1}{s}\left[\frac{s}{g(s)}\right]^{\frac{1}{\alpha}} \mathrm{d}s < \infty \qquad (\delta > 0).$$

We will also need the following modification of Vidossich theorem (cf. [19], [16]).

Let K be a bounded convex subset of a normed space, and let E be a Banach space. Denote by $C = C(K,E)$ the space of all bounded continuous functions $K \to E$ with the usual supremum norm.

Theorem 2. *Assume that $F\colon C \to C$ is a continuous mapping such that*

1° *$F(C)$ is an equiuniformly continuous set;*

2° *for every $\varepsilon > 0$, $x|K_\varepsilon = y|K_\varepsilon \Longrightarrow F(x)|K_\varepsilon = F(y)|K_\varepsilon$, $(x,y) \in C$, where $K_\varepsilon = K \cap B(t_0, \varepsilon)$;*

3° *there exists $t_0 \in K$ and $x_0 \in E$ such that $F(x)(t_0) = x_0$ for all $x \in C$;*

4° *every sequence (x_n) in C such that $\lim\limits_{n\to\infty} (x_n - F(x_n)) = 0$ has a limit point,*

then the set S of fixed points of F is an R_δ, i.e. it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.

The main tool used in our considerations is the concept of a measure of noncompactness. Denote by α the Kuratowski measure of noncompactness in E. For any bounded subset A of a Banach space E we define by $\alpha(A)$ the infimum of all $\varepsilon > 0$ such that there exists a finite covering of A by sets of diameter $\leq \varepsilon$.

The measure of noncompactness has several properties which will be useful in the sequel.

The following lemma can be found in [2].

Lemma 1. *For any bounded sets A, $B \subset E$, we have*

1° $\alpha(A) = 0 \iff A$ *is relatively compact;*

2° $A \subset B \implies \alpha(A) \leq \alpha(B)$*;*

3° $\alpha(\bar{A}) = \alpha(A)$*;*

4° $\alpha(\lambda A) = |\lambda| \alpha(A) \quad (\lambda \in \mathbb{R})$*;*

5° $\alpha(A \cup B) = \max(\alpha(A), \alpha(B))$*;*

6° $\alpha(A + B) = \alpha(A) + \alpha(B)$*;*

7° $\alpha(\operatorname{conv} A) = \alpha(A)$*;*

8° $\alpha\Big(\bigcup_{|\lambda| \leq h} \lambda A\Big) = h\alpha(A)$.

For a given set V of functions from J into E we define a function v by $v(t) = \alpha(V(t))$ for $t \in J$ (under the convention that $\alpha(X) = \infty$ if X is unbounded), where $V(t) = \big\{x(t) : \ x \in V\big\}$.

The principal tool used in this paper is the following result of Heinz [6].

Lemma 2. *Let V be a countable set of strongly measurable functions $J \to E$ such that there exists an integrable function $\mu \colon J \to R$ such that $\|x(t)\| \leq \mu(t)$ for all $x \in V$ and $t \in J$. Then the corresponding function v is integrable and*

$$\alpha\bigg(\bigg\{\int_J x(t)\,\mathrm{d}t : \ x \in V\bigg\}\bigg) \leq 2 \int_J v(t)\,\mathrm{d}t.$$

Now we present the main result of the paper.

Theorem 3. *Let $\omega \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous nondecreasing function such that $\omega(0) = 0$, $\omega(t) > 0$ for $t > 0$ and*

$$\int_0^\delta \frac{1}{s} \left[\frac{s}{\omega(s)}\right]^{\frac{1}{\beta}} \mathrm{d}s = \infty \qquad (\delta > 0). \tag{2}$$

If

$$\alpha(f(t,X)) \le \omega(\alpha(X)) \qquad \text{for} \quad t \in I \quad \text{and} \quad X \subset B,$$

then there exists an interval $J = [0,d]$ *such that the set of all solutions of* (1) *defined on* J *is a compact* R_δ.

Let us notice that methods and arguments presented here are continuation and extension of those from [18]. Recall that in our earlier paper [18] we proved Aronszajn type theorems for the nth order ordinary differential equation, where $n \in \mathbb{N}$. In the present paper we extend these results to the differential equation of fractional order, where $\beta \in (0,\infty)$.

P r o o f. Put $M = \sup\bigl\{\|f(t,x)\| : \ t \in I, \ \ x \in B\bigr\}$. Choose a positive number d such that $d \le a$ and

$$\sum_{k=1}^{\lceil \beta \rceil - 1} |x_0^{(k)}| \frac{d^k}{k!} + M \frac{d^\beta}{\Gamma(\beta+1)} \le b.$$

Let $C = C(J,E)$ be the Banach space of continuous functions $J \to E$ with the usual supremum norm $\|\cdot\|_C$, where $J = [0,d]$.

Put

$$r(x) = \begin{cases} x & \text{if } x \in B \\ x_0 + \frac{b(x-x_0)}{\|x-x_0\|} & \text{if } x \in E \setminus B, \end{cases}$$

then r is a continuous function $E \to B$ and from the inclusion

$$r(A) \subset x_0 + \bigcup_{0 \le \lambda \le 1} \lambda(A - x_0)$$

and the corresponding properties of α (see Lemma 1) it follows that

$$\alpha(r(A)) \le \alpha\Bigl(\bigcup_{0 \le \lambda \le 1} \lambda(A - x_0)\Bigr) \le \alpha(A - x_0) = \alpha(A)$$

for each bounded $A \subset E$.

We define the function g by

$$g(t,x) = f(t, r(x)) \qquad \text{for} \quad t \in I, \quad x \in E.$$

Then $g\colon I \times E \to E$ is a continuous function such that $\|g(t,x)\| \le M$ for $t \in I$, $x \in E$ and

$$\alpha(g(t,X)) \le \omega(\alpha(X)) \qquad \text{for} \quad t \in I$$

and for each bounded $X \subset E$.

It is well known that the initial value problem (1) is equivalent to the Volterra integral equation (cf. [4])

$$x(t) = \sum_{k=0}^{\lceil\beta\rceil-1} x_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s))\,\mathrm{d}s \qquad (t \in I).$$

We define a mapping F by

$$F(x)(t) = \sum_{k=0}^{\lceil\beta\rceil-1} x_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x(s))\,\mathrm{d}s \qquad (t \in J, \quad x \in C(J,E)).$$

Since $\int_0^t \frac{\mathrm{d}s}{(t-s)^{1-\beta}} = \frac{t^\beta}{\beta}$, we have

$$\begin{aligned} \|F(x)(t) - x_0\| &\le \sum_{k=1}^{\lceil\beta\rceil-1} |x_0^{(k)}| \frac{t^k}{k!} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} M\,\mathrm{d}s \\ &\le \sum_{k=1}^{\lceil\beta\rceil-1} |x_0^{(k)}| \frac{d^k}{k!} + M \frac{d^\beta}{\Gamma(\beta+1)} \le b. \end{aligned}$$

In the case where $0 < \beta < 1$ we obtain the integral equation with weakly singular kernel. Arguing similarly as in [7] we can prove that the set $F(C)$ is equiuniformly continuous and F is a continuous mapping $C \to C$.

For $\beta \ge 1$ we have the integral equation with the continuous kernel. In the same way as in [17] we can also prove that the set $F(C)$ is equiuniformly continuous and $F\colon C \to C$ is a continuous mapping.

Moreover, the problem (1) is equivalent to the equation $x = F(x)$.

Notice, that F has the following properties:

1′ $F(x)(0) = x_0$ for all $x \in C$;

2′ $F(C)$ is an equiuniformly continuous set;

3′ for every $\varepsilon > 0$, $x|[0,\varepsilon] = y|[0,\varepsilon] \Longrightarrow F(x)|[0,\varepsilon] = F(y)|[0,\varepsilon]$, $(x,y) \in C$.

Now, we shall show that

4′ each sequence (v_n) in C such that $\lim_{n\to\infty}(v_n - F(v_n)) = 0$ has a limit point.

Let (v_n) be a sequence in C such that

$$\lim_{n\to\infty}(v_n - F(v_n)) = 0. \tag{3}$$

Put $V = \{v_n :\ n \in \mathbb{N}\}$ and $V(t) = \big\{v_n(t) :\ n \in \mathbb{N}\big\}$. From (3) it follows that the set $\big\{v_n - F(v_n) :\ n \in \mathbb{N}\big\}$ is equiuniformly continuous. Moreover, the set $F(V)$ is equiuniformly continuous.

As $V \subset \big\{v_n - F(v_n) :\ n \in \mathbb{N}\big\} + F(V)$, from the above it follows that the set V is equiuniformly continuous. Therefore the function $t \mapsto v(t) = \alpha(V(t))$ is continuous on J.

Since $V(t) \subset \big\{v_n(t) - F(v_n)(t) :\ n \in \mathbb{N}\big\} + F(V)(t)$, we have $v(t) \leq \alpha(F(V)(t))$.

Let us notice that

$$\left|\frac{1}{(t-s)^{1-\beta}} g(s, x(s))\right| \leq \frac{M}{|t-s|^{1-\beta}}.$$

Then by Lemma 2 and the corresponding properties of α we obtain

$$\begin{aligned} v(t) &= \alpha(V(t)) \leq \alpha(F(V)(t)) \\ &= \alpha\left(\left\{\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, v_n(s))\,\mathrm{d}s :\ n \in \mathbb{N}\right\}\right) \\ &\leq \frac{2}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \alpha\left(\big\{g(s, v_n(s)) :\ n \in \mathbb{N}\big\}\right) \mathrm{d}s \\ &\leq \frac{2}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \omega(\alpha(V(s))\,\mathrm{d}s \qquad \text{for} \quad t \in J, \end{aligned}$$

i.e.

$$v(t) \leq \frac{2}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \omega(v(s))\,\mathrm{d}s \qquad \text{for} \quad t \in J. \tag{4}$$

By the Theorem 1 and assumption (2) the integral equation

$$z(t) = \frac{2}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \omega(z(s))\,\mathrm{d}s \qquad \text{for} \quad t \in J$$

has the unique solution $z(t) \equiv 0$, which is also the maximal solution. Applying now the theorem on integral inequalities ([1, Th. 2]), from (4) we deduce that $v(t) \leq z(t)$ for $t \in J$. Thus $v(t) = 0$ for $t \in J$ and consequently $\alpha(V(t)) = 0$ for $t \in J$. Therefore for each $t \in J$ the set $V(t)$ is relatively compact in E and by Ascoli's theorem the set V is relatively compact in C. Hence we can find a subsequence (v_{n_k}) of (v_n) which converges in C. This proves $(4')$.

We see that F satisfies all assumptions of Theorem 2. Hence the set $S = \operatorname{Fix} F$ is a compact R_δ. $\square$

Acknowledgement. The author expresses her appreciation to the referee for his suggestions to amend the earlier version of the paper.

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Received 16. 2. 2009
Accepted 13. 7. 2009

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