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COMMUTATIVE EXTENDED BCK-ALGEBRAS

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ABSTRACT. We introduce an *extended cone algebra*, which generalises a Bosbach's cone algebra within the framework of extended BCK-algebras and show that every such an algebra is a direct product of an ℓ-group and a cone algebra of Bosbach.

Introduction

The aim of this paper is to generalise Bosbach's cone algebra [2] within the framework of extended BCK-algebras [10] which were recently introduced. An extended BCK-algebra is a generalisation of a pseudo BCK-algebra of Georgescu and Iorgulescu [6], which itself is an extension of a BCK-algebra due to Imai and Iséki [7]. An integral extended BCK-algebra is equivalent to a pseudo BCK-algebra [10, Corollary 1.15] and a symmetric pseudo BCK-algebra is a BCK-algebra of Imai and Iséki ([6, Remark 1.2]).

From [2], we gather that a *cone algebra* of Bosbach, is an algebra $(C; *, :)$ of type $(2,2)$ satisfying the following equations: $(a * a) * b = b = b : (a : a)$; $(a * b) : c = a * (b : c)$; $a : (b * a) = (b : a) * b$; $(a * b) * (a * c) = (b * a) * (b * c)$ and $(c : b) : (a : b) = (c : a) : (b : a)$. If $(G; \leq, \cdot, e)$ is an ℓ-group and G^+ is its positive cone, then it is easily verified that $(G^+; *, :)$ is a cone algebra (called the *cone algebra of the ℓ-group cone* G^+) if we define $a * b := (a^{-1}b) \vee e$ and $a : b := (ab^{-1}) \vee e$. Further, Bosbach has proved that every cone algebra is a subalgebra of the cone algebra of some ℓ-group cone [2].

BCK-algebras were called *commutative* [8] if they are meet semilattices under a *specified* operation and Kühr [9] has extended this definition to pseudo BCK-algebras ([9, Section 3]). Also see [6, Remark 1.19, Corollary 1.20]. In this paper,

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we define a commutative extended BCK-algebra by adopting the approach of Kühr [9].

A certain (relative) cancellation property for a commutative BCK-algebra [3] has been of immense use to obtain a representation of a commutative BCK-algebra in terms of the positive cone of a commutative ℓ-group. This result has been generalised to a commutative pseudo BCK-algebra by Kühr [9, Section 6].

A commutative pseudo BCK-algebra is known to be equivalent to a precone algebra ([12, Definition 1.1] and [9, Theorem 4.3] and comments following Lemma 1.2 in [12]); and hence by [12, Theorem 1.5], a commutative pseudo BCK-algebra with relative cancellation property is a cone algebra of Bosbach [2]. We generalise the notion of relative cancellation to a commutative extended BCK-algebra and introduce the concept of an extended cone algebra; and prove the following theorem:

Decomposition Theorem. *Every extended cone algebra is a direct product of an ℓ-group and a cone algebra of Bosbach.*

This paper is organised as follows: in Section 1, we present the basic preliminaries, together with the notion of a commutative extended BCK-algebra and some of its elementary properties. The second section contains a discussion on the relative cancellation properties of a commutative extended BCK-algebra and the notion of an extended cone algebra. Finally, in the last Section 3, we prove a decomposition theorem for extended cone algebras.

1. Commutative extended BCK-algebras

We now recall from [10], the definition of an extended BCK-algebra and some of its basic properties, which we need in the sequel.

Definition 1.1. An *extended BCK-algebra* is a structure $(C; \leq, *, :, e)$ where $\leq$ is a binary relation on C, $*$, $:$ are binary operations on C and $e \in C$, satisfying the following axioms: for all $a, b, c \in C$,

(i) $(a * c) : (b * c) \leq a * b$,

(ii) $(c : b) * (c : a) \leq b : a$,

(iii) $a : (b * a) \leq b$,

(iv) $(a : b) * a \leq b$,

(v) $a \leq a$,

(vi) if $a \leq b$ and $b \leq a$, then $a = b$,

(vii) if $a \leq b$ and $b \leq e$, then $a \leq e$, and

(viii) $a \leq b \Longleftrightarrow a : b \leq e \Longleftrightarrow b * a \leq e$.

An extended BCK-algebra is said to be *symmetric* if it satisfies the *identity* $a * b = b : a$.

If $(C; \leq, *, :, e)$ is an extended BCK-algebra, then so is the structure $(C; \leq, \circledast, \odot, e)$, where $a \circledast b := b : a$ and $a \odot b := b * a$, which we call the dual of $(C; \leq, *, :, e)$. Clearly, the dual of the dual of an extended BCK-algebra C is C itself, and C is symmetric if and only if C *equals* its dual. We will find it useful in the following to appeal to the duality of an extended BCK-algebra for deriving similar results. The following lemma collects some basic properties of an extended BCK-algebra.

Lemma 1.2.

(i) $a * a \leq e$ *and* $a : a \leq e$ *for all* $a \in C$,

(ii) $a \leq b \Longrightarrow b * c \leq a * c$ *and* $c : b \leq c : a$,

(iii) $a \leq b$ *and* $b \leq c \Longrightarrow a \leq c$,

(iv) $(C; \leq)$ *is a poset,*

(v) $c : b \leq a \Longleftrightarrow a * c \leq b$,

(vi) $a * (b : c) = (a * b) : c$

(vii) $e * a = a = a : e$, *and*

(viii) $b \leq c \Longrightarrow a * b \leq a * c$ *and* $b : a \leq c : a$.

If $(C; \leq, *, :, e)$ is an extended BCK-algebra and $a \in C$, then a is called *integral* if and only if $a * e = e = e : a$; and C is called *integral* if and only if every $a \in C$ is integral. By [10, Lemma 1.14], an extended BCK-algebra $(C; \leq, *, :, e)$ is integral if and only if $e \leq a$ for all $a \in C$. Also, if $K = K(C)$ is the set of all integral elements of an extended BCK-algebra C, then K is a subalgebra of C by [10, Theorem 1.13(1)] and is an integral extended BCK-algebra by [10, Theorem 1.13(3), Lemma 1.14, Corollary 1.15]. Hence, an extended BCK-algebra C with $e \leq x$ for all $x \in C$ = an integral extended BCK-algebra = a pseudo BCK-algebra of Georgescu and Iorgulescu.

We now assume that $(C; \leq, *, :, e)$ is an extended BCK-algebra and $a, b, c, \ldots$ are elements of C. If $a, b \in C$ then $\inf\{a, b\}$ and $\sup\{a, b\}$ will be denoted by $a \wedge b$ and $a \vee b$, respectively, whenever they exist, and to indicate their existence we write $a \wedge b \in C$ (respectively $a \vee b \in C$).

Lemma 1.3. *If* $a \wedge b \in C$, *then* $(a * c) \vee (b * c) \in C$, $(c : a) \vee (c : b) \in C$ *and the following identities hold:*

$$(a \wedge b) * c = (a * c) \vee (b * c) \qquad \text{and} \qquad c : (a \wedge b) = (c : a) \vee (c : b).$$

Proof. For $u \in C$, $a * c \leq u$ and $b * c \leq u \Longleftrightarrow c : u \leq a$ and $c : u \leq b$ (Lemma 1.2(v)) $\Longleftrightarrow c : u \leq a \wedge b \Longleftrightarrow (a \wedge b) * c \leq u$ (Lemma 1.2(v)). Hence $(a \wedge b) * c = (a * c) \vee (b * c)$; and the other part is similar. □

We now recall that a structure $(C; \leq, *, :, \cdot, e)$ is called a *BCK-monoid* [10, Lemma 2.1, Definition 2.2] if and only if

(1) $(C; \leq, *, :, e)$ is an extended BCK-algebra and

(2) the following equation is satisfied: $(ab) * c = b * (a * c)$.

Hence if $(C; \leq, *, :, \cdot, e)$ is a BCK-monoid, then its reduct $(C; \leq, *, :, e)$ is an extended BCK-algebra. By [10, Theorem 2.3], a BCK-monoid is equivalent to the dual of a residuated pomonoid. An extended BCK-algebra, even if it is integral, need not occur as the reduct of some BCK-monoid. (See [10, Remark 6.5, Example 6.6].)

We now recall that a BCK-monoid $(C; \leq, *, :, \cdot, e)$ is called *semiintegral* [10, Definition 2.18] if and only if for all $a \in C$, $a \leq e$ implies a is invertible, and by [10, Lemma 2.20(i)] we have $a * a = a : a = e$ for all a of such a monoid. We now call an *arbitrary* extended BCK-algebra $(C; \leq, *, :, e)$ *semiintegral* if and only if $a * a = a : a = e$ for all $a \in C$ so that the reduct $(C; \leq, *, :, e)$ of a semiintegral BCK-monoid $(C; \leq, *, :, \cdot, e)$ is a semiintegral extended BCK-algebra. Hence if C is a semiintegral extended BCK-algebra we also have, as in [10, Lemma 2.20(ii)],

$$a * e = a * (a : a) = (a * a) : a = e : a.$$

Corollary 1.4. *If $a \wedge b \in C$ and C is semiintegral, then $(a \wedge b) * b = (a * b) \vee e$ and $a : (a \wedge b) = (a : b) \vee e$.*

Proof. By Lemma 1.3. □

Corollary 1.5. *If $a \wedge e \in C$ and C is semiintegral, then $a \vee e \in C$.*

Proof. By Corollary 1.4 and Lemma 1.2(vii). □

Theorem 1.6. *Let $(C; \leq, *, :, e)$ be a semiintegral extended BCK-algebra, then the following are equivalent:*

(α) *$(C; \leq)$ is a meet semilattice satisfying the equations*

$$a : ((a \wedge b) * a) = a \wedge b = (a : (a \wedge b)) * a.$$

(β) *$a \vee e$ exists for all $a \in C$ and*

$$a \leq b \Longrightarrow (b : a) * b = a = b : (a * b).$$

Proof. Assume (α); then, by Corollary 1.5, $a \vee e$ exists for all $a \in C$ and now $a \leq b \Longrightarrow b : (a * b) = b : ((a \wedge b) * b) = a \wedge b = a$ and similarly $(b : a) * b = a$. Hence $(\alpha) \Longrightarrow (\beta)$.

Conversely, assume (β); then, by [10, Lemma 5.6], $(C; \leq)$ is a meet semilattice with $a \wedge b = ((a : b) \vee e) * a = a : ((b * a) \vee e)$. Hence, by Corollary 1.5, we have $a \wedge b = (a : (a \wedge b)) * a = a : ((a \wedge b) * a)$. Hence $(\beta) \Longrightarrow (\alpha)$. □

We now introduce the following definition.

Definition 1.7. A semiintegral extended BCK-algebra satisfying either of the conditions (α) and (β) of Theorem 1.6 will be called a *commutative extended BCK-algebra*.

Since e is the least element of an integral extended BCK-algebra C and $a * a \leq e$, $a : a \leq e$ for all $a \in C$ (by Lemma 1.2), it follows that $a * a = a : a = e$ for all a in an integral extended BCK-algebra C. Hence an integral extended BCK-algebra is clearly semiintegral and hence an integral extended BCK-algebra (equivalently, a pseudo BCK-algebra) is commutative if and only if

$$a \leq b \Longrightarrow b : (a * b) = a = (b : a) * b$$

by Theorem 1.6. (See [10, Corollary 5.8] and [12, page 3].)

By [9, Lemma 3.3], it follows that commutative extended BCK-algebras generalise commutative pseudo BCK-algebras. We now extend Lemmas 3.4 through 3.9 of [9] to commutative extended BCK-algebras. We therefore assume that $(C; \leq, *, :, e)$ is a commutative extended BCK-algebra and $a, b, c, \ldots \in C$.

Lemma 1.8. *If $a \leq s$ and $b \leq s$, then $a \vee b$ exists in C and*

$$a \vee b = ((s : a) \wedge (s : b)) * s = s : ((a * s) \wedge (b * s)).$$

Proof. Since $a \leq s \Longrightarrow (s : a) * s = a = s : (a * s)$, the lemma follows from Lemma 1.3. □

Observe that by Lemma 1.8, $a \vee b$ exists if and only if a and b have a common upper bound. Hence $(C; \leq)$ is a lattice if and only if $(C; \leq)$ is directed above.

Lemma 1.9. *If $a \vee b$ exists, then $c * (a \vee b) = (c * a) \vee (c * b)$ and $(a \vee b) : c = (a : c) \vee (b : c)$.*

Proof. Write $a \vee b = s$; then $a \leq s$ and $b \leq s$. Hence by Lemma 1.8, $c * (a \vee b) = c * (s : ((a * s) \wedge (b * s))) = (c * s) : ((a * s) \wedge (b * s)) = ((c * s) : (a * s)) \vee ((c * s) : (b * s))$ (by Lemma 1.3) $= (c * (s : (a * s))) \vee (c * (s : (b * s))) = (c * a) \vee (c * b)$. The other equation is similar. □

Lemma 1.10. *Let $(C; \leq, *, :, e)$ is a commutative extended BCK-algebra, $a, b \in C$ and assume $a \vee b$ exists in C. Then*

$$(a \vee b) : ((a \wedge b) * a) = b = (a : (a \wedge b)) * (a \vee b).$$

Proof. Assume $a \vee b$ exists; then $b = b \wedge (a \vee b) = (a \vee b) : ((b * (a \vee b)) \vee e) = (a \vee b) : ((b * a) \vee e) = (a \vee b) : ((a \wedge b) * a)$ by Lemma 1.9 and Corollary 1.4. Similarly, $b = (a : (a \wedge b)) * (a \vee b)$. □

Theorem 1.11. *Let $(C;\leq,*,:,e)$ be a commutative extended BCK-algebra, then the following are equivalent:*

(α) *$(C;\leq)$ is a distributive meet semilattice.*

(β) *$(C;\leq)$ is directed above.*

(γ) *$(C;\leq)$ is a distributive lattice.*

Proof. $(\alpha) \Longrightarrow (\beta)$ since any distributive meet semilattice is directed above; and $(\beta) \Longrightarrow (\gamma)$ by Lemmas 1.8 and 1.10. Finally, $(\gamma) \Longrightarrow (\alpha)$ is clear. □

Hence a commutative extended BCK-algebra is a lattice if it is directed above; and is a distributive lattice if it is a lattice.

Kühr [9] has shown that a commutative pseudo BCK-algebra can be equationally defined ([9, Theorem 4.2]) and whether a commutative extended BCK-algebra can be so defined is an open question.

2. Extended cone algebras

Let $(C;\leq,*,:,e)$ be a commutative extended BCK-algebra, then we say that C has *relative right cancellation property* (RRCP) if and only if, for $a,b,c \in C$,

(α) $c \leq a \wedge b$ and $c * a = c * b \Longrightarrow a = b$.

Dually, if for $a,b,c \in C$,

(β) $c \leq a \wedge b$ and $a : c = b : c \Longrightarrow a = b$,

then we say that C has *relative left cancellation property* (RLCP).

If C is integral (equivalently, a commutative pseudo BCK-algebra), then it is easy to see that (α) (= RRCP) and (β) (= RLCP) are equivalent ([9, Section 6] and [12]) and C is a cone algebra; however, the equivalence of (α) and (β) for commutative extended BCK-algebras is an open question.

We now generalise [12, Theorem 1.5] to the case of commutative extended BCK-algebras.

Theorem 2.1. *If $(C;\leq,*,:,e)$ is a commutative extended BCK-algebra, then C has RRCP if and only if C satisfies any of the following identities:*

(α) $((a*b)\vee e)\wedge((b*a)\vee e) = e$,

(β) $c*(a\wedge b) = (c*a)\wedge(c*b)$, *and*

(γ) $(((a*b)\wedge(a*c))\vee e)*((a*c)\vee e) = (((b*a)\wedge(b*c))\vee e)*((b*c)\vee e)$.

P r o o f. Observe that if $(C;\leq,*,:,e)$ is a semiintegral extended BCK-algebra, $a,b\in C$ and $a\leq b$, then by Lemma 1.2(viii), $e=a*a\leq a*b$ and similarly $e\leq b:a$. Now assume that C is a commutative extended BCK-algebra having RRCP, $a,b,c\in C$ and $c=a\wedge b$; then by above remark, $e\leq(c*a)\wedge(c*b)$ and hence $((c*a)\wedge(c*b))*(c*a)\leq e*(c*a)=c*a$. Hence if $x=a:(((c*a)\wedge(c*b))*(c*a))$ then $c*x=(c*a):(((c*a)\wedge(c*b))*(c*a))=(c*a)\wedge(c*b)$. Also $c\leq a$ and hence $c=a:(c*a)\leq x$ by Lemma 1.2(ii). Further, $e\leq((c*a)\wedge(c*b))*(c*a)$ and hence $x=a:(((c*a)\wedge(c*b))*(c*a))\leq a:e$ (by Lemma 1.2(ii)) $=a$. Thus $c\leq x\leq a$. Similarly, if $y=b:(((c*a)\wedge(c*b))*(c*b))$, then $c*y=(c*a)\wedge(c*b)$ and $c\leq y\leq b$. Hence by RRCP, $x=y$ (since $c*x=c*y$) and hence $c\leq x\leq a\wedge b=c$ so that $c=x$. Hence $(c*a)\wedge(c*b)=c*x=e$, and thus

$$((a*b)\vee e)\wedge((b*a)\vee e)=(c*a)\wedge(c*b)=c*x=e.$$

Hence RRCP $\Longrightarrow(\alpha)$. Now assume (α). By Lemma 1.2(viii), we have

$$c*(a\wedge b)\leq(c*a)\wedge(c*b).$$

Hence

$$\begin{aligned}
&(c*(a\wedge b))*((c*a)\wedge(c*b))\\
\leq\;&((c*(a\wedge b))*(c*a))\wedge((c*(a\wedge b))*(c*b))\\
=\;&\big[\big(c*(a:((b*a)\vee e))\big)*(c*a)\big]\wedge\big[(c*(b:((a*b)\vee e)))*(c*b)\big]\\
=\;&\big[\big((c*a):((b*a)\vee e)\big)*(c*a)\big]\wedge\big[\big((c*b):((a*b)\vee e)\big)*(c*b)\big]\\
\leq\;&((b*a)\vee e)\wedge((a*b)\vee e)\qquad\text{(by Definition 1.1(iv))}\\
=\;&e\qquad\text{(by }(\alpha)).
\end{aligned}$$

Hence $(c*a)\wedge(c*b)\leq c*(a\wedge b)$ and hence

$$c*(a\wedge b)=(c*a)\wedge(c*b).$$

Thus $(\alpha)\Longrightarrow(\beta)$; and now assume (β).

If $a\leq c$ and $b\leq c$, then $(a*c)\wedge(b*c)=\big(((a*c):(b*c))\vee e\big)*(a*c)=\big[(a*(c:(b*c)))\vee e\big]*(a*c)=((a*b)\vee e)*(a*c)$. Hence if $a,b,c\in C$ are arbitrary, then

$$\begin{aligned}((a\wedge c)*c)\wedge((b\wedge c)*c)&=(((a\wedge c)*(b\wedge c))\vee e)*((a\wedge c)*c)\\&=(((a\wedge c)*(b\wedge c))\vee e)*((a*c)\vee e).\end{aligned}\tag{A}$$

Now

$$\begin{aligned}(a\wedge c)*(b\wedge c)&=(a*(b\wedge c))\vee(c*(b\wedge c))\\&=((a*b)\wedge(a*c))\vee((c*b)\wedge e)\qquad\text{(by }(\beta))\end{aligned}$$

Hence

$$\begin{aligned}((a\wedge c)*(b\wedge c))\vee e&=((a*b)\wedge(a*c))\vee((c*b)\wedge e)\vee e\\&=((a*b)\wedge(a*c))\vee e.\end{aligned}$$

Hence from the symmetry of a and b in the left side of (A) and the equality (A) we obtain

$$\big(((a*b)\wedge(a*c))\vee e\big)*((a*c)\vee e)=\big(((b*a)\wedge(b*c))\vee e\big)*((b*c)\vee e).$$

Hence $(\beta)\Longrightarrow(\gamma)$ and now assume (γ). Let $c\le a\wedge b$ and $c*a=c*b$; then by the equation (γ),

$$\big(((c*a)\wedge(c*b))\vee e\big)*((c*b)\vee e)=\big(((a*c)\wedge(a*b))\vee e\big)*((a*b)\vee e)$$

This implies, $e=e*((a*b)\vee e)$ (since $a*c\le e$), so that $e=(a*b)\vee e$. Hence $a*b\le e$ and by symmetry $b*a\le e$. Hence $a=b$ and $(\gamma)\Longrightarrow$ RRCP. Thus RRCP, (α), (β) and (γ) are all equivalent. □

Dually, we have the following theorem.

Theorem 2.2. *If $(C;\le,*,:,e)$ is a commutative extended BCK-algebra, then C has RLCP if and only if C satisfies any of the following identities:*

(α) $((a:b)\vee e)\wedge((b:a)\vee e)=e$,

(β) $(a\wedge b):c=(a:c)\wedge(b:c)$, *and*

(γ) $((c:a)\vee e):(((c:a)\wedge(b:a))\vee e)=((c:b)\vee e):(((c:b)\wedge(a:b))\vee e)$.

This theorem follows from Theorem 2.1 by duality. We now assume that $(C;\le,*,:,e)$ is a commutative extended BCK-algebra enjoying both RRCP and RLCP. Now we prove

Lemma 2.3. *If $a*x=b=a*y$ and $x:b=a=y:b$, then $x=y$.*

P r o o f. Since C has RRCP, the equation (γ) of Theorem 2.1 holds in C; hence

$$\big(((a*x)\wedge(a*y))\vee e\big)*((a*y)\vee e)=\big(((x*a)\wedge(x*y))\vee e\big)*((x*y)\vee e)$$

$$\begin{aligned}\Longrightarrow\quad e &= \big(((x*a)\wedge(x*y))\vee e\big)*((x*y)\vee e)\\ \Longrightarrow\quad (x*y)\vee e &\le ((x*a)\wedge(x*y))\vee e\\ &\le (x*y)\vee e\\ \Longrightarrow\quad (x*y)\vee e &= ((x*a)\wedge(x*y))\vee e\end{aligned}\tag{B}$$

Now $x*a=x*(x:b)=e:b=b*e$ and also, $x*a=x*(y:b)=(x*y):b$. Hence

$$\begin{aligned}\big((x*a)\wedge(x*y)\big)\vee e &= \big[(x*a):\big(((x*y)*(x*a))\vee e\big)\big]\vee e\\ &= \big[(b*e):\big(((x*y)*((x*y):b))\vee e\big)\big]\vee e\\ &= \big[(b*e):((e:b)\vee e)\big]\vee e\\ &= \big[(b*e):((b*e)\vee e)\big]\vee e=e\end{aligned}$$

since $b*e\le(b*e)\vee e$ and hence $(b*e):((b*e)\vee e)\le e$. Hence $(x*y)\vee e=e$ by (B) and hence $x*y\le e$. By symmetry, $y*x\le e$ and hence $x=y$. □

By means of this Lemma 2.3, we can define a *partial* "multiplicative structure" on C with domain

$$D = \{(a,b) \in C \times C \mid a * x = b \text{ and } x : b = a \text{ for some } x \in C\}.$$

For $a, b \in D$, we define ab to be the unique solution of the pair of equations $a * x = b$ and $x : b = a$.

Suppose now that $(C; \leq, *, :, \cdot, e)$ is a BCK-monoid, $a, b \in C$ and there exists a unique $x \in C$ such that $a * x = b$ and $x : b = a$. Then $x \leq ab$ and hence by [10, Lemma 1.11] $b = a * x \leq a * ab$ [10, Lemma 2.4] $\leq b$ and hence $a * x = a * ab = b$; and dually, $x : b = ab : b = a$. Hence $x = ab$.

Consequently, if $(C; \leq, *, :, \cdot, e)$ is a BCK-monoid such that its reduct $(C; \leq, *, :, e)$ is a commutative extended BCK-algebra satisfying RRCP and RLCP, then by Lemma 2.3, the equations $a * x = b$ and $x : b = a$ have a unique common solution if one exists at all and hence if $(a, b) \in D$, then the partial product defined above must be the "total" product ab.

Lemma 2.4. *The following are equivalent in every commutative extended BCK-algebra enjoying RRCP and RLCP:*

(α) $x = ab$.

(β) $a * x = b$ *and* $x : b = a$.

Since $(a : (a \wedge b)) * a = a \wedge b$ and $a : (a \wedge b) = a : (a \wedge b)$ we have, by Lemma 2.4, the equation $a = (a : (a \wedge b))(a \wedge b)$. Similarly, we get the dual equation $a = (a \wedge b)((a \wedge b) * a)$. Thus we have

Lemma 2.5. *Under the hypothesis of* Lemma 2.4, *we have the equations*

$$(a \wedge b)((a \wedge b) * a) = a = (a : (a \wedge b))(a \wedge b).$$

Taking $a = b$ in the above, we get $a = a(a * a) = ae$ and $a = (a : a)a = ea$. Hence $ae = ea = a$ for all $a \in C$.

Lemma 2.6. *Under the hypothesis of* Lemma 2.4,

(1) *if* ab *and* ac *exist, then* $a(b \wedge c)$ *exists and* $a(b \wedge c) = (ab) \wedge (ac)$;

(2) *if* ba *and* ca *exist, then* $(b \wedge c)a$ *exists and* $(b \wedge c)a = (ba) \wedge (ca)$.

Proof. Assume ab and ac exist; then $a * ((ab) \wedge (ac)) = (a * ab) \wedge (a * ac) = b \wedge c$. Also, by Lemma 1.3,

$$\begin{aligned} (ab \wedge ac) : (b \wedge c) &= ((ab \wedge ac) : b) \vee ((ab \wedge ac) : c) \qquad \text{(by Theorem 2.2)} \\ &= ((ab : b) \wedge (ac : b)) \vee ((ab : c) \wedge (ac : c)) \\ &= (a \wedge (ac : b)) \vee ((ab : c) \wedge a) \\ &\leq a \vee a = a \end{aligned}$$

On the other hand, $(ab \wedge ac) : (b \wedge c)$ (by Theorem 2.2) $= (ab : (b \wedge c)) \wedge (ac : (b \wedge c))$ $=$ (by Lemma 1.3) $((ab : b) \vee (ab : c)) \wedge ((ac : b) \vee (ac : c)) = (a \vee (ab : c)) \wedge ((ac : b) \vee a) \geq a$. Hence $(ab \wedge ac) : (b \wedge c) = a$; and hence by Lemma 2.4, $a(b \wedge c) = (ab) \wedge (ac)$. The second part is similarly proved. □

Corollary 2.7. *If ab and ac exist, then $ab \leq ac \Longleftrightarrow b \leq c$; and if ba, ca exist, then $ba \leq ca \Longleftrightarrow b \leq c$.*

Proof. By Lemma 2.6, $b \leq c \Longrightarrow ab \wedge ac = a(b \wedge c) = ab$ and so $ab \leq ac$. Conversely, if $ab \leq ac$, then $b = a * ab \leq a * ac = c$. The second part is dual. □

Now let $(C; \leq, *, :, e)$ be as above and assume $a, b \in C$ and ab exists; then, by Lemma 1.2(viii), $c \leq ab$ implies $a*c \leq a*ab = b$ and also implies $c : b \leq ab : b = a$. If further, C is integral, then C is a cone algebra and hence $ab \in C$ and $a*c \leq b \Longrightarrow e = b*(a*c) = (a*ab)*(a*c) = (ab*a)*(ab*c) = e*(ab*c) = ab*c \Longrightarrow c \leq ab$. Hence in a cone algebra $c \leq ab \Longleftrightarrow a * c \leq b$ whenever ab is defined. However, if $(C; \leq, *, :, e)$ is a commutative extended BCK-algebra enjoying both RRCP and RLCP, it is an open question whether $ab \in C$ and $a * c \leq b \Longrightarrow c \leq ab$. We now assume further this happens and introduce the following definition.

Definition 2.8. A commutative extended BCK-algebra $(C; \leq, *, :, e)$ having RRCP and RLCP is called an *extended cone algebra* if and only if, for $a, b, c \in C$,

(K) ab exists and $a * c \leq b \Longrightarrow c \leq ab$.

By Lemma 1.2(v), the condition (K) is equivalent to:

ab exists and $c : b \leq a \Longrightarrow c \leq ab$.

Hence in an extended cone algebra $(C; \leq, *, :, e)$, for $ab \in C$,

$$c \leq ab \Longleftrightarrow a * c \leq b \Longleftrightarrow c : b \leq a.$$

We now assume that $(C; \leq, *, :, e)$ is an extended cone algebra and $a, b, \ldots \in C$.

Lemma 2.9. *If $ab \in C$ and $c \in C$, then $ab * c = b * (a * c)$ and $c : ab = (c : b) : a$.*

Proof. $ab * c \leq u \Longleftrightarrow ab * (c : u) \leq e \Longleftrightarrow c : u \leq ab \Longleftrightarrow a * (c : u) \leq b \Longleftrightarrow (a * c) : u \leq b \Longleftrightarrow b * (a * c) \leq u$. Hence $(ab) * c = b * (a * c)$; and the other equation follows similarly. □

Lemma 2.10. *ab and $(ab)c$ exist if and only if bc and $a(bc)$ exist; and in such a case, $(ab)c = a(bc)$.*

Proof. Assume $x = ab$ and $y = (ab)c$ exist; then $a * x = b$, $x : b = a$, $x * y = c$ and $y : c = x$. Now $(a*y) : c = a*(y : c) = a*x = b$ and $b*(a*y) = (ab)*y$ (Lemma 2.9) $= x * y = c$. Hence bc exists and $a * y = bc$. Also $y : (bc) = (y : c) : b = x : b = a$ (Lemma 2.9) and hence $y = a(bc)$. Hence $(ab)c = a(bc)$; the converse follows similarly. □

We recall that a BCK-monoid $(C; \leq, *, :, \cdot, e)$ is called a *Wajsberg monoid* [10, Definition 5.1] (or simply, a *W-monoid*) if and only if, for all $a, b \in C$,

$$a \leq b \Longrightarrow b : (a * b) = a = (b : a) * b.$$

By [10, Remark 5.4] and [5, Definition 2.3(ii), Lemma 2.4], a W-monoid $(C; \leq, *, :, \cdot, e)$, where $(C; \leq)$ is a lattice, is the dual of a GMV-algebra of Galatos and Tsinakis. Hence a GMV-algebra is equivalent to the dual of a residuated lattice satisfying the equations (α) in Theorem 1.6. It now follows that if $(C; \leq, *, :, \cdot, e)$ is a W-monoid which is a lattice, then its reduct $(C; \leq, *, :, e)$ is an extended cone algebra. Conversely, assume that $(C; \leq, *, :, \cdot, e)$ is a BCK-monoid and its reduct $(C; \leq, *, :, e)$ is a commutative extended BCK-algebra. Then by Definition 1.7 and Theorem 1.6(β), $(C; \leq, *, :, \cdot, e)$ is a W-monoid in which $a \vee e$ exists for all $a \in C$. Hence by [10, Theorem 5.5], $(C; \leq)$ is a lattice. Hence, in particular, if $(C; \leq, *, :, \cdot, e)$ is a W-monoid and its reduct $(C; \leq, *, :, e)$ is an extended cone algebra, then $(C; \leq)$ is necessarily a lattice.

Also, every cone algebra is an (integral) extended cone algebra and an extended cone algebra which is *not* a cone algebra is given in the following example.

Example 2.11. Let $(G; \leq, \cdot, e)$ be an ℓ-group; then $(G; \leq, *, :, e)$, where $a * b = a^{-1}b$ and $a : b = ab^{-1}$ is an extended cone algebra but *not* a cone algebra.

3. Decomposition theorem for an extended cone algebra

Let $(C; \leq, *, :, e)$ be an extended cone algebra and $a, b, c, \ldots \in C$.

Lemma 3.1. *If ac exists and $b \leq c$, then ab exists; and if ca exists and $b \leq c$, then ba exists.*

Proof. By Lemma 2.5, $b(b * c) = c = (c : b)b$ since $b \leq c$. Hence if ac exists, then $ac = a(b(b * c)) = (ab)(b * c)$ by Lemma 2.10 and so ab exists. The other assertion is proved similarly. □

An element $a \in C$ is said to be *invertible* if and only if there exists $b \in C$ such that $ab = ba = e$.

Lemma 3.2. *$a \in C$ is invertible if and only if $a(a * e) = (a * e)a = e$.*

Proof. Assume a is invertible, then there exists $b \in C$ such that $ab = ba = e$. Hence by the definition of *product* ab, we have $a * e = b$ and $b * e = e : b = a$. Hence $a(a * e) = ab = e$ and $(a * e)a = ba = e$; the converse is trivial. □

By Lemma 3.2, the inverse a^{-1} of an invertible element a equals $a * e = e : a$.

Lemma 3.3. *If a is invertible, then ax and xa exist for all $x \in C$.*

Proof. If $ab = ba = e$, then $x(ab) = xe = x = ex = (ba)x$. Hence xa and ax exist by Lemma 2.10. □

Lemma 3.4. *If $a \in C$, then $a \wedge e$ is invertible.*

Proof. By Lemma 2.5 and since $e : x = x * e$ for all $x \in C$, we have $(a \wedge e)((a \wedge e) * e) = e = (e : (a \wedge e))(a \wedge e) = ((a \wedge e) * e)(a \wedge e)$. Hence $a \wedge e$ is invertible. □

Corollary 3.5. *If $a \leq e$, then a is invertible.*

Lemma 3.6. *If a and b are invertible, then so is ab. (Observe that by* Lemma 3.3, *ab exists).*

Proof. Routine. □

Lemma 3.7. *If $a \leq b$ and b is invertible, then a is invertible.*

Proof. Since b is invertible, there exists $c \in C$ such that $bc = cb = e$. By Corollary 2.7, $ac \leq bc = e$ (Observe that ac exists by Lemma 3.3); and hence ac is invertible. Hence $a = a(cb) = (ac)b$ is invertible by Lemma 3.6. □

Corollary 3.8. *If $a \in C$, then $a * e$ is invertible.*

Proof. $a * e \leq (a \wedge e) * e$, which is invertible since $(a \wedge e) * e$ is the inverse of $a \wedge e$. Hence $a * e$ is invertible by Lemma 3.7. □

Lemma 3.9. *If a is invertible, then $a * x = a^{-1}x$ and $x : a = xa^{-1}$.*

Proof. If $u \in C$, then au is defined by Lemma 3.3. Hence $a * x \leq u \iff x \leq au \iff a^{-1}x \leq a^{-1}(au) = (a^{-1}a)u = u$. Hence $a * x = a^{-1}x$; and similarly $x : a = xa^{-1}$. □

We denote by J the set of all invertible elements of C; then J is a subalgebra of C by Lemma 3.9 and we have

Lemma 3.10. *$(J; \leq, \cdot, e)$ is an ℓ-group.*

Proof. By Lemma 3.3, the product function "$\cdot$" restricted to $J \times J$ is total. Hence ab is defined for $a, b \in J$. By Lemma 3.6, J is stable under $\cdot$ and by Lemma 2.10, $(J; \cdot, e)$ is a semigroup which is a group under routine verification. By Corollary 2.7, $(J; \leq, \cdot, e)$ is a pogroup and is an ℓ-group since $a \wedge e$ exists for all $a \in J$. □

Lemma 3.11. *If $a \vee b$ exists in C, then*

$$a \vee b = a((a * b) \vee e) = ((a : b) * e)b.$$

P r o o f. $a * (a \vee b) = (a * b) \vee e$ (Lemma 1.9) and $(a \vee b) : ((a * b) \vee e) = (a \vee b) : (a * (a \vee b)) = a$ since $a \leq a \vee b$. Hence $a \vee b = a((a * b) \vee e)$ and similarly, $a \vee b = ((a : b) \vee e)b$. □

Lemma 3.12. *If $a \wedge b = e$ and $a \vee b$ exists, then $a \vee b = ab = ba$.*

P r o o f. We have $b = e * b = (a \wedge b) * b = (a * b) \vee e$ and hence $a \vee b = a((a * b) \vee e)$ (Lemma 3.11) $= ab = ba$ by symmetry. □

Lemma 3.13. *If u is invertible and g is integral, then $gu = ug$.*

P r o o f. First assume that $e \leq u$; then $g \wedge u \leq u$ and hence is invertible by Lemma 3.7. Also, $e : (g \wedge u) = (g \wedge u) * e = (g * e) \vee (u * e) = e \vee (u * e) = e$ since $u * e \leq e$. Since $(g \wedge u) * e$ is inverse of $g \wedge u$, we have $g \wedge u = e$.

Now $e \leq u$ and $e \leq g$ and hence $g \leq gu$ and $u \leq gu$ so that $g \vee u$ exists. Hence by Lemma 3.12, $gu = ug = g \vee u$.

If $u \leq e$, then $e \leq u^{-1}$ and so by the above $gu^{-1} = u^{-1}g$ and hence $gu = ug$.

Now assume that u is an arbitrary invertible element; then by Lemma 2.5,

$$u = (u \wedge e)((u \wedge e) * u) = (u \wedge e)(u \vee e).$$

Hence $u \vee e = (u \wedge e)^{-1}u$ which is invertible by Lemma 3.6. Since $u \wedge e \leq e$ and $e \leq u \vee e$, g commutes with each of $u \wedge e$ and $u \vee e$ by the above and so commutes with their product u. Hence $gu = ug$. □

Lemma 3.14. $a(a * e) = (a * e)a$ *for all* $a \in C$.

P r o o f. By Corollary 3.8, $a * e$ is invertible and since $e : (a(a * e)) = (a(a * e)) * e = (a * e) * (a * e) = e$, $a(a * e)$ is integral. Hence by Lemma 3.13, $(a * e)a(a * e) = a(a * e)(a * e)$; and this implies $a(a * e) = (a * e)a$ since $a * e$ is invertible. □

Lemma 3.15. *If $a \in C$, there exists a unique pair u, g where u is invertible, g is integral and $a = ug = gu$.*

P r o o f. Let $a \in C$ and write $u = (a * e)^{-1}$ and $g = a(a * e) = (a * e)a$. Then u is invertible and g is integral; and $a = ug = gu$. Conversely, assume $a = ug$ ($= gu$) where u is invertible and g is integral. Then $a * e = (gu) * e = u * (g * e) = u * e = u^{-1}$ and hence $u = (a * e)^{-1}$ and hence $g = u^{-1}a = au^{-1}$ which gives $g = (a * e)a = a(a * e)$. □

We now prove the following theorem:

Theorem 3.16. *Every extended cone algebra is the direct product of the ℓ-group of its invertible elements and the cone algebra of its integral elements.*

Remark 3.17. [5] contains a similar theorem saying that every GMV-algebra is a direct product of an ℓ-group and an integral GMV-algebra.

Proof of Theorem 3.16. Let $a_1, a_2 \in C$ and write $a_1 = u_1 g_1$ and $a_2 = u_2 g_2$ (as in Lemma 3.15), where u_1, u_2 are invertible and g_1, g_2 are integral. We now prove

(1) $a_1 a_2$ exists if and only if $g_1 g_2$ exists and in such a case $a_1 a_2 = (u_1 u_2)(g_1 g_2)$,
(2) $a_1 \le a_2$ if and only if $u_1 \le u_2$ and $g_1 \le g_2$,
(3) $a_1 * a_2 = (u_1 * u_2)(g_1 * g_2)$,
(4) $a_1 : a_2 = (u_1 : u_2)(g_1 : g_2)$,
(5) $a_1 \wedge a_2 = (u_1 \wedge u_2)(g_1 \wedge g_2)$, and
(6) $a_1 \vee a_2$ exists if and only if $g_1 \vee g_2$ exists and in such a case $a_1 \vee a_2 = (u_1 \vee u_2)(g_1 \vee g_2)$.

(1) Assume $a_1 a_2$ exists; then $a_1 a_2 = (u_1 g_1)(u_2 g_2) = ((u_1 g_1) u_2) g_2 = (u_1 (g_1 u_2)) g_2 = (u_1 (u_2 g_1)) g_2 = ((u_1 u_2) g_1) g_2 = (u_1 u_2)(g_1 g_2)$ (by Lemmas 2.10 and 3.13) and hence $g_1 g_2$ exists (by Lemma 2.10). Conversely if $g_1 g_2$ exists, then $(u_1 u_2)(g_1 g_2)$ exists by Lemma 3.3 and equals $(u_1 g_1)(u_2 g_2)$ (by Lemmas 2.10 and 3.13 as above) $= a_1 a_2$.
(2) $a_1 \le a_2 \Longrightarrow u_1 g_1 \le u_2 g_2 \Longrightarrow g_1 \le u_1^{-1}(u_2 g_2) = (u_1^{-1} u_2) g_2 \Longrightarrow g_1 : g_2 \le u_1^{-1} u_2$. Hence $g_1 : g_2$ is invertible since $u_1^{-1} u_2$ is invertible (Lemma 3.7). But $g_1 : g_2$ is also integral since g_1 and g_2 are integral. Hence $(g_1 : g_2) * e = e$ and so $g_1 : g_2 = e$ since $(g_1 : g_2) * e$ is the inverse of $g_1 : g_2$. Hence $g_1 \le g_2$ and $e \le u_1^{-1} u_2$ and so $u_1 \le u_2$. Thus $a_1 \le a_2 \Longrightarrow u_1 \le u_2$ and $g_1 \le g_2$. The reverse implication is obvious.
(3) $a_1 * a_2 = (u_1 g_1) * a_2 = g_1 * (u_1 * a_2)$ (Lemma 2.9) $= g_1 * (u_1^{-1} a_2)$ (Lemma 3.9) $= g_1 * ((u_1^{-1} u_2) g_2) = g_1 * ((u_1^{-1} u_2)^{-1} * g_2)$ (Lemma 3.9) $= ((u_2^{-1} u_1) g_1) * g_2$ (Lemma 2.9) $= (g_1 (u_2^{-1} u_1)) * g_2$ (Lemma 3.13) $= (u_2^{-1} u_1) * (g_1 * g_2) = (u_1^{-1} u_2)(g_1 * g_2)$ (Lemma 3.9) $= (u_1 * u_2)(g_1 * g_2)$ (Lemma 3.9).
(4) Similar to (3).
(5) Follows from (1) and (2) or (3).
(6) By (1) and (2), $a_1 \vee a_2$ exists if and only if $u_1 \vee u_2$ and $g_1 \vee g_2$ both exist. However, $u_1 \vee u_2$ always exists by Lemma 3.10, hence (6). □

Corollary 3.18. *Let $(C; \le, *, :, e)$ be an extended cone algebra satisfying the following condition:*

(AC) *given $a, b \in C$, there exists $x \in C$ such that $a * x = b$ and $x : b = a$.*

Then C is isomorphic to the direct product of an ℓ-group and the cone algebra of an ℓ-group cone.

P r o o f. By the above theorem, the integral part of C is a cone algebra satisfying the condition (AC). Hence by the statement [2, 1.20], the integral part of C is the cone algebra of an ℓ-group cone. (Observe that if $a * x = b$ and C is a cone algebra then $x : b = a \iff x \wedge a = a \iff a \leq x \iff x * a = e$.) □

Remark 3.19. If $(C; \leq, *, :, e)$ is an extended cone algebra, then $(C; \leq)$ is a meet semilattice and by Lemmas 2.5, 2.10 and 3.12 and Corollary 2.7, C has the structure of a Bosbach's semiclan [1]. Whether every semiclan is an extended cone algebra is an open question.

Remark 3.20. Let C be a semiintegral extended BCK-algebra with greatest element u; then if $a \in C$, we have $a * e = a * (u : u) = (a * u) : u \leq e$ since $a * u \leq u$. Hence $e \leq a$ for all $a \in C$ and so by Theorem [10, 1.14], C is integral. Hence if $(C; \leq, *, :, e)$ is a commutative extended BCK-algebra bounded above, then C is a commutative pseudo BCK-algebra bounded above and hence, by Theorem 1.11, is a distributive lattice. Further, by Corollary [10, 5.8], C is a bounded precone algebra and hence is a brick ([2, 12]) and hence is equivalent to a pseudo MV-algebra ([11]). See Theorem [6, 3.14].

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