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VISCOSITY APPROXIMATION METHODS FOR MONOTONE MAPPINGS AND A COUNTABLE FAMILY OF NONEXPANSIVE MAPPINGS

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ABSTRACT. We use viscosity approximation methods to obtain strong convergence to common fixed points of monotone mappings and a countable family of nonexpansive mappings. Let C be a nonempty closed convex subset of a Hilbert space H and P_C is a metric projection. We consider the iteration process $\{x_n\}$ of C defined by $x_1 = x \in C$ is arbitrary and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S_n P_C(x_n + \lambda_n A x_n),$$

where f is a contraction on C, $\{S_n\}$ is a sequence of nonexpansive self-mappings of a closed convex subset C of H, and A is an inverse-strongly-monotone mapping of C into H. It is shown that $\{x_n\}$ converges strongly to a common element of the set of common fixed points of a countable family of nonexpansive mappings and the set of solutions of the variational inequality for an inverse-strongly-monotone mapping which solves some variational inequality. Finally, the ideas of our results are applied to find a common element of the set of equilibrium problems and the set of solutions of the variational inequality problem, a zero of a maximal monotone operator and a strictly pseudocontractive mapping in a real Hilbert space. The results of this paper extend and improve the results of Chen, Zhang and Fan.

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1. Introduction

Let C be a closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C. A mapping S of C into itself is called *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \tag{1.1}$$

for all $x, y \in C$. We denote by $\mathrm{F}(S)$ the set of fixed points of S. A mapping f of C into H is called *contraction* if there exists a constant $k \in (0,1)$ such that

$$\|fx - fy\| \leq k\|x - y\|,$$

for all $x, y \in C$. A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0,$$

for all $u, v \in C$, The *variational inequality problem* is to find $u \in C$ with that

$$\langle Au, v - u\rangle \geq 0,$$

for all $v \in C$. The set of solutions of variational inequality problem is denoted by $VI(C, A)$. A mapping A of C into H is called *α-inverse-strongly-monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \tag{1.2}$$

for all $u, v \in C$. It is obvious that any α-inverse-strongly-monotone mapping A is monotone and Lipschitz continuous. For finding an element of $\mathrm{F}(S) \cap VI(C, A)$ under the assumption that a set $C \subset H$ is closed and convex, a mapping S of C into itself is nonexpansive and a mapping A of C into H is α-inverse-strongly-monotone, Takahashi and Toyoda [17] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n) \tag{1.3}$$

for every $n = 0, 1, 2, \ldots$, where $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They shown that, if $\mathrm{F}(S) \cap VI(C, A) \neq \emptyset$, then such a sequence $\{x_n\}$ converges weakly to some $z \in P_{\mathrm{F}(S)\cap VI(C,A)}x$.

On the other hand, Moudafi [9] introduced the viscosity approximation method for nonexpansive mappings (see [19] for further developments in Banach spaces). Let f be a contraction on H, S is a nonexpansive self-mapping on a Hilbert space H. Starting with an arbitrary $x_0 \in H$, he defined a sequence $\{x_n\}$ generated by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Sx_n, \qquad n \geq 0, \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. It is proved ([9, 19]) that under certain appropriate condition imposed on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.4) converges strongly to the unique solution x^* in C of the variational inequality

$$\langle (I - f)x^*, x - x^*\rangle \geq 0 \qquad \text{for all} \quad x \in C. \tag{1.5}$$

Recently, Plubtieng and Punpaeng [13] introduced a new viscosity iterative scheme and proved some strong convergence theorems to a common element of the set of common fixed points of nonexpansive semigroups and the set of solutions of the variational inequality in Hilbert spaces.

In 2007, Chen, Zhang and Fan [4] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n) \tag{1.6}$$

for every $n = 0, 1, 2, \ldots$, where $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$, f is a contraction on C, S is a nonexpansive self-mapping of a closed convex subset C of a Hilbert space H. They proved that such a sequence $\{x_n\}$ converges strongly to a common element of the set of fixed points of nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly-monotone mapping which solves some variational inequality. Very recently, Aoyama et. al. [1] introduced an iterative sequence $\{x_n\}$ of C defined by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_n x_n, \qquad n \in \mathbb{N}, \tag{1.7}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$, C is a closed convex subset of H and $\{S_n\}$ is a sequence of nonexpansive mappings of C into itself with $\bigcap_{n=1}^{\infty} \mathrm{F}(S_n) \neq \emptyset$. They also proved that such a sequence converges strongly to a common fixed point of nonexpansive mappings. Plubtieng and Kumam [14] introduced an iterative process for finding the common element of the set of common fixed points of a countable family of nonexpansive mappings and the set of solutions of the variational inequality problem for an inverse-strongly-monotone mapping. They obtain a weak convergence theorem for a sequence generated by this process.

In this paper, we consider and analyze the following viscosity iterative method for an inverse-strongly-monotone mapping and a countable family of nonexpansive mappings. Let $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S_n P_C(x_n - \lambda_n A x_n) \tag{1.8}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$, P_C is a metric projection, f is a contraction mapping on C, $\{S_n\}$ is a sequence of nonexpansive mappings of a closed convex subset C of H, A is an α inverse-strongly-monotone mapping of C into H and $\{\lambda_n\} \subset (a, b) \subset (0, 2\alpha)$. We will prove in Section 3 that if the sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ of parameters satisfy appropriate conditions, then the sequence $\{x_n\}$ generated by (1.8) converges strongly to a common element of the set of common fixed points of a countable family of nonexpansive mappings and the set of solutions of the variational inequality for an inverse-strongly-monotone mapping which solves some variational inequality. Moreover, we apply our result to the problem for finding a common element of the set of equilibrium problems and the set solutions of the variational inequality problems for a monotone mapping.

2. Preliminaries

Let H be a real Hilbert space. Then

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle \tag{2.1}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in [0,1]$. It is also known that H satisfies

(1) the *Opial's condition* ([12]), that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

(2) the *Kadec-Klee property* ([7, 15]), that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$.

Let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \qquad \text{for all} \quad y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{2.3}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.4}$$

$$\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.5}$$

for all $x \in H$, $y \in C$.

In the context of the variational inequality problem, this implies that

$$u \in VI(C,A) \iff (\forall \lambda > 0)\big(u = P_C(u - \lambda A u)\big). \tag{2.6}$$

We note that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned}\|(I-\lambda A)u - (I-\lambda A)v\|^2 &= \|(u-v) - \lambda(Au - Av)\|^2 \\ &= \|u-v\|^2 - 2\lambda\langle u-v, Au-Av\rangle + \lambda^2\|Au-Av\|^2 \\ &\leq \|u-v\|^2 + \lambda(\lambda - 2\alpha)\|Au-Av\|^2. \end{aligned} \tag{2.7}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.1. ([18]) *Assume* $\{\alpha_n\}$ *is a sequences of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \delta_n, \qquad n \ge 0,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \ge 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n$ *exists.*

Lemma 2.2. ([1, Lemma 3.2]) *Let* C *be a nonempty closed subset of a Banach space and let* $\{T_n\}$ *be a sequence of nonexpansive mappings of* C *into itself. Suppose that* $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in C\} < \infty$. *Then, for each* $y \in C$, $\{T_n y\}$ *converges strongly to some point of* C. *Moreover, let* T *be a mapping of* C *into itself defined by*

$$Ty = \lim_{n\to\infty} T_n y \qquad \text{for all} \quad y \in C.$$

Then $\lim_{n\to\infty} \sup\{\|T_n z - Tz\| : z \in C\} = 0$.

3. Strong convergence theorems

In this section, we prove some strong convergence theorems for monotone mappings and a countable family of nonexpansive mappings.

Theorem 3.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\alpha > 0$ *and let* A *be an* α*-inverse-strongly monotone mapping of* C *into* H. *Let* $\{S_n\}$ *be a sequence of nonexpansive mappings from* C *into itself such that* $\bigcap_{n=1}^{\infty} \mathrm{F}(S_n) \cap VI(C,A) \neq \emptyset$, *and* f *be a contraction of* C *into itself. Suppose* $x_1 = x \in C$ *and let* $\{x_n\}$ *be the iterative sequence defined by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S_n P_C(x_n - \lambda_n A x_n)$$

for all $n = 0, 1, 2, \ldots$, *where* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{\lambda_n\} \subset [a,b] \subset (0,2\alpha)$ *satisfy* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : \ z \in B\} < \infty$ *for any bounded subset* B *of* C*. Let* S *be a mapping of* C *into itself defined by* $Sz = \lim_{n\to\infty} S_n z$ *for all* $z \in C$ *and suppose that* $\mathrm{F}(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(S_n)$*. Then* $\{x_n\}$ *converges strongly to* $z \in \mathrm{F}(S) \cap VI(C, A)$*, where* $z = P_{\mathrm{F}(S)\cap VI(C,A)} f(z)$*.*

P r o o f. Let $Q = P_{\mathrm{F}(S)\cap VI(C,A)}$. Then Qf is a contraction of H into C. In fact, there exists $k \in [0, 1)$ such that $\|f(x) - f(y)\| \le k\|x - y\|$ for all $x, y \in H$. So, we have that

$$\|Qf(x) - Qf(y)\| \le \|f(x) - f(y)\| \le k\|x - y\|$$

for all $x, y \in H$. This implies that Qf is a contraction on H into C. Since H is complete, there exists a unique element of $z \in H$, such that $z = Qf(z)$. Such a $z \in H$ is an element of C.

Put $y_n = P_C(x_n - \lambda_n Ax_n)$ for every $n \in \mathbb{N} \cup \{0\}$. Let $u \in \mathrm{F}(S) \cap VI(C, A)$. Since $I - \lambda_n A$ is nonexpansive and $u = P_C(u - \lambda_n Au)$ from (2.6), we have

$$\begin{aligned}
\|y_n - u\| &= \|P_C(x_n - \lambda_n Ax_n) - P_C(u - \lambda_n Au)\| \\
&\le \|(x_n - \lambda_n Ax_n) - (u - \lambda_n Au)\| \\
&\le \|(I - \lambda_n A)x_n - (I - \lambda_n A)u\| \\
&\le \|x_n - u\|
\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. From (2.7), we note that

$$\begin{aligned}
\|x_{n+1} - u\| &= \|\alpha_n(f(x_n) - u) + (1 - \alpha_n)(S_n y_n - u)\| \\
&\le \alpha_n\|f(x_n) - u\| + (1 - \alpha_n)\|S_n y_n - u\| \\
&\le \alpha_n(\|f(x_n) - f(u)\| + \|f(u) - u\|) + (1 - \alpha_n)\|x_n - u\| \\
&\le \alpha_n k\|x_n - u\| + \alpha_n\|f(u) - u\| + (1 - \alpha_n)\|x_n - u\| \\
&= (1 - (1 - k)\alpha_n)\|x_n - u\| + (1 - k)\alpha_n \left(\frac{1}{1 - a}\|f(u) - u\|\right) \\
&\le \max\Big\{\|x_n - u\|, \frac{1}{1 - k}\|f(u) - u\|\Big\}
\end{aligned}$$

for all $n \in \mathbb{N}$. By induction, we get

$$\|x_{n+1} - u\| \le \max\Big\{\|x_1 - u\|, \frac{1}{1 - k}\|f(u) - u\|\Big\}, \qquad n \ge 1. \tag{3.1}$$

Therefore $\{x_n\}$ is bounded. Hence, we also obtain that $\{y_n\}$, $\{S_n y_n\}$ and $\{f(x_n)\}$ are bounded.

Since $I-\lambda_n A$ is nonexpansive, we have

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|P_C(x_{n+1}-\lambda_n Ax_{n+1})-P_C(x_n-\lambda_n Ax_n)\|\\ &\le \|(I-\lambda_{n+1}A)x_{n+1}-(I-\lambda_n A)x_n\|\\ &\le \|(I-\lambda_{n+1}A)x_{n+1}-(I-\lambda_{n+1}A)x_n\|+|\lambda_n-\lambda_{n+1}|\|Ax_n\|\\ &\le \|(x_{n+1}-x_n)\|+|\lambda_n-\lambda_{n+1}|\|Ax_n\| \qquad (3.2)\end{aligned}$$

for all $n\in\mathbb{N}$. So, we obtain

$$\begin{aligned}&\|x_{n+1}-x_n\|\\ &=\|[\alpha_n f(x_n)+(1-\alpha_n)S_ny_n]-[\alpha_{n-1}f(x_{n-1})+(1-\alpha_{n-1})S_{n-1}y_{n-1}]\|\\ &=\|\alpha_n[f(x_n)-f(x_{n-1})]+(\alpha_n-\alpha_{n-1})f(x_{n-1})\\ &\quad+(1-\alpha_n)(S_ny_n-S_{n-1}y_{n-1})+(\alpha_{n-1}-\alpha_n)S_{n-1}y_{n-1}\|\\ &\le \alpha_n k\|x_n-x_{n-1}\|+|(\alpha_n-\alpha_{n-1}|(\|f(x_{n-1})\|+\|S_{n-1}y_{n-1}\|)\\ &\quad+(1-\alpha_n)(\|S_ny_n-S_ny_{n-1}\|+\|S_ny_{n-1}-S_{n-1}y_{n-1}\|)\\ &\le \alpha_n k\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|(\|f(x_{n-1})\|+\|S_{n-1}y_n\|)\\ &\quad+(1-\alpha_n)\|y_n-y_{n-1}\|+(1-\alpha_n)\sup\big\{\|S_nz-S_{n-1}z\|:\ z\in\{y_{n-1}\}\big\}\\ &\le \alpha_n k\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|(\|fx_{n-1}\|+\|S_{n-1}y_{n-1}\|)\\ &\quad+(1-\alpha_n)\big(\|x_n-x_{n-1}\|+|\lambda_{n-1}-\lambda_n|\|Ax_{n-1}\|\big) \qquad \text{(by 3.2)}\\ &\quad+(1-\alpha_n)\sup\big\{\|S_nz-S_{n-1}z\|:\ z\in\{y_n\}\big\}\\ &\le (1-(1-k)\alpha_n)\|x_n-x_{n-1}\|+|\lambda_{n-1}-\lambda_n|M+|\alpha_n-\alpha_{n-1}|L\\ &\quad+(1-\alpha_n)\sup\big\{\|S_nz-S_{n-1}z\|:\ z\in\{y_n\}\big\}\end{aligned}$$

for every $n\in\mathbb{N}$, where

$$L:=\sup_{n\ge1}\big\{\|fx_{n-1}\|+\|S_{n-1}y_{n-1}\|\big\} \quad\text{and}\quad M:=\sup_{n\ge1}\big\{\|Ax_n\|\big\}.$$

Since

$$\sum_{n=1}^{\infty}\alpha_n=\infty,\quad \sum_{n=1}^{\infty}|\lambda_n-\lambda_{n-1}|<\infty,\quad \sum_{n=1}^{\infty}|\alpha_n-\alpha_{n-1}|<\infty$$

and

$$\sum_{n=1}^{\infty}\big\{\|S_nz-S_{n+1}z\|:\ z\in\{y_n\}\big\}<\infty,$$

it follows by Lemma 2.1 that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \qquad (3.3)$$

Then we also obtain $\lim_{n\to\infty}\|y_{n+1}-y_n\|=0$. Moreover, we note that

$$\begin{aligned}
&\|x_n - S_n y_n\| \\
&\le \|x_n - S_{n-1}y_{n-1}\| + \|S_{n-1}y_{n-1} - S_n y_{n-1}\| + \|S_n y_{n-1} - S_n y_n\| \\
&\le \alpha_{n-1}\|f(x_{n-1}) - S_{n-1}y_{n-1}\| + \sup\big\{\|S_{n-1}z - S_n z\| :\ z \in \{y_{n-1}\}\big\} \\
&\quad + \|y_{n-1} - y_n\| \qquad \text{for all} \quad n \in \mathbb{N}.
\end{aligned}$$

Thus, we have

$$\lim_{n\to\infty}\|x_n - S_n y_n\| = 0. \tag{3.4}$$

From (2.7), we obtain

$$\begin{aligned}
&\|x_{n+1} - u\|^2 \\
&\le \alpha_n\|f(x_n) - u\|^2 + (1-\alpha_n)\|S_n y_n - u\|^2 \\
&\le \alpha_n\|f(x_n) - u\|^2 + (1-\alpha_n)\|y_n - u\|^2 \\
&= \alpha_n\|f(x_n) - u\|^2 + (1-\alpha_n)\|P_C(x_n - \lambda_n Ax_n) - u\|^2 \\
&\le \alpha_n\|f(x_n) - u\|^2 + (1-\alpha_n)\|(I-\lambda_n A)x_n - (I-\lambda_n A)u\|^2 \\
&\le \alpha_n\|f(x_n) - u\|^2 + (1-\alpha_n)(\|x_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Au\|^2) \\
&\le \alpha_n\|f(x_n) - u\|^2 + \|x_n - u\|^2 + (1-\alpha_n)a(b-2\alpha)\|Ax_n - Au\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
&-(1-\alpha_n)a(b-2\alpha)\|Ax_n - Au\|^2 \\
&\le \alpha_n\|f(x_n) - u\|^2 + \|x_n - u\|^2 - \|x_{n+1} - u\|^2 \\
&\le \alpha_n\|f(x_n) - u\|^2 + (\|x_n - u\| + \|x_{n+1} - u\|)\|x_{n+1} - x_n\|.
\end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$, it follows that $\|Ax_n - Au\| \to 0$. Further from (2.3), we obtain

$$\begin{aligned}
\|y_n - u\|^2 &= \|P_C(x_n - \lambda_n Ax_n) - P_C(u - \lambda_n Au)\|^2 \\
&\le \langle (x_n - \lambda_n Ax_n) - (u - \lambda_n Au), y_n - u\rangle \\
&= (1/2)\big\{\|(x_n - \lambda_n Ax_n) - (u - \lambda_n Au)\|^2 + \|y_n - u\|^2 \\
&\qquad - \|[(x_n - \lambda_n Ax_n) - (u - \lambda_n Au)] - (y_n - u)\|^2\big\} \\
&\le (1/2)\big\{\|x_n - u\|^2 + \|y_n - u\|^2 - \|(x_n - y_n) - \lambda_n(Ax_n - Au)\|^2\big\} \\
&= (1/2)\big\{\|x_n - u\|^2 + \|y_n - u\|^2 - \|(x_n - y_n)\|^2 \\
&\qquad + 2\lambda_n\langle x_n - y_n, Ax_n - Au\rangle - \lambda_n^2\|Ax_n - Au\|^2\big\}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}\|y_n - u\|^2 &\le \|x_n - u\|^2 - \|x_n - y_n\|^2 \\ &\quad + 2\lambda_n\langle x_n - y_n, Ax_n - Au\rangle - \lambda_n^2\|Ax_n - Au\|^2\end{aligned}$$

and hence

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)S_n y_n - u\|^2 \\ &\le \alpha_n\|f(x_n) - u\|^2 + (1-\alpha_n)\|y_n - u\|^2 \\ &\le \alpha_n\|f(x_n) - u\|^2 + \|x_n - u\|^2 - \|x_n - y_n\|^2 \\ &\quad + 2\lambda_n\langle x_n - y_n, Ax_n - Au\rangle - \lambda_n^2\|Ax_n - Au\|^2.\end{aligned}$$

Since $\alpha_n \to 0$, $\|x_{n+1} - x_n\| \to 0$ and $\|Ax_n - Au\| \to 0$, we have

$$\|x_n - y_n\| \to 0. \tag{3.5}$$

From $\|S_n y_n - y_n\| \le \|S_n y_n - x_n\| + \|x_n - y_n\|$, we obtain

$$\|S_n y_n - y_n\| \to 0. \tag{3.6}$$

Next we show that

$$\limsup_{n\to\infty}\langle f(z) - z, S_n y_n - z\rangle \le 0,$$

where $z = P_{\mathrm{F}(S)\cap VI(C,A)} f(z)$. To show it, choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z) - z, S_n y_n - z\rangle = \lim_{i\to\infty}\langle f(z) - z, S_n y_{n_i} - z\rangle.$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ converges weakly to w. We may assume without loss of generality that $y_{n_i} \rightharpoonup w$. Since $\|S_n y_n - y_n\| \to 0$, we obtain $S_{n_i} y_{n_i} \rightharpoonup w$. We now show that $w \in \mathrm{F}(S) \cap VI(C,A)$.

First, it follows by the same argument as in the proof of [8, Theorem 3.1, pp. 346-347] that $z \in VI(C,A)$. Let us show that $w \in \mathrm{F}(S)$. Assume $w \notin \mathrm{F}(S)$. From Opial's condition, we have

$$\begin{aligned}\liminf_{n\to\infty}\|y_{n_i} - w\| &< \liminf_{i\to\infty}\|y_{n_i} - Sw\| \\ &= \liminf_{i\to\infty}\|y_{n_i} - S_{n_i}y_{n_i} + S_{n_i}y_{n_i} - Sy_{n_i} + Sy_{n_i} - Sw\| \\ &\le \liminf_{i\to\infty}\|Sy_{n_i} - Sw\| \\ &\le \liminf_{i\to\infty}\|y_{n_i} - w\|\end{aligned}$$

This is a contradiction. Thus, we obtain $w \in \mathrm{F}(S)$. Therefore $w \in \mathrm{F}(S) \cap VI(C, A)$. Since $z = P_{\mathrm{F}(S)\cap VI(C,A)} f(z)$, we have

$$\limsup_{n\to\infty} \langle f(z) - z), S_n y_n - z\rangle = \lim_{i\to\infty} \langle f(z) - z, S_n y_{n_i} - z\rangle$$
$$= \langle f(z) - z, w - z\rangle \le 0$$

for all $n \ge m$. For all $n \ge m$, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n) S_n y_n - z\|^2 \\
&= \|\alpha_n (f(x_n) - z) + (1-\alpha_n)(S_n y_n - z)\|^2 \\
&\le \alpha_n^2 \|(f(x_n) - z)\|^2 + 2\alpha_n (1-\alpha_n)\langle f(x_n) - z, S_n y_n - z\rangle \\
&\quad + (1-\alpha_n)^2 \|S_n y_n - z\|^2 \\
&\le \alpha_n^2 \|(f(x_n) - z)\|^2 + (1-\alpha_n)^2 \|x_n - z\|^2 \\
&\quad + 2\alpha_n (1-\alpha_n)\langle f(x_n) - f(z), S_n y_n - z\rangle \\
&\quad + 2\alpha_n (1-\alpha_n)\langle f(z) - z, S_n y_n - z\rangle \\
&\le \alpha_n^2 \|(f(x_n) - z)\|^2 + (1 - 2\alpha_n + \alpha_n^2)\|x_n - z\|^2 \\
&\quad + 2\alpha_n (1-\alpha_n) k\|x_n - z\|^2 + 2\alpha_n (1-\alpha_n)\langle f(z) - z, S_n y_n - z\rangle \\
&= [1 - 2\alpha_n + \alpha_n^2 + 2k\alpha_n(1-\alpha_n)]\|x_n - z\|^2 + \alpha_n^2 \|f(x_n) - z\|^2 \\
&\quad + 2\alpha_n (1-\alpha_n)\langle f(z) - z, S_n y_n - z\rangle \\
&= (1 - \bar{\alpha}_n)\|x_n - z\|^2 + \bar{\alpha}_n \bar{\beta}_n,
\end{aligned}$$

where

$$\bar{\alpha}_n = 2\alpha_n + \alpha_n^2 + 2k\alpha_n(1-\alpha_n),$$
$$\bar{\beta}_n = \frac{\alpha_n \|f(x_n) - z\|^2 + 2(1-\alpha_n)\langle f(z) - z, S_n y_n - z\rangle}{2 + \alpha_n + 2k(1-\alpha_n)}.$$

It is easily to see that $\bar{\alpha}_n \to 0$, $\sum_{n=1}^{\infty} \bar{\alpha}_n = \infty$ and $\limsup_{n\to\infty} \bar{\beta}_n \le 0$. Hence, by Lemma 2.1, we obtain $x_n \to z = P_{\mathrm{F}(S)\cap VI(C,A)} f(z)$. This completes the proof. □

Putting $f(y) = x \in C$ for all $y \in H$ in Theorem 3.1, we have the following result.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} \mathrm{F}(S_n) \cap VI(C, A) \ne \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) S_n P_C (x_n - \lambda_n A x_n)$$

for all $n = 0, 1, 2, \ldots$, *where* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 2\alpha)$ *with* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ *for any bounded subset* B *of* C *and* S *be a mapping of* C *into itself defined by* $Sz = \lim_{n\to\infty} S_n z$ *for all* $z \in C$ *and suppose that* $\mathrm{F}(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(S_n)$. *Then* $\{x_n\}$ *converges strongly to* $z \in \mathrm{F}(S) \cap VI(C,A)$, *where* $z = P_{\mathrm{F}(S)\cap VI(C,A)} x_1$.

P r o o f. It follows by Theorem 3.1 that $x_n \to z$, where $z = P_{\mathrm{F}(S)\cap VI(C,A)} x_1$. $\square$

Setting $S_n \equiv S$ in Theorem 3.1 and 3.2, we have the following results.

Corollary 3.3. (Chen, Zhang and Fan [4]) *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\alpha > 0$ *and let* A *be an* α*-inverse-strongly monotone mapping of* C *into* H. *Let* S *be a sequence of nonexpansive mappings from* C *into itself such that* $\mathrm{F}(S) \cap VI(C,A) \neq \emptyset$. *Let* f *be a contraction of* C *into itself. Suppose* $x_1 = x \in C$ *and* $\{x_n\}$ *is given by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S P_C(x_n - \lambda_n A x_n)$$

for all $n = 0, 1, 2, \ldots$, *where* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 2\alpha)$ *with* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then $\{x_n\}$ *converges strongly to* $z \in \mathrm{F}(S) \cap VI(C,A)$, *where*

$$z = P_{\mathrm{F}(S)\cap VI(C,A)} f(z).$$

By using the same argument in the proof of Theorem 3.1, we have the following theorem.

Theorem 3.4. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\{S_n\}$ *be a sequence of nonexpansive mappings from* C *into itself such that* $\bigcap_{n=1}^{\infty} \mathrm{F}(S_n) \neq \emptyset$ *and* f *be a contraction of* C *into itself. Suppose* $x_1 = x \in C$ *and let* $\{x_n\}$ *be the iterative sequence defined by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S_n x_n$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0,1)$. Suppose that

$$\sum_{n=1}^{\infty} \sup\big\{\|S_{n+1}z - S_n z\| :\ z \in B\big\} < \infty$$

for any bounded subset B of C. Let S be a mapping defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in C$ and suppose that $\mathrm{F}(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(S_n)$. Then $\{x_n\}$ converges strongly to $z \in \mathrm{F}(S)$, where $z = P_{\mathrm{F}(S)} f(z)$.

P r o o f. Put $Q = P_{\mathrm{F}(S)}$ and $y_n = x_n$ in the proof of Theorem 3.1. By using the same argument as in the proof of Theorem 3.1, we can show that $\{x_n\}$ converges strongly to a point $z \in \mathrm{F}(S)$, where $z = P_{\mathrm{F}(S)} f(z)$. □

4. Applications

4.1. Equilibrium problems

Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F\colon C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x,y) \ge 0 \qquad \text{for all} \quad y \in C. \tag{4.1}$$

The set of solutions of (4.1) is denoted by $\mathrm{EP}(F)$. Numerous problems in physics, optimization, and economics reduce to find a solution of (4.1). Some methods have been proposed to solve the equilibrium problem (see [2, 6, 10, 16]). In 2005, Combettes and Hirstoaga [5] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(F)$ is nonempty and they also proved a strong convergence theorem.

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions (see [2]):

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \le 0$ for any $x, y \in C$;

(A3) F is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t\to 0^+} F(tz + (1-t)x, y) \le F(x,y);$$

(A4) $F(x,\cdot)$ is convex and lower semicontinuous for each $x \in C$.

By [2, Corollary 1], [5, Lemma 2.12] and [11, Theorem 16], we have the following lemmas.

Lemma 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)–(A4) *and let $r > 0$ and $x \in H$. Then there exists unique $x^* \in C$ such that*

$$F(x^*, y) + \frac{1}{r}\langle y - x^*, x^* - x\rangle \geq 0 \qquad \text{for all} \quad y \in C.$$

Moreover, let T_r be a mapping of H into C defined by

$$T_r(x) = x^*$$

for all $x \in H$. Then, the following hold:

(i) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(ii) $\mathrm{F}(T_r) = \mathrm{EP}(F)$;

(iii) $\mathrm{EP}(F)$ *is closed and convex.*

Lemma 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)–(A4). *Let $\{r_n\}$ be a sequence of positive integers and T_{r_n} be mapping defined as in* Lemma 4.1. *Let $\{r_n\}$ be a sequence in $(0, \infty)$ such that* $\inf\{r_n : n \in \mathbb{N}\} > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, *then the following hold:*

(i) $\sum_{n=1}^{\infty} \sup\{\|T_{r_{n+1}} z - T_{r_n} z\| : z \in B\} < \infty$ *for any bounded subset B of C,*

(ii) $F(T) = \bigcap_{n=1}^{\infty} \mathrm{F}(T_{r_n})$ *where T is a mapping defined by $Tx = \lim_{n\to\infty} T_{r_n} x$ for all $x \in C$.*

Using Theorem 3.1 and Lemma 4.2, we have the following theorem.

Theorem 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)–(A4). *Let A be an α-inverse-strongly monotone mapping of C into H such that $VI(C, A) \cap \mathrm{EP}(F) \neq \emptyset$. Let f be a contraction of H into itself. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and*

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n) \\ F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq 0 \qquad \text{for all} \quad y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) u_n \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$ satisfy $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ with $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|$

$< \infty$ and $\{r_n\}$ is a sequence in $(0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ converges strongly to $w \in VI(C,A) \cap \mathrm{EP}(F)$, moreover $w = P_{\mathrm{EP}(F)\cap VI(C,A)} f(w)$.

Using Theorem 3.4 and Lemma 4.2, we have the following theorem.

Theorem 4.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4) with $EP(F) \neq \emptyset$ and let f be a contraction of C into itself. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x \in C$ and*

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \text{for all} \quad y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)u_n \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{r_n\}$ is a sequence in $(0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ converges strongly to $w \in \mathrm{EP}(F)$, moreover $w = P_{\mathrm{EP}(F)} f(w)$.

4.2. Accretive operator

In this section, we consider the problem of finding a zero of an accretive operator. Let E be a real Banach space. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0,2]$. Observe that every p-uniform convex Banach space is uniformly convex. One should note that no Banach space is p-uniform convex for $1 < p < 2$. It is well known that a Hilbert space is 2-*uniformly convex*, uniformly smooth. For each $p > 1$, the *generalized duality mapping* $J_p \colon E \to 2^{E^*}$ is defined by $J_p(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^p,\ \|x^*\| = \|x\|^{p-1}\}$ for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping.* If E is a Hilbert space, then $J = I$, where I is the identity mapping. An operator $A \subset E \times E$ is said to be accretive if for each (x_1, y_1) and $(x_2, y_2) \in A$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$. An accretive operator A is said to satisfy the range condition of $\overline{D(A)} \subset R(I + \lambda A)$ for all $\lambda > 0$, where $D(A)$ is the domain of A, $R(I + \lambda A)$ is the range of $I + \lambda A$, and $\overline{D(A)}$ is the closure of $D(A)$. If A is an accretive operator which satisfies the range condition, then we can define, for each $\lambda > 0$, a mapping $J_\lambda \colon R(I + \lambda A) \to D(A)$ by $J_\lambda = (I - \lambda A)^{-1}$, which is called the resolvent of A. We know that J_λ is nonexpansive and $F(J_\lambda) = A^{-1}(0)$ for all $\lambda > 0$. An accretive operator A is said to be *m-accretive* if $R(I + \lambda A) = E$ for all $\lambda > 0$ (see also [1]).

Lemma 4.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T \subset H \times H$ be an accretive operator such that $T^{-1}(0) \neq \emptyset$ and $\overline{D(T)} \subset C \subset \bigcap_{r>0} R(I + rT)$, and $\{r_n\}$ be a sequence in $(0, \infty)$. If $\inf\{r_n : n \in \mathbb{N}\} > 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, then the followings hold:*

(i) $\sum_{n=1}^{\infty} \sup\{\|J_{r_{n+1}} z - J_{r_n} z\| : z \in B\} < \infty$ *for any bounded subset B of C,*

(ii) $\mathrm{F}(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(J_{r_n})$, *where S is a mapping defined by $Sx = \lim_{n\to\infty} J_{r_n} x$ for all $x \in C$.*

Using Theorem 3.1 and Lemma 4.5, we have the following theorem.

Theorem 4.6. *Let $T \subset H \times H$ be an m-accretive operator with $T^{-1}(0) \neq \emptyset$ and let $C := \overline{D(T)}$. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H and let f be a contraction of C into itself. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) J_{r_n} P_C(x_n - \lambda_n A x_n)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$, $\{r_n\}$ is a sequence in $(0, \infty)$ and $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$ satisfy $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. Suppose that S is a mapping defined by $Sx = \lim_{n\to\infty} J_{r_n} x$ for all $x \in C$. If $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\inf\{r_n : n \in \mathbb{N}\} > 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, then $\{x_n\}$ converges strongly to $z \in T^{-1}(0) \cap VI(C, A)$, where $z = P_{T^{-1}(0) \cap VI(C,A)} f(z)$.

P r o o f. Since H is Hilbert space, $C = \overline{D(T)}$ is closed and convex. By Lemma 4.5, we have the following

$$\mathrm{F}(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(J_{r_n}) = T^{-1}(0) \neq \emptyset.$$

Therefore, by Theorem 3.1, we obtain $\{x_n\}$ converges strongly to $z = P_{\mathrm{F}(S) \cap T^{-1}(0)} f(z)$. □

Using Theorem 3.4 and Lemma 4.5, we have the following theorem.

Theorem 4.7. (Chen and Zhu [3, Theorem 3.2]) *Let $T \subset H \times H$ be an m-accretive operator with $T^{-1}(0) \neq \emptyset$ and let $C := \overline{D(T)}$. Let f be a contraction of C into itself. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J_{r_n} x_n \qquad \textit{for all} \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$, $\{r_n\}$ is a sequence in $(0,\infty)$. If $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\inf\{r_n : n \in \mathbb{N}\} > 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, then $\{x_n\}$ converges strongly to $z \in T^{-1}(0)$, where $z = P_{T^{-1}(0)} f(z)$.

4.3. Strictly pseudocontractive mapping

A mapping $T\colon C \to C$ is called strictly pseudocontractive on C if there exists k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x + (I - T)y\|^2 \qquad \text{for all} \quad x, y \in C.$$

If $k = 0$, then T is nonexpansive. Put $A = I - T$, where $T\colon C \to C$ is a strictly pseudocontractive mapping with k. We know that, A is $\frac{1-k}{2}$-inverse strongly monotone and $A^{-1}(0) = \mathrm{F}(T)$ (see [8]).

Now, using Theorem 3.1 we state a strong convergence theorem for a pair of a nonexpansive mapping and strictly pseudocontractive mapping as follows.

Theorem 4.8. *Let C be a closed convex subset of a real Hilbert space H. Let $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself. Let T be a strictly pseudocontractive mapping with constant k of C into itself such that $\bigcap_{n=1}^{\infty} \mathrm{F}(S_n) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S_n P_C((1-\lambda_n)x_n + \lambda_n T x_n)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$, $\{r_n\}$ is a sequence in $(0,\infty)$ and $\{\lambda_n\} \subset [a,b] \subset (0, 2\alpha)$ satisfy $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. Suppose that S is a mapping defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in C$. If $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\inf\{r_n : n \in \mathbb{N}\} > 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, then $\{x_n\}$ converges strongly to $z \in \mathrm{F}(S) \cap \mathrm{F}(T)$, where $z = P_{\mathrm{F}(S)\cap\mathrm{F}(T)} f(z)$.

Proof. Put $A = I - T$. Then A is $\frac{1-k}{2}$–inverse-strongly monotone. We have that $\mathrm{F}(T)$ is the solution set of $VI(A, C)$ i.e., $\mathrm{F}(T) = VI(A, C)$ and

$$P_C(x_n - \lambda_n A x_n) = (1-\lambda_n)x_n + \lambda_n T x_n.$$

Therefore, by Theorem 3.1, the conclusion follows. □

Setting $f(y) = x$ for all $y \in C$ in Theorem 4.8, we have the following corollary.

Corollary 4.9. *Let C be a closed convex subset of a real Hilbert space H. Let $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself and let T be a strictly pseudocontractive mapping with constant k of C into itself such that $\bigcap_{n=1}^{\infty} \mathrm{F}(S_n) \cap \mathrm{F}(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n) S_n P_C((1-\lambda_n)x_n - \lambda_n T x_n)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$, $\{r_n\}$ is a sequence in $(0,\infty)$ and $\{\lambda_n\} \subset [a,b] \subset (0,2\alpha)$ satisfy $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. Suppose that S is a mapping defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in C$. If $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\inf\{r_n :\ n \in \mathbb{N}\} > 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, then $\{x_n\}$ converges strongly to $z \in \mathrm{F}(S) \cap \mathrm{F}(T)$, where $z = P_{\mathrm{F}(S)\cap \mathrm{F}(T)} x_1$.

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