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BINOMIAL SUMS INVOLVING HARMONIC NUMBERS

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ABSTRACT. This paper develops the approach to the evaluation of a class of infinite series that involve special products of binomial type, generalized harmonic numbers of order 1 and rational functions. We give new summation results for certain infinite series of non-hypergeometric type. New formulas for the number π are included.

1. Introduction

The number π has been in the centre of view of mathematicians for centuries. Lambert proved the irrationality of π in 1761 and in 1794 Legendre proved that π is not a square root of a rational number. See also [10]. In 1882 Lindemann comes with the proof that π is a transcendental number. Up to date there exist more than 50 different proofs of the transcendence of π. Among them let us mention e.g. [9]. There is a lot of ways how to express the number π. The purpose of this note is to develop a new class of series summing up to π. These series involve the central binomial coefficient $\binom{2n}{n}$, the generalized harmonic number of order 1, i.e. $\sum_{j=1}^{n} \frac{1}{j\alpha^j}$, $(\alpha, n) \in \mathbb{R}^+ \times \mathbb{N}$ and a special kind of rational functions.

The central binomial coefficient $\binom{2n}{n}$ plays in the theory of the infinite series an important role by their transformations and accelerations. In 1979 Apéry [2] proved the irrationality of $\zeta(3)$ — and in the same manner, the irrationality of $\zeta(2)$ — by use of the formulae $\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3\binom{2n}{n}}$ and $\zeta(2) = 3 \sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}}$.

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Such types of Cantor and factorial series can be also found in [11]–[15] or in [19]. Other important results with evaluations and transformations of infinite series can be seen in [1], [5], [7], [16], [18], [21] or [22].

Similar importance play harmonic numbers of order 1 defined by $H_n := \sum\limits_{j=1}^{n} \frac{1}{j}$ for every positive integer n which can be expressed with the help of various definite integrals or infinite series. Harmonic numbers arise by the transformations of the Riemann function $\zeta(x)$. Significant results involving infinite series with the term H_n were presented by many authors, c.f. [6] or [8]. An interesting consideration connected with harmonic numbers is presented in [20].

It is not typical to analyse the infinite series with harmonic numbers, binomial coefficients and a rational function. We generalize the construction of such infinite series by considering the generalized harmonic numbers of order 1 and a product of the binomial type introduced in the next definitions. A similar type of this infinite series occurs by the computation of the hyperbolic volume V of the Haleman Ferguson's sculpture *Figure — Eight Complement II*, see [4].

Definition 1.1. For every ordered pair $(\alpha, n) \in \mathbb{R}^+ \times \mathbb{N}$, we define the generalized harmonic number $H_n(\alpha)$ by

$$H_n(\alpha) := \sum_{j=1}^{n} \frac{1}{j\alpha^j}.$$

Similarly as the usual harmonic numbers $H_n(1)$, there exists a known integral representation in the form $H_n(\alpha) = \int\limits_{L(\alpha)}^{1} \frac{1-(1-x)^n}{x}\,\mathrm{d}x$ with $L(\alpha) := 1 - \frac{1}{\alpha}$. This fact can be easily used to show that

$$\sum_{n=1}^{\infty} \frac{H_n(\alpha)}{q^n} = \frac{q}{q-1} \cdot \ln \frac{\alpha q}{\alpha q - 1}, \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{n \cdot H_n(\alpha)}{q^n} = \frac{q}{(q-1)^2} \cdot \left(\ln \frac{\alpha q}{\alpha q - 1} + \frac{q-1}{\alpha q - 1} \right). \tag{2}$$

The identities are valid for all $(\alpha, q) \in \mathbb{R}^2$ with $|q| > 1$ and $\alpha \geq 1$.

Definition 1.2. Assume that $(a, b, m_1, m_2) \in (\mathbb{R}_0^+)^2 \times \mathbb{N}^2$. For $(j, n) \in \mathbb{N}_0 \times \mathbb{N}$ define

$$\varphi_n \begin{pmatrix} m_1, & m_2 \\ a, & b \end{pmatrix} := \prod_{s=1}^{m_1 n} (s+a) \prod_{s=1}^{m_2 n} (s+b) \prod_{s=1}^{(m_1+m_2)n+1} (s+a+b)^{-1},$$

$$\varphi_{n,j} \begin{pmatrix} m_1, & m_2 \\ a, & b \end{pmatrix} := \prod_{s=m_1 n+1}^{m_1(n+j)} (s+a) \prod_{s=m_2 n+1}^{m_2(n+j)} (s+b) \prod_{s=(m_1+m_2)n+2}^{(m_1+m_2)(n+j)+1} (s+a+b)^{-1}.$$

Using the definition of Beta function $\mathrm{B}(x, y)$ by means of the Eulerian integral, we obtain the integral representation of the term φ_n in the form

$$\varphi_n \begin{pmatrix} m_1, & m_2 \\ a, & b \end{pmatrix} = \frac{1}{(a+b+1)\,\mathrm{B}(a+1, b+1)} \int\limits_0^1 t^{m_1 n + a}(1-t)^{m_2 n + b}\,\mathrm{d}t \tag{3}$$

for every positive integer n.

In the later considerations of the infinite series with harmonic numbers, central binomial coefficient $\binom{2n}{n}$ and a rational function, we introduce the notation $\phi_1(q)$ and $\phi_2(q)$ for the infinite series

$$\phi_1(q) := \sum_{n=1}^{\infty} \frac{H_n(1)}{q^n(2n+1)\binom{2n}{n}} \qquad \text{and} \qquad \phi_2(q) := \sum_{n=1}^{\infty} \frac{H_n(1)}{q^n\binom{2n}{n}}$$

which sums are not known in closed form.

2. Main results

In this section we give the overview of the main results connected with infinite series involving generalized harmonic numbers $H_n(\alpha)$, products of binomial type and rational functions.

Theorem 2.1. *Assume that* $(\alpha, a, b, q, m_1, m_2, k) \in (\mathbb{R}_0^+)^4 \times \mathbb{N}^2 \times \mathbb{N}_0$ *with* $\alpha \geq 1$ *and* $|q| > \frac{1}{4}$. *Define*

$$\sigma_1 := \sum_{n=1}^{\infty} \frac{H_n(\alpha)}{q^n} \cdot \varphi_n \begin{pmatrix} m_1, & m_2 \\ a, & b \end{pmatrix} \sum_{j=0}^{k} \frac{\binom{k}{j}}{(-q)^j} \cdot \varphi_{n,j} \begin{pmatrix} m_1, & m_2 \\ a, & b \end{pmatrix},$$

$$\sigma_2 := \sum_{n=1}^{\infty} \frac{n \cdot H_n(\alpha)}{q^n} \cdot \varphi_n \begin{pmatrix} m_1, & m_2 \\ a, & b \end{pmatrix} \sum_{j=0}^{k} \frac{\binom{k}{j}}{(-q)^j} \cdot \varphi_{n,j} \begin{pmatrix} m_1, & m_2 \\ a, & b \end{pmatrix}.$$

Then

$$\sigma_1 = \frac{1}{(a+b+1)\,\mathrm{B}(a+1, b+1)} \int\limits_0^1 x^a(1-x)^b \left(1 - \frac{x^{m_1}(1-x)^{m_2}}{q}\right)^{k-1} \times \\ \times \ln \frac{\alpha q}{\alpha q - x^{m_1}(1-x)^{m_2}}\,\mathrm{d}x \tag{4}$$

and

$$\sigma_2 = \frac{q}{(a+b+1)\,\mathrm{B}(a+1,b+1)} \int_0^1 x^{a+m_1}(1-x)^{b+m_2}\left(1-\frac{x^{m_1}(1-x)^{m_2}}{q}\right)^{k-2} \times \left(\ln\frac{\alpha q}{\alpha q - x^{m_1}(1-x)^{m_2}} + \frac{q - x^{m_1}(1-x)^{m_2}}{\alpha q - x^{m_1}(1-x)^{m_2}}\right) \mathrm{d}x. \tag{5}$$

Example 2.1.

$$\sum_{n=1}^{\infty} \frac{H_n(\alpha)}{\binom{2n}{n}} \cdot \frac{6n^2+25n+25}{8n^3+36n^2+46n+15} = \begin{cases} 4 - \dfrac{2}{\sqrt{3}}\pi & \text{for } \alpha = 1, \\ 4 - \dfrac{7+4\sqrt{3}}{3(2+\sqrt{3})}\pi & \text{for } \alpha = 2+\sqrt{3}, \\ 4 - \dfrac{3+2\sqrt{2}}{2(1+\sqrt{2})}\pi & \text{for } \alpha = 1 + \frac{1}{\sqrt{2}}. \end{cases}$$

Example 2.2.

$$\sum_{n=1}^{\infty} \frac{H_n(\alpha)}{\binom{2n}{n}} \cdot \frac{9n^3+39n^2+42n}{8n^3+36n^2+46n+15} = \begin{cases} \dfrac{40}{9} - \dfrac{2}{\sqrt{3}}\pi & \text{for } \alpha = 1, \\ \dfrac{1}{9}\left(100+60\sqrt{3} - 7\pi(4+3\sqrt{3})\right) & \text{for } \alpha = 2+\sqrt{3}, \\ \dfrac{1}{18}\left(80+60\sqrt{2} - 3\pi(11+4\sqrt{2})\right) & \text{for } \alpha = 1+\frac{1}{\sqrt{2}}. \end{cases}$$

Theorem 2.2. *Assume that* $q > \frac{1}{4}$ *and define*

$$\delta(q) := \left(\mathrm{Li}_2 \frac{\sqrt{4q-1}+\mathrm{i}}{2\sqrt{4q-1}} - \mathrm{Li}_2 \frac{\sqrt{4q-1}-\mathrm{i}}{2\sqrt{4q-1}}\right)\mathrm{i}. \tag{6}$$

Then

$$\phi_1(q) = \frac{-2q}{\sqrt{4q-1}}\left(\arctan\frac{1}{\sqrt{4q-1}} \cdot \ln\frac{4q-1}{q} + \delta(q)\right), \tag{7}$$

$$\phi_2(q) = \frac{-2q}{\sqrt{(4q-1)^3}}\left(\delta(q) - \left(4 - \ln\frac{4q-1}{q}\right) \cdot \arctan\frac{1}{\sqrt{4q-1}}\right). \tag{8}$$

Example 2.3. The following identities hold

$$\sum_{n=1}^{\infty} \frac{H_n(1)}{\binom{2n}{n}} = (4-\ln 3)\frac{\pi}{27}\sqrt{3} - \frac{2}{9}\delta(1)\sqrt{3}, \tag{9}$$

$$\sum_{n=1}^{\infty} \frac{H_n(1)}{\binom{2n}{n}(2n+1)} = -\frac{\pi}{3\sqrt{3}}\ln 3 - \frac{2}{\sqrt{3}}\delta(1). \tag{10}$$

Remark 2.1. Using the concept of Clausen's function $\mathrm{Cl}_2(x)$ defined by

$$\mathrm{Cl}_2(x) := \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$$

and the relations [17, (4.18), (5.5)] we get

$$\delta(1) = \frac{\pi}{6}\ln 3 - \frac{4}{3}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) = \frac{\pi}{6}\ln 3 - \sqrt{3}\,\mathrm{L}_{-3}(2).$$

Here $\mathrm{L}_{-3}(2)$ denotes the Dirichlet L-series defined as

$$\mathrm{L}_{-3}(2) := \sum_{n=1}^{\infty} \left(\frac{-3}{n}\right) \cdot \frac{1}{n^2},$$

where $\left(\frac{-3}{n}\right)$ is the Kronecker symbol, see [3]. Thus the formulae in (9) and (10) can be written in a more compact form as

$$\sum_{n=1}^{\infty} \frac{H_n(1)}{\binom{2n}{n}} = (2-\ln 3)\frac{2\pi}{27}\sqrt{3} + \frac{2}{3}\,\mathrm{L}_{-3}(2),$$

$$\sum_{n=1}^{\infty} \frac{H_n(1)}{\binom{2n}{n}(2n+1)} = -\frac{2\pi}{3\sqrt{3}}\ln 3 + 2\,\mathrm{L}_{-3}(2).$$

This identities shows that the irrationality of the values $\phi_1(1)$ and $\phi_2(1)$ depends essentially on the algebraic character of the value $\mathrm{L}_{-3}(2)$. However, the irrationality of $\mathrm{L}_{-3}(2)$ is still an open problem.

Example 2.4.

$$\sum_{n=1}^{\infty} \frac{H_n(1)}{\binom{2n}{n}} \cdot \frac{R(n)}{2n+1} = \begin{cases} \dfrac{2}{3\sqrt{3}}\pi & \text{for } R(n) = 3n+1, \\[2ex] \dfrac{\pi}{2} & \text{for } R(n) = n \cdot 2^n, \\[2ex] \dfrac{4}{\sqrt{3}}\pi & \text{for } R(n) = (n-1)\cdot 3^n. \end{cases}$$

Corollary 2.1. *Let $p(n) \in \overline{\mathbb{Q}}[n]$, $\deg p(n) \le 3$, $q(n) := 8n^3 + 36n^2 + 46n + 15$ and q be a real algebraic number with $q > \frac{1}{4}$. Then*

$$\sum_{n=1}^{\infty} \frac{H_1(n)}{q^n \binom{2n}{n}} \cdot \frac{p(n)}{q(n)} = \overline{Q} + \sum_{j=1}^{J_1} \lambda_j^{(1)} \ln \alpha_j^{(1)} + \sum_{j=1}^{J_2} \lambda_j^{(2)} \ln \alpha_j^{(21)} \ln \alpha_j^{(22)} + \sum_{j=1}^{J_3} \lambda_j^{(3)} \mathrm{Li}_2(\alpha_j^{(3)}) \tag{11}$$

where $(\overline{Q}, \alpha_j^{(k)}, \lambda_j^{(k)}) \in \overline{\mathbb{Q}}^3$.

Example 2.5.

$$\sum_{n=1}^{\infty} \frac{H_n(1)}{\binom{2n}{n}} \cdot \frac{81n^3 + 291n^2 + 128n - 250}{8n^3 + 36n^2 + 46n + 15} = \frac{2}{\sqrt{3}}\pi, \tag{12}$$

$$\sum_{n=1}^{\infty} \frac{H_n(1)}{\binom{2n}{n}} \cdot \frac{9n^3 + 33n^2 + 17n - 25}{8n^3 + 36n^2 + 46n + 15} = \frac{4}{9}. \tag{13}$$

3. Proofs

Proof of Theorem 2.1. Using the integral representation in (3) we get

$$\begin{aligned}
\sigma_1 &= q^{-k} \sum_{n=1}^{\infty} \frac{H_n(\alpha)}{q^n} \sum_{j=0}^{k} (-1)^j \binom{k}{j} q^{k-j} \varphi_{n+j} \begin{pmatrix} m_1, & m_2 \\ a, & b \end{pmatrix} \\
&= \frac{q^{-k}}{(a+b+1)\,\mathrm{B}(a+1,b+1)} \sum_{n=1}^{\infty} \frac{H_n(\alpha)}{q^n} \sum_{j=0}^{k} \frac{(-1)^j}{q^{j-k}} \binom{k}{j} \times \\
&\qquad \times \int_0^1 x^{m_1(n+j)+a}(1-x)^{m_2(n+j)+b}\,\mathrm{d}x \\
&= \frac{q^{-k}}{(a+b+1)\,\mathrm{B}(a+1,b+1)} \sum_{n=1}^{\infty} \frac{H_n(\alpha)}{q^n} \times \\
&\qquad \times \int_0^1 x^{m_1 n+a}(1-x)^{m_2 n+b}(q - x^{m_1}(1-x)^{m_2})^k\,\mathrm{d}x \\
&= \frac{q^{-k}}{(a+b+1)\,\mathrm{B}(a+1,b+1)} \int_0^1 x^a(1-x)^b (q - x^{m_1}(1-x)^{m_2})^k \times \\
&\qquad \times \sum_{n=1}^{\infty} \frac{H_n(\alpha)}{\left(\frac{q}{x^{m_1}(1-x)^{m_2}}\right)^n}\,\mathrm{d}x
\end{aligned}$$

$$= \frac{q^{-k}}{(a+b+1)\,\mathrm{B}(a+1,b+1)} \int\limits_0^1 x^a(1-x)^b\,(q-x^{m_1}(1-x)^{m_2})^k \times$$
$$\times \frac{\frac{q}{x^{m_1}(1-x)^{m_2}}}{\frac{q}{x^{m_1}(1-x)^{m_2}}-1} \cdot \ln \frac{\frac{\alpha q}{x^{m_1}(1-x)^{m_2}}}{\frac{\alpha q}{x^{m_1}(1-x)^{m_2}}-1}\,\mathrm{d}x$$

where the interchange of the integral and the infinite summation in $\sum\limits_{n=1}^{\infty}\int\limits_0^1 a_n(x)\,\mathrm{d}x$ with

$$a_n(x) := \frac{H_n(\alpha)}{q^n} x^{m_1 n+a}(1-x)^{m_2 n+b}(q-x^{m_1}(1-x)^{m_2})^k, \qquad |q| > \frac{1}{4},\ \ x\in[0,1],$$

follows easy. It suffices to consider that for $M, N \in \mathbb{N}$, $M < N$,

$$\begin{aligned} 0 &\le \sum_{l=M}^{N} |a_l(x)| \\ &\le \max_{0\le x\le 1}\Big(\big(x^a(1-x)^b\big)\,|q-x^{m_1}(1-x)^{m_2}|^k\Big)\sum_{l=M}^{N}\frac{H_l(\alpha)}{\Big(\frac{|q|}{x^{m_1}(1-x)^{m_2}}\Big)^l}. \end{aligned} \tag{14}$$

For $|q| > \frac{1}{4}$, $x \in [0,1]$ and all positive integers m_1, m_2 we have $\frac{|q|}{x^{m_1}(1-x)^{m_2}} > 1$ and thus by (1) there exists a positive integer N_0 so that for every $M, N \in \mathbb{N}$, $N_0 < M < N$, the finite sum on the left side of (14) can be made arbitrarily small, independent of $x \in [0,1]$. Finally, the simplification of the last integral gives immediately the formula (4).

The proof of (5) uses the same technique and the formula in (2). □

Remark 3.1. The validity of Theorem 2.1 can be extended by taking $m_i \in \mathbb{N}_0$, $i = 1, 2$, with $m_1 + m_2 > 1$.

Proof of Example 2.1. Take $a = b = 0$, $m_1 = m_2 = q = k = 1$ in (4) with $\alpha = 1$, $\alpha = 2+\sqrt{3}$ and $\alpha = 1+\frac{1}{\sqrt{2}}$ respectively. □

Proof of Example 2.2. Take $a = b = 0$, $m_1 = m_2 = q = 1$ and $k = 2$ in (5) with $\alpha = 1$, $\alpha = 2+\sqrt{3}$ and $\alpha = 1+\frac{1}{\sqrt{2}}$ respectively. □

Proof of Theorem 2.2. Using the fact that for every positive integer n we have

$$\frac{1}{(2n+1)\binom{2n}{n}} = \int\limits_0^1 x^n(1-x)^n\,\mathrm{d}x,$$

we obtain for $q > \frac{1}{4}$

$$\begin{aligned}\phi_1(q) &= \sum_{n=1}^{\infty} \frac{H_n(1)}{q^n} \int_0^1 x^n (1-x)^n \,\mathrm{d}x \\ &= q \int_0^1 \frac{1}{q - x(1-x)} \cdot \ln \frac{q}{q - x(1-x)} \,\mathrm{d}x \\ &= \frac{4q \ln q}{\sqrt{4q-1}} \arctan \frac{1}{\sqrt{4q-1}} - \frac{4q}{\sqrt{4q-1}} \int_0^{\frac{1}{\sqrt{4q-1}}} \frac{\ln \frac{4q-1}{4} + \ln(t^2+1)}{t^2+1} \,\mathrm{d}t.\end{aligned}$$

The evaluation of the last integral is based on the evaluation of the integral $J(t) := \int \frac{\ln(t^2+1)}{t^2+1} \,\mathrm{d}t$. However,

$$\begin{aligned}J(t) &= \frac{\mathrm{i}}{2} \left(\mathrm{Li}_2 \left(\frac{1 + \mathrm{i}t}{2} \right) - \mathrm{Li}_2 \left(\frac{1 - \mathrm{i}t}{2} \right) \right) + \frac{\mathrm{i}}{2} \ln(t^2+1) \left(\ln(t+\mathrm{i}) - \ln(t-\mathrm{i}) \right) \\ &\quad - \frac{\mathrm{i}}{4} \left(\ln^2(t+\mathrm{i}) - \ln^2(t-\mathrm{i}) \right) + \frac{\mathrm{i}}{2} \ln(t-\mathrm{i}) \cdot \ln \left(\frac{-\mathrm{i}}{2}(t+\mathrm{i}) \right) \\ &\quad - \frac{\mathrm{i}}{2} \ln(t+\mathrm{i}) \cdot \ln \left(\frac{\mathrm{i}}{2}(t-\mathrm{i}) \right)\end{aligned}$$

where $\ln z$, $z \in \mathbb{C} \setminus \{0\}$, is defined by $\ln z := \ln |z| + \mathrm{i} \cdot \arg z$ with $-\pi < \arg z \leq \pi$. This can be verified by the fact that $\frac{\mathrm{d}}{\mathrm{d}x} \mathrm{Li}_2\, x = -\frac{\ln(1-x)}{x}$.

Hence, for $q > \frac{1}{4}$ we have

$$\int_0^{\frac{1}{\sqrt{4q-1}}} \frac{\ln \frac{4q-1}{4} + \ln(t^2+1)}{t^2+1} \,\mathrm{d}t = \ln \frac{4q-1}{4} \arctan \frac{1}{\sqrt{4q-1}} + J \left(\frac{1}{\sqrt{4q-1}} \right) - J(0). \tag{15}$$

It is easy to see that $J(0) = -\frac{\pi}{2} \ln 2$. For $J\left(\frac{1}{\sqrt{4q-1}}\right)$, with $q > \frac{1}{4}$, the situation is more complicated. Set

$$\eta_1 := \ln \left(\frac{1}{\sqrt{4q-1}} - \mathrm{i} \right), \qquad \eta_2 := \ln \left(\frac{1}{\sqrt{4q-1}} + \mathrm{i} \right),$$
$$\eta_3 := \ln \left(\frac{-\mathrm{i}}{2} \left(\frac{1}{\sqrt{4q-1}} + \mathrm{i} \right) \right), \qquad \eta_4 := \ln \left(\frac{\mathrm{i}}{2} \left(\frac{1}{\sqrt{4q-1}} - \mathrm{i} \right) \right)$$

then

$$\eta_1 = \frac{1}{2}\ln\frac{4q}{4q-1} - \mathrm{i}\arctan\sqrt{4q-1}, \qquad \eta_2 = \frac{1}{2}\ln\frac{4q}{4q-1} + \mathrm{i}\arctan\sqrt{4q-1},$$

$$\eta_3 = \frac{1}{2}\ln\frac{q}{4q-1} - \mathrm{i}\arctan\frac{1}{\sqrt{4q-1}}, \qquad \eta_4 = \frac{1}{2}\ln\frac{q}{4q-1} + \mathrm{i}\arctan\frac{1}{\sqrt{4q-1}}.$$

Putting

$$\delta(q) := \left(\mathrm{Li}_2\frac{\sqrt{4q-1}+\mathrm{i}}{2\sqrt{4q-1}} - \mathrm{Li}_2\frac{\sqrt{4q-1}-\mathrm{i}}{2\sqrt{4q-1}}\right)\mathrm{i}, \qquad q > \frac{1}{4},$$

we obtain

$$J\left(\frac{1}{\sqrt{4q-1}}\right) = \frac{\delta(q)}{2} + \frac{\mathrm{i}(\eta_2-\eta_1)}{2}\cdot\ln\frac{4q}{4q-1} - \frac{\mathrm{i}(\eta_2^2-\eta_1^2)}{4} + \frac{\mathrm{i}\eta_1\eta_3}{2} - \frac{\mathrm{i}\eta_2\eta_4}{2}$$

which can be simplified using the identities for η_i, $1 \le i \le 4$, as follows

$$J\left(\frac{1}{\sqrt{4q-1}}\right) = \frac{\delta(q)}{2} + \ln\frac{\sqrt{4q-1}}{4\sqrt{q}}\arctan\sqrt{4q-1} + \frac{\pi}{4}\cdot\ln\frac{4q}{4q-1}.$$

We deduce that

$$\begin{aligned}
&J\left(\frac{1}{\sqrt{4q-1}}\right) - J(0)\\
&= -2\arctan\sqrt{4q-1}\cdot\ln 2 - \frac{1}{2}\arctan\sqrt{4q-1}\cdot\ln q\\
&\quad + \frac{1}{2}\arctan\sqrt{4q-1}\cdot\ln(4q-1) + \pi\ln 2 + \frac{\pi}{4}\ln q - \frac{\pi}{4}\ln(4q-1) + \frac{\delta(q)}{2}\\
&= -\frac{1}{2}\ln\frac{16q}{4q-1}\cdot\left(\arctan\sqrt{4q-1} - \frac{\pi}{2}\right) + \frac{\delta(q)}{2}\\
&= \frac{1}{2}\ln\frac{16q}{4q-1}\cdot\arctan\frac{1}{\sqrt{4q-1}} + \frac{\delta(q)}{2}.
\end{aligned}$$

This leads to

$$\begin{aligned}
&\int_0^{\frac{1}{\sqrt{4q-1}}} \frac{\ln\frac{4q-1}{4} + \ln(t^2+1)}{t^2+1}\,\mathrm{d}t\\
&= \ln\frac{4q-1}{4}\arctan\frac{1}{\sqrt{4q-1}} + \frac{1}{2}\ln\frac{16q}{4q-1}\arctan\frac{1}{\sqrt{4q-1}} + \frac{\delta(q)}{2}\\
&= \arctan\frac{1}{\sqrt{4q-1}}\cdot\left(\ln\frac{4q-1}{4} + \ln\frac{4\sqrt{q}}{\sqrt{4q-1}}\right) + \frac{\delta(q)}{2}\\
&= \frac{1}{2}\left(\arctan\frac{1}{\sqrt{4q-1}}\cdot\ln(q(4q-1)) + \delta(q)\right).
\end{aligned}$$

Finally, this can be used for the evaluation of the infinite series $\phi_1(q)$ in the following way

$$\begin{aligned}
\phi_1(q) &= \frac{4q\ln q}{\sqrt{4q-1}}\cdot\arctan\frac{1}{\sqrt{4q-1}}\\
&\quad-\frac{4q}{\sqrt{4q-1}}\cdot\frac{1}{2}\left(\arctan\frac{1}{\sqrt{4q-1}}\cdot\ln(q(4q-1))+\delta(q)\right)\\
&=\frac{4q}{\sqrt{4q-1}}\cdot\arctan\frac{1}{\sqrt{4q-1}}\cdot\ln\frac{q}{\sqrt{q(4q-1)}}-\frac{2q}{\sqrt{4q-1}}\delta(q)\\
&=\frac{4q}{\sqrt{4q-1}}\cdot\arctan\frac{1}{\sqrt{4q-1}}\cdot\ln\sqrt{\frac{q}{4q-1}}-\frac{2q\delta(q)}{\sqrt{4q-1}}\\
&=\frac{-2q}{\sqrt{4q-1}}\left(\arctan\frac{1}{\sqrt{4q-1}}\cdot\ln\frac{4q-1}{q}+\delta(q)\right).
\end{aligned}$$

This completes the proof of (7). The proof of the identity (8) is based on the same technique and it uses the obvious differential equation $\phi_1(q)-\phi_2(q)=2q\frac{\mathrm{d}}{\mathrm{d}q}\phi_1(q)$, with $q>\frac{1}{4}$. □

Proof of Example 2.3. In Theorem 2.2 set $q=1$ in the sums $\phi_1(q)$ and $\phi_2(q)$. □

Proof of Example 2.4. To prove the identities use Theorem 2.2 and evaluate the terms $3\phi_2(1)-\phi_1(1)$, $\phi_2\left(\frac{1}{2}\right)-\phi_1\left(\frac{1}{2}\right)$ and $\phi_2\left(\frac{1}{3}\right)-3\phi_1\left(\frac{1}{3}\right)$ respectively. □

Proof of Corollary 2.1. Using the obvious identity $8n^3+36n^2+46n+15=(2n+1)(2n+3)(2n+5)$, we have

$$\begin{aligned}
\phi_1(q)&=\sum_{n=1}^{\infty}\frac{H_n(1)}{q^n\binom{2n}{n}}\cdot\frac{4n^2+16n+15}{8n^3+36n^2+46n+15},\\
\phi_2(q)&=\sum_{n=1}^{\infty}\frac{H_n(1)}{q^n\binom{2n}{n}}\cdot\frac{8n^3+36n^2+46n+15}{8n^3+36n^2+46n+15}.
\end{aligned}\tag{16}$$

Taking $a=b=0$ and $m_1=m_2=1$ in the series σ_1 and σ_2 defined in Theorem 2.1 we can write

$$\begin{aligned}
\sigma_1&=\sum_{n=1}^{\infty}\frac{H_n(1)}{q^n\binom{2n}{n}}\sum_{j=0}^{1}(-1)^jq^{-j}\binom{1}{j}\cdot\frac{\prod\limits_{s=1}^{j}(n+s)^2}{\prod\limits_{s=1}^{2j+1}(2n+s)}\\
&=\frac{1}{2q}\sum_{n=1}^{\infty}\frac{H_n(1)}{q^n\binom{2n}{n}}\cdot\frac{(8q-2)n^2+(32q-7)n+30q-5}{8n^3+36n^2+46n+15}
\end{aligned}\tag{17}$$

and

$$\sigma_2 = \sum_{n=1}^{\infty} \frac{n \cdot H_n(1)}{q^n \binom{2n}{n}} \sum_{j=0}^{2} (-1)^j q^{-j} \binom{2}{j} \cdot \frac{\prod_{s=1}^{j} (n+s)^2}{\prod_{s=1}^{2j+1} (2n+s)}$$
$$= \frac{1}{4q^2} \sum_{n=1}^{\infty} \frac{H_n(1)}{q^n \binom{2n}{n}} \cdot \frac{(16q^2 - 8q + 1)n^3 + (64q^2 - 28q + 3)n^2 + (60q^2 - 20q + 2)n}{8n^3 + 36n^2 + 46n + 15}. \tag{18}$$

Next, note

$$\mathbf{P} = (p_{ij})_{\substack{1 \le i \le 4 \\ 1 \le j \le 4}} := \begin{pmatrix} 0 & 8 & 0 & 16q^2 - 8q + 1 \\ 4 & 36 & 8q - 2 & 64q^2 - 28q + 3 \\ 16 & 46 & 32q - 7 & 60q^2 - 20q + 2 \\ 15 & 15 & 30q - 5 & 0 \end{pmatrix}.$$

Then by easy computation we obtain $\det \mathbf{P} \equiv 60 \neq 0$. Thus every polynomial $p(n) \in \mathbb{R}[n]$, with $\deg p(n) \le 3$, can be expressed as a linear combination of the polynomials $P_i(n) := \sum_{j=1}^{4} p_{ji} n^{j-1}$, $1 \le i \le 4$, say $p(n) = \sum_{i=1}^{4} \sum_{j=1}^{4} k_i p_{ji} n^{j-1}$. Hence,

$$\sum_{n=1}^{\infty} \frac{H_1(n)}{q^n \binom{2n}{n}} \cdot \frac{p(n)}{q(n)} = \sum_{n=1}^{\infty} \frac{H_n(1)}{q^n \binom{2n}{n}} \cdot \frac{\sum_{i=1}^{4} \sum_{j=1}^{4} k_i p_{ji} n^{j-1}}{q(n)}$$
$$= \sum_{i=1}^{4} k_i \sum_{n=1}^{\infty} \frac{H_n(1)}{q^n \binom{2n}{n}} \cdot \frac{P_i(n)}{q(n)}$$
$$= k_1 \phi_1(q) + k_2 \phi_2(q) + 2q k_3 \sigma_1 + 4q^2 k_4 \sigma_2.$$

Now use the formulas for σ_i, $\phi_i(q)$, $i = 1, 2$, in Theorem 2.1 and Theorem 2.2 to deduce (11). □

P r o o f o f E x a m p l e 2.5. Let σ_1 and σ_2 denotes the sums in Example 2.1 and Example 2.2 with $\alpha = 1$ respectively. Then $9\sigma_2 - 10\sigma_1 = \frac{2}{\sqrt{3}}\pi$ and $\sigma_2 - \sigma_1 = \frac{4}{9}$. These results agree with the identities in (12) and (13). □

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