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A UNIQUENESS RESULT RELATED TO CERTAIN NON-LINEAR DIFFERENTIAL POLYNOMIALS SHARING THE SAME 1-POINTS

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ABSTRACT. The purpose of the paper is to study the uniqueness of meromorphic function when certain non-linear differential polynomials share the same 1-points. As a consequence of the main result we improve and supplement the following recent result: [LAHIRI, I.—PAL, R.: *Nonlinear differential polynomials sharing* 1*-points*, Bull. Korean Math. Soc. **43** (2006), 161–168].

1. Introduction definitions and results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. We shall use the standard notations of value distribution theory:

$$T(r,f),\ \ m(r,f),\ \ N(r,\infty;f),\ \ \overline{N}(r,\infty;f),\ \ \dots$$

(see [7]).

For $a \in \mathbb{C} \cup \{\infty\}$, we define

$$\Theta(a;f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

If for some $a \in \mathbb{C} \cup \{\infty\}$, $f-a$ and $g-a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities). Let m be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_{m)}(a;f)$ the set of all a-points of f with multiplicities not exceeding

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m, where an a-point is counted according to its multiplicity. If for some $a \in \mathbb{C} \cup \{\infty\}$, $E_{\infty)}(a; f) = E_{\infty)}(a; g)$ we say that f, g share the value a CM.

In 1999, Lahiri [8] studied the problem of uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points. In the same paper [8] regarding the nonlinear differential polynomials Lahiri asked the following question.

What can be said if two nonlinear differential polynomials generated by two meromorphic functions share 1 CM?

Since then the progress to investigate the uniqueness of meromorphic functions which are the generating functions of different types of non-linear differential polynomials is remarkable and continuous efforts are being put in to relax the hypothesis of the results. (cf. [1]–[6], [12]–[18]).

In 2001, Fang and Hong [6] proved the following result.

Theorem A. *Let f and g be two transcendental entire functions and n (≥ 11) be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.*

In 2002, Fang and Fang [5] improved and supplemented the above theorem by proving the following theorems.

Theorem B. *Let f and g be two non-constant entire functions and m (≥ 3), n (≥ 8) be two positive integers. If $E_{m)}(1; f^n(f-1)f') = E_{m)}(1; g^n(g-1)g')$, then $f \equiv g$.*

Theorem C. *Let f and g be two non-constant entire functions and n (≥ 9) be an integer. If $E_{2)}(1; f^n(f-1)f') = E_{2)}(1; g^n(g-1)g')$, then $f \equiv g$.*

Theorem D. *Let f and g be two non-constant entire functions and n (≥ 14) be an integer. If $E_{1)}(1; f^n(f-1)f') = E_{1)}(1; g^n(g-1)g')$, then $f \equiv g$.*

In 2004, Lin and Yi [16] further improved Theorem A as follows.

Theorem E. *Let f and g be two transcendental entire functions and n (≥ 7) be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.*

The following example shows that the above theorems are not valid when f and g are two meromorphic functions.

Example 1.1.

$$f(z) = \frac{(n+2)}{(n+1)} \frac{e^z + \cdots + e^{(n+1)z}}{1 + e^z + \cdots + e^{(n+1)z}}$$

and

$$g(z) = \frac{(n+2)}{(n+1)} \frac{1 + e^z + \cdots + e^{nz}}{1 + e^z + \cdots + e^{(n+1)z}}.$$

Clearly $f(z) = e^z g(z)$. Also $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM but $f \not\equiv g$.

We note that in the above example $\Theta(\infty; f) = \Theta(\infty; g) = 0$.

So to replace entire functions by meromorphic functions in the above mentioned theorems definitely some extra conditions are required.

Further investigations in the above directions have already been executed by many contemporary mathematicians and consequently some elegant results have been obtained in this aspect (see [3], [12], [14], [16]). But in all the papers just mentioned, to prove the uniqueness of the meromorphic functions some restrictions on the ramification indexes of f and g has to be imposed by all the authors.

Recently, Lahiri-Pal [13] has proved the following theorem.

Theorem F. *Let f and g be two non-constant meromorphic functions and n (≥ 14) be an integer. If $E_{3)}(1; f^n(f^3-1)f') = E_{3)}(1; g^n(g^3-1)g')$, then $f \equiv g$.*

In the paper we will consider the value sharing of more generalised differential polynomial than that was considered in Theorem F and we will show that the same conclusion can be obtained as a corollary of our main result. Following theorem is the main result of the paper.

Theorem 1.1. *Let f and g be two transcendental meromorphic functions and n, k (≥ 1), m (≥ 2) be three positive integers. Suppose for two non zero constants a and b, $E_{l)}\left(1; [f^n(af^m+b)]^{(k)}\right) = E_{l)}\left(1; [g^n(ag^m+b)]^{(k)}\right)$. Then $f \equiv g$ or $f \equiv -g$ or $[f^n(af^m+b)]^{(k)}[g^n(ag^m+b)]^{(k)} \equiv 1$ provided one of the following holds.*

(i) $l \geq 3$ *and* $n > 3k+m+8$;

(ii) $l = 2$ *and* $n > 4k+\frac{3m}{2}+9$;

(iii) $l = 1$ *and* $n > 7k+3m+12$.

When $k = 1$ the possibility $[f^n(af^m+b)]^{(k)}[g^n(ag^m+b)]^{(k)} \equiv 1$ does not occur. Also the possibility $f \equiv -g$ arises only if n and m are both even.

Putting $n = s+1$, $m = 3$, $a = \frac{1}{s+4}$, $b = -\frac{1}{s+1}$ and $k = 1$ in the above theorem we can immediately deduce the following corollary.

Corollary 1.1. *Let f and g be two non-constant meromorphic functions and s be a positive integer. Suppose $E_{l)}\left(1; f^s(f^3-1)f'\right) = E_{l)}\left(1; g^s(g^3-1)g'\right)$. Then $f \equiv g$ provided one of the following holds.*

(i) $l \geq 3$ *and* $s \geq 14$;

(ii) $l = 2$ *and* $s \geq 17$;

(iii) $l = 1$ *and* $s \geq 28$.

Remark 1.1. Since Theorem F can be obtained as a special case of Theorem 1.1, clearly Theorem 1.1 improves and supplements Theorem F.

Though we use the standard notations and definitions of the value distribution theory available in [7], we explain some definitions and notations which are used in the paper.

Definition 1.1. ([14]) For $a \in \mathbb{C} \cup \{\infty\}$we denote by $N(r, a; f \mid =1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r, a; f \mid \leq m)$ ($N(r, a; f \mid \geq m)$) the counting function of those a-points of f whose multiplicities are not greater (less) than m where each a-point is counted according to its multiplicity.

$\overline{N}(r, a; f \mid \leq m)$ ($\overline{N}(r, a; f \mid \geq m)$) are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r, a; f \mid < m)$, $N(r, a; f \mid > m)$, $\overline{N}(r, a; f \mid < m)$ and $\overline{N}(r, a; f \mid > m)$ are defined analogously.

Definition 1.2. Let m be a positive integer and for $a \in \mathbb{C}$, $E_{m)}(a; f) = E_{m)}(a; g)$. Let z_0 be a zero of $f(z) - a$ of multiplicity p and a zero of $g(z) - a$ of multiplicity q. We denote by $\overline{N}_L(r, a; f)$ ($\overline{N}_L(r, a; g)$) the reduced counting function of those a-points of f and g for which $p > q \geq m+1$ ($q > p \geq m+1$), by $\overline{N}_E^{(m+1)}(r, a; f)$ the reduced counting function of those a-points of f and g for which $p = q \geq m+1$, by $\overline{N}_{f>m+1}(r, 1; g)$ the reduced counting function of f and g for which $p \geq m+2$ and $q = m+1$. Also by $\overline{N}_{f \geq m+1}(r, a; f \mid g \neq a)$ ($\overline{N}_{g \geq m+1}(r, a; g \mid f \neq a)$) we denote the reduced counting functions of those a-points of f and g for which $p \geq m+1$ and $q = 0$ ($q \geq m+1$ and $p = 0$).

Definition 1.3. We denote by $\overline{N}(r, a; f \mid =k)$ the reduced counting function of those a-points of f whose multiplicities is exactly k where $k \geq 2$ is an integer. For $k = 1$ we refer Definition 1.1.

Definition 1.4. ([10]) Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g = b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.

Definition 1.5. ([10]) Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g \neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.

Definition 1.6. ([11], cf. [19]) For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid \geq 2) + \cdots + \overline{N}(r, a; f \mid \geq p)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 1.7. Let $a, b \in \mathbb{C} \cup \{\infty\}$. Let p be a positive integer. We denote by $\overline{N}(r, a; f \mid \geq p \mid g = b)$ ($\overline{N}(r, a; f \mid \geq p \mid g \neq b)$) the reduced counting function of those a-points of f with multiplicities $\geq p$, which are the b-points (not the b-points) of g.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function.

$$H = \left(\frac{F^{(k+2)}}{F^{(k+1)}} - \frac{2F^{(k+1)}}{F^{(k)}-1}\right) - \left(\frac{G^{(k+2)}}{G^{(k+1)}} - \frac{2G^{(k+1)}}{G^{(k)}-1}\right). \tag{2.1}$$

Lemma 2.1. ([7]) *Let f be a non-constant meromorphic function, k a positive integer and let c be a non-zero finite complex number. Then*

$$\begin{aligned} T(r,f) &\le \overline{N}(r,\infty;f) + N(r,0;f) + N\left(r,c;f^{(k)}\right) - N\left(r,0;f^{(k+1)}\right) + S(r,f) \\ &\le \overline{N}(r,\infty;f) + N_{k+1}(r,0;f) + \overline{N}\left(r,c;f^{(k)}\right) \\ &\qquad\qquad - N_0\left(r,0;f^{(k+1)}\right) + S(r,f), \end{aligned}$$

where $N_0\left(r,0;f^{(k+1)}\right)$ is the counting function of the zeros of $f^{(k+1)}$ which are not the zeros of $f(f^{(k)}-c)$

Lemma 2.2. ([11]) *If $N(r,0;f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)} \mid f \neq 0) \le k\overline{N}(r,\infty;f) + N(r,0;f \mid < k) + k\overline{N}(r,0;f \mid \ge k) + S(r,f).$$

Lemma 2.3. ([20]) *Let f be a non-constant meromorphic function and p, k be positive integers, then*

$$N_p(r,0;f^{(k)}) \le N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 2.4. *Let $E_{m)}(1;f) = E_{m)}(1;g)$ and $2 \le m < \infty$. Then*

$$\begin{aligned} &\overline{N}(r,1;f \mid =2) + 2\overline{N}(r,1;f \mid =3) + \cdots + (m-1)\overline{N}(r,1;f \mid =m) \\ &+ m\overline{N}_E^{(m+1)}(r,1;f) + m\overline{N}_L(r,1;f) + (m+1)\overline{N}_L(r,1;g) \\ &+ m\overline{N}_{g\ge m+1}(r,1;g \mid f \neq 1) \\ \le\; & N(r,1;g) - \overline{N}(r,1;g). \end{aligned}$$

P r o o f. Since $E_{(m)}(1;f) = E_{m)}(1;g)$, we note that common zeros of $f-1$ and $g-1$ up to multiplicity m are same. Clearly a 1-point of f and g with multiplicity $i \le m$ is counted exactly $(i-1)$ times in both sides of the inequality. Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. If $q = m+1$ the possible values of p are as follows

(i) $p = m + 1$,
(ii) $p \geq m + 2$,
(iii) $p = 0$.

Similarly when $q = m + 2$ the possible values of p are

(i) $p = m + 1$,
(ii) $p = m + 2$,
(iii) $p \geq m + 3$,
(iv) $p = 0$.

If $q \geq m+3$ we can similarly find the possible values of p. When $q \geq m+1$, the common 1-points of f and g with the same multiplicities, the common 1-points of f and g where the multiplicities for f are greater than those for g, the 1-points of f which are not the 1-points of g are counted at least m times in the right hand side of the above inequality which is evident from the possible values of p when $q = m + 1$. Also we note that the 1-points of g whose multiplicities are greater than those of f are counted at least $m + 1$ times and this case can only happen when $q \geq m + 2$. The rest of the proof follows easily. □

Lemma 2.5. *Let* $E_{1)}(1; f) = E_{1)}(1; g)$. *Then*

$$\begin{aligned} &2\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + \overline{N}^{(2}_E(r,1;f) \\ &\quad + \overline{N}_{g\geq 2}(r,1;g \mid f \neq 1) - \overline{N}_{f>2}(r,1;g) \\ &\leq N(r,1;g) - \overline{N}(r,1;g), \end{aligned}$$

Proof. Since $E_{1)}(1; f) = E_{1)}(1; g)$ the simple 1-points of f and g are same. Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. If $q = 2$ the possible values of p are as follows

(i) $p = 2$
(ii) $p \geq 3$
(iii) $p = 0$.

Similarly when $q = 3$ the possible values of p are

(i) $p = 2$
(ii) $p = 3$
(iii) $p \geq 4$
(iv) $p = 0$.

If $q \geq 4$ we can similarly find the possible values of p. Now the lemma follows from above discussion and the explanation in the previous lemma. □

Lemma 2.6. *Let* $E_{2)}(1; f) = E_{2)}(1; g)$. *Then*

$$\overline{N}_{f\geq 3}(r, 1; f \mid g \neq 1) \leq \frac{1}{2}\overline{N}(r, 0; f) + \frac{1}{2}\overline{N}(r, \infty; f) - \frac{1}{2}N_{\oplus}(r, 0; f') + S(r, f),$$

where $N_{\oplus}(r, 0; f')$ *is the counting function of those zeros of* f' *which are not the zeros of* $f(f-1)$, *each point is counted according to its multiplicity.*

P r o o f. Using Lemma 2.2 we get

$$\begin{aligned} &\overline{N}_{f\geq 3}(r, 1; f \mid g \neq 1) \\ \leq\; &\overline{N}(r, 1; f \mid \geq 3) \\ \leq\; &\frac{1}{2}N(r, 0; f' \mid f = 1) \\ \leq\; &\frac{1}{2}N(r, 0; f' \mid f \neq 0) - \frac{1}{2}N_{\oplus}(r, 0; f') \\ \leq\; &\frac{1}{2}\overline{N}(r, 0; f) + \frac{1}{2}\overline{N}(r, \infty; f) - \frac{1}{2}N_{\oplus}(r, 0; f') + S(r, f). \end{aligned}$$

□

Lemma 2.7. *Let* $E_{1)}(1; f) = E_{1)}(1; g)$. *Then*

$$\begin{aligned} &\overline{N}_{f>2}(r, 1; g) + \overline{N}_{f\geq 2}(r, 1; f \mid g \neq 1) \\ \leq\; &\overline{N}(r, 0; f) + \overline{N}(r, \infty; f) - N_{\oplus}(r, 0; f') + S(r, f). \end{aligned}$$

P r o o f. We note that a 1-point of f with multiplicity 2 is counted at most once in the counting function $\overline{N}_{f\geq 2}(r, 1; f \mid g \neq 1)$. Also since a 1-point of f with multiplicity ≥ 3 may or may not be a 1 point of g, those 1-points of f are counted only once, either in $\overline{N}_{f>2}(r, 1; g)$ or $\overline{N}_{f\geq 2}(r, 1; f \mid g \neq 1)$. So using Lemma 2.2 we get

$$\begin{aligned} &\overline{N}_{f>2}(r, 1; g) + \overline{N}_{f\geq 2}(r, 1; f \mid g \neq 1) \\ \leq\; &\overline{N}(r, 1; f \mid \geq 2) \\ \leq\; &N(r, 0; f' \mid f = 1) \\ \leq\; &\overline{N}(r, 0; f' \mid f \neq 0) - N_{\oplus}(r, 0; f') \\ \leq\; &\overline{N}(r, 0; f) + \overline{N}(r, \infty; f) - N_{\oplus}(r, 0; f') + S(r, f). \end{aligned}$$

□

Lemma 2.8. *Let* $E_{1)}(1; f) = E_{1)}(1; g)$. *Then*

$$\overline{N}_{f\geq 2}(r, 1; f \mid g \neq 1) \leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) - N_{\oplus}(r, 0; f') + S(r, f).$$

P r o o f. Using Lemma 2.2 and following the same procedure as in Lemma 2.7 we get

$$\begin{aligned}\overline{N}_{f\geq 2}(r,1;f\mid g\neq 1) &\leq \overline{N}(r,1;f\mid\geq 2)\\ &\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)-N_{\oplus}(r,0;f')+S(r,f).\end{aligned}$$

□

Lemma 2.9. ([17]) *Let f be a non-constant meromorphic function and let*

$$R(f)=\frac{\sum\limits_{k=0}^{n}a_kf^k}{\sum\limits_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n\neq 0$ and $b_m\neq 0$. Then

$$T(r,R(f))=dT(r,f)+S(r,f),$$

where $d=\max\{n,m\}$.

Lemma 2.10. *Let f and g be two non-constant meromorphic functions and a, b be two non zero constants. Then*

$$[f^n(af^m+b)]'[g^n(ag^m+b)]'\not\equiv 1,$$

where n, $m\geq 2$ be two positive integers and n $(\geq m+3)$.

P r o o f. We note that according to the statement of the lemma we have to prove

$$[f^{n-1}\left(a(n+m)f^m+bn\right)f'][g^{n-1}\left(a(n+m)g^m+bn\right)g']\not\equiv 1.$$

If possible let us suppose that

$$[f^{n-1}\left(a(n+m)f^m+bn\right)f'][g^{n-1}\left(a(n+m)g^m+bn\right)g']\equiv 1. \tag{2.2}$$

Let z_0 be a zero of f with multiplicity p (≥ 1). So from (2.2) we get z_0 be a pole of g with multiplicity q (≥ 1) such that

$$np-1=(n+m)q+1, \tag{2.3}$$

i.e.

$$mq=n(p-q)-2\geq n-2.$$

Again from (2.3) we get

$$np=(n+m)q+2\geq (n+m)\frac{n-2}{m}+2,$$

i.e.,

$$p\geq\frac{n+m-2}{m}.$$

Therefore
$$\Theta(0;f)\geq 1-\frac{m}{n+m-2}.$$

Suppose $a(n+m)f^m+bn=a(n+m)(f-\alpha_1)(f-\alpha_2)\dots(f-\alpha_m)$. Let z_1 be a zero of $(f-\alpha_i)$, $i=1,2,\dots,m$, with multiplicity p. Then from (2.2) we have z_1 be a pole of g with multiplicity q (≥ 1) such that
$$2p-1=(n+m)q+1$$
i.e.,
$$p\geq\frac{n+m+2}{2}.$$
Hence
$$\Theta(\alpha_i;f)\geq 1-\frac{2}{n+m+2}.$$
Since
$$\Theta(0;f)+\sum_{i=1}^{m}\Theta(\alpha_i;f)\leq 2,$$
it follows that
$$\frac{2m}{n+m+2}+\frac{m}{n+m-2}\geq m-1,$$
which is a contradiction. □

Lemma 2.11. *Let f and g be two non-constant meromorphic functions such that $F=f^n\left(af^m+b\right)$ and $G=g^n\left(ag^m+b\right)$, where $m\geq 2$ and $n+m\geq 9$ is an integer and a, b are non-zero constants. Then*
$$F\equiv G$$
implies either $f\equiv g$ or $f\equiv -g$. Also only if n and m are both even then the possibility $f\equiv -g$ occurs.

Proof. Clearly if n and m are both odd or if n is odd and m is even or if n is even and m is odd then $f\equiv -g$ contradicts $F\equiv G$. Let neither $f\equiv g$ nor $f\equiv -g$. We put $h=\frac{g}{f}$. Then $h\not\equiv 1$ and $h\not\equiv -1$. Also $F\equiv G$ implies
$$f^m=-\frac{b}{a}\frac{h^n-1}{h^{n+m}-1}.$$
If n and m are both even then the numerator and the denominator have two common factors namely $h+1$ and $h-1$. Also we observe that since a non-constant meromorphic function can not have more than two Picard exceptional values h can take at least $n+m-4$ values among $u_j=\exp\left(\frac{2j\pi\mathrm{i}}{n+m}\right)$, where $j=1,2,\dots,n+m-1$. Since f is non-constant it follows that h is non constant. Again since f^m has no simple pole $h-u_j$ has no simple zero for at least $n+m-4$ values of u_j, for $j=1,2,\dots,n+m-1$ and for these values of j we have

$\Theta(u_j;h)\geq\frac{1}{2}$, which leads to a contradiction. Therefore either $f\equiv g$ or $f\equiv -g$. This proves the lemma. □

3. Proofs of the theorems

Proof of Theorem 1.1. Let $F=f^n(af^m+b)$ and $G=g^n(ag^m+b)$. It follows that $E_{l)}\big(1;F^{(k)}\big)=E_{l)}\big(1;G^{(k)}\big)$.

Case 1. Let $H\not\equiv 0$.

From (2.1) we get

$$\begin{aligned}&N(r,\infty;H)\\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_L\big(r,1;F^{(k)}\big)\\ &\quad+\overline{N}_L\big(r,1;G^{(k)}\big)+\overline{N}_{F^{(k)}\geq l+1}\big(r,1;F^{(k)}\mid G^{(k)}\neq 1\big)\\ &\quad+\overline{N}_{G^{(k)}\geq l+1}\big(r,1;G^{(k)}\mid F^{(k)}\neq 1\big)+\overline{N}\big(r,0;F^{(k)}\mid\geq 2\big)\\ &\quad+\overline{N}\big(r,0;G^{(k)}\mid\geq 2\big)+\overline{N}_{\otimes}\big(r,0;F^{(k+1)}\big)+\overline{N}_{\otimes}\big(r,0;G^{(k+1)}\big),\end{aligned}\tag{3.1}$$

where $\overline{N}_{\otimes}\big(r,0;F^{(k+1)}\big)$ is the reduced counting function of those zeros of $F^{(k+1)}$ which are not the zeros of $F^{(k)}\big(F^{(k)}-1\big)$ and $\overline{N}_{\otimes}\big(r,0;G^{(k+1)}\big)$ is similarly defined.

Let z_0 be a simple zero of $F^{(k)}-1$. Then z_0 is a simple zero of $G^{(k)}-1$ and a zero of H. So

$$N\big(r,1;F^{(k)}\mid=1\big)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,F)+S(r,G).\tag{3.2}$$

Subcase 1.1. $l\geq 2$.

Using Lemma 2.4, (3.1) and (3.2) we get

$$\begin{aligned}&\overline{N}\big(r,1;F^{(k)}\big)+\overline{N}\big(r,1;G^{(k)}\big)\\ &\leq N\big(r,1;F^{(k)}\mid=1\big)+\overline{N}\big(r,1;F^{(k)}\mid=2\big)+\cdots+\overline{N}\big(r,1;F^{(k)}\mid=l\big)\\ &\quad+\overline{N}_L\big(r,1;F^{(k)}\big)+\overline{N}_L\big(r,1;G^{(k)}\big)+\overline{N}_{F^{(k)}\geq l+1}\big(r,1;F^{(k)}\mid G^{(k)}\neq 1\big)\\ &\quad+\overline{N}_E^{(l+1}\big(r,1;G^{(k)}\big)+\overline{N}\big(r,1;G^{(k)}\big)\\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}\big(r,0;F^{(k)}\mid\geq 2\big)+\overline{N}\big(r,0;G^{(k)}\mid\geq 2\big)\\ &\quad+T\big(r,G^{(k)}\big)+2\overline{N}_{F^{(k)}\geq l+1}\big(r,1;F^{(k)}\mid G^{(k)}\neq 1\big)\\ &\quad-(l-1)\overline{N}_{G^{(k)}\geq l+1}\big(r,1;G^{(k)}\mid F^{(k)}\neq 1\big)+\overline{N}_{\otimes}\big(r,0;F^{(k+1)}\big)\\ &\quad+\overline{N}_{\otimes}\big(r,0;G^{(k+1)}\big)+S(r,F)+S(r,G).\end{aligned}\tag{3.3}$$

So in view of (3.3), from Lemma 2.1 we have

$$\begin{aligned}
&T(r,F)+T(r,G)\\
&\leq 2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+N_{k+1}(r,0;F)+N_{k+1}(r,0;G)\\
&\quad+\overline{N}\big(r,0;F^{(k)}\mid\geq 2\big)+\overline{N}\big(r,0;G^{(k)}\mid\geq 2\big)+T(r,G)+k\overline{N}(r,\infty;G)\\
&\quad+2\overline{N}_{F^{(k)}\geq l+1}\big(r,1;F^{(k)}\mid G^{(k)}\neq 1\big)\\
&\quad-(l-1)\overline{N}_{G^{(k)}\geq l+1}\big(r,1;G^{(k)}\mid F^{(k)}\neq 1\big)\\
&\quad+\overline{N}_{\otimes}\big(r,0;F^{(k+1)}\big)+\overline{N}_{\otimes}\big(r,0;G^{(k+1)}\big)\\
&\quad-N_0\big(r,0;F^{(k+1)}\big)-N_0\big(r,0;G^{(k+1)}\big)+S(r,F)+S(r,G).
\end{aligned}\tag{3.4}$$

We note that

$$\begin{aligned}
&N_{k+1}(r,0;F)+\overline{N}\big(r,0;F^{(k)}\mid\geq 2\big)+\overline{N}_{\otimes}\big(r,0;F^{(k+1)}\big)\\
&\leq N_{k+1}(r,0;F)+\overline{N}\big(r,0;F^{(k)}\mid\geq 2\mid F=0\big)\\
&\quad+\overline{N}\big(r,0;F^{(k)}\mid\geq 2\mid F\neq 0\big)+\overline{N}_{\otimes}\big(r,0;F^{(k+1)}\big)\\
&\leq N_{k+1}(r,0;F)+\overline{N}\big(r,0;F\mid\geq k+2\big)+\overline{N}_0\big(r,0;F^{(k+1)}\big)\\
&\leq N_{k+2}(r,0;F)+\overline{N}_0\big(r,0;F^{(k+1)}\big).
\end{aligned}\tag{3.5}$$

Clearly similar expression holds for G also.

Using (3.5) in (3.4) we get

$$\begin{aligned}
T(r,F)&\leq 2\overline{N}(r,\infty;F)+(k+2)\overline{N}(r,\infty;G)+N_{k+2}(r,0;F)\\
&\quad+N_{k+2}(r,0;G)+2\overline{N}_{F^{(k)}\geq l+1}\big(r,1;F^{(k)}\mid G^{(k)}\neq 1\big)\\
&\quad-(l-1)\overline{N}_{G^{(k)}\geq l+1}\big(r,1;G^{(k)}\mid F^{(k)}\neq 1\big)+S(r,F)+S(r,G).
\end{aligned}\tag{3.6}$$

In a similar way we can obtain

$$\begin{aligned}
T(r,G)&\leq (k+2)\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+N_{k+2}(r,0;F)\\
&\quad+N_{k+2}(r,0;G)+2\overline{N}_{G^{(k)}\geq l+1}\big(r,1;G^{(k)}\mid F^{(k)}\neq 1\big)\\
&\quad-(l-1)\overline{N}_{F^{(k)}\geq l+1}\big(r,1;F^{(k)}\mid G^{(k)}\neq 1\big)+S(r,F)+S(r,G).
\end{aligned}\tag{3.7}$$

While $l\geq 3$, in view of Lemma 2.9, adding (3.6) and (3.7) we get for $\varepsilon>0$

$$\begin{aligned}
&(n+m)\{T(r,f)+T(r,g)\}\\
&\leq (k+4)\overline{N}(r,\infty;f)+2\{(k+2)\overline{N}(r,0;f)+N_{k+2}(r,0;af^m+b)\}\\
&\quad+(k+4)\overline{N}(r,\infty;g)+2\{(k+2)\overline{N}(r,0;g)+N_{k+2}(r,0;ag^m+b)\}\\
&\quad+S(r,f)+S(r,g)\\
&\leq\ (3k+2m+8)\,T(r,f)+(3k+2m+8)\,T(r,g)+S(r,f)+S(r,g).
\end{aligned}\tag{3.8}$$

That is

$$(n-3k-m-8)\,T(r,f)+(n-3k-m-8)\,T(r,g)\le S(r,f)+S(r,g).$$

Since $n>3k+m+8$, we get a contradiction from above.

While $l=2$, in view of Lemmas 2.3, 2.6 and 2.9, adding (3.6) and (3.7) we get

$$\begin{aligned}&(n+m)\{T(r,f)+T(r,g)\}\\ \le\;& \left(\frac{3k}{2}+\frac{9}{2}\right)\overline{N}(r,\infty;f)+2\{(k+2)\overline{N}(r,0;f)+N_{k+2}(r,0;af^m+b)\}\\ &+\frac{1}{2}\big((k+1)\overline{N}(r,0;f)+N_{k+2}(r,0;af^m+b)\big)+\left(\frac{3k}{2}+\frac{9}{2}\right)\overline{N}(r,\infty;g)\\ &+2\{(k+2)\overline{N}(r,0;g)+N_{k+2}(r,0;ag^m+b)\}\\ &+\frac{1}{2}\big((k+1)\overline{N}(r,0;g)+N_{k+2}(r,0;ag^m+b)\big)+S(r,f)+S(r,g)\\ \le\;& \left(4k+\frac{5m}{2}+9\right)T(r,f)+\left(4k+\frac{5m}{2}+9\right)T(r,g)+S(r,f)+S(r,g),\end{aligned}\tag{3.9}$$

which is a contradiction since $n>4k+\frac{3m}{2}+9$.

Subcase 1.2. $l=1$.

Using Lemma 2.5, (3.1) and (3.2) we get

$$\begin{aligned}&\overline{N}\left(r,1;F^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right)\\ \le\;& N\left(r,1;F^{(k)}\mid=1\right)+\overline{N}_L\left(r,1;F^{(k)}\right)+\overline{N}_L\left(r,1;G^{(k)}\right)\\ &+\overline{N}_{F^{(k)}\ge 2}\left(r,1;F^{(k)}\mid G^{(k)}\ne 1\right)+\overline{N}_E^{(2}\left(r,1;G^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right)\\ \le\;& \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}\left(r,0;F^{(k)}\mid\ge 2\right)+\overline{N}\left(r,0;G^{(k)}\mid\ge 2\right)\\ &+T\left(r,G^{(k)}\right)+2\overline{N}_{F^{(k)}\ge 2}\left(r,1;F^{(k)}\mid G^{(k)}\ne 1\right)+\overline{N}_{F^{(k)}>2}\left(r,1;G^{(k)}\right)\\ &+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)+\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)+S(r,F)+S(r,G).\end{aligned}\tag{3.10}$$

So in view of (3.5) and (3.10) from Lemmas 2.1, 2.3, 2.7 and 2.8 we have

$$\begin{aligned}&T(r,F)+T(r,G)\\ \le\;& 4\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+N_{k+2}(r,0;F)+N_{k+2}(r,0;G)\\ &+T(r,G)+k\overline{N}(r,\infty;G)+2\overline{N}\left(r,0;F^{(k)}\right)+S(r,F)+S(r,G)\\ \le\;& (2k+4)\overline{N}(r,\infty;F)+(k+2)\overline{N}(r,\infty;G)+N_{k+2}(r,0;F)\\ &+2N_{k+1}(r,0;F)+N_{k+2}(r,0;G)+T(r,G)+S(r,F)+S(r,G).\end{aligned}$$

Using Lemma 2.9 we get from above

$$\begin{aligned}&(n+m)T(r,f) \\ \leq\ &(5k+3m+8)\,T(r,f)+(2k+m+4)\,T(r,g)+S(r,f)+S(r,g).\end{aligned} \tag{3.11}$$

In a similar manner we can obtain

$$\begin{aligned}&(n+m)T(r,f) \\ \leq\ &(2k+m+4)\,T(r,f)+(5k+3m+8)\,T(r,g)+S(r,f)+S(r,g).\end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) we get

$$(n-7k-3m-12)T(r,f)+(n-7k-3m-12)T(r,g)\leq S(r,f)+S(r,g). \tag{3.13}$$

Since $n>7k+3m+12$, (3.13) implies a contradiction.
Case 2. Let $H\equiv 0$.
Then by integration we get from (2.1)

$$\frac{1}{F^{(k)}-1}\equiv\frac{bG^{(k)}+a-b}{G^{(k)}-1}, \tag{3.14}$$

where a, b are constants and $a\neq 0$. From (3.14) it is clear that $F^{(k)}$ and $G^{(k)}$ share 1 CM and hence $E_{3)}\left(1;F^{(k)}\right)=E_{3)}\left(1;G^{(k)}\right)$. So in this case always $n>3k+m+8$. We now consider the following subcases.
Subcase 2.1. Let $b\neq 0$ and $a\neq b$.
If $b=-1$, then from (3.14) we have

$$F^{(k)}=\frac{-a}{G^{(k)}-a-1}.$$

Therefore

$$\overline{N}\left(r,a+1;G^{(k)}\right)=\overline{N}\left(r,\infty;F^{(k)}\right)=\overline{N}(r,\infty;f).$$

Since $a\neq b=-1$, from Lemma 2.1 we have

$$\begin{aligned}&(n+m)T(r,g)+O(1)=T(r,G)\\ &\leq\overline{N}(r,\infty;G)+N_{k+1}(r,0;G)+\overline{N}\big(r,a+1;G^{(k)}\big)+S(r,G)\\ &\leq\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+N_{k+1}(r,0;G)+S(r,G)\\ &\leq T(r,f)+(k+2+m)T(r,g)+S(r,g).\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f)\leq T(r,g)$ for $r\in I$.

So for $r\in I$ we have

$$(n-k-3)T(r,g)\leq S(r,g),$$

which is a contradiction for $n>3k+m+8$.

If $b \neq -1$, from (3.14) we obtain that

$$F^{(k)} - \left(1 + \frac{1}{b}\right) = \frac{-a}{b^2[G^{(k)} + (a-b)/b]}.$$

Therefore

$$\overline{N}\left(r, (b-a)/b; G^{(k)}\right) = \overline{N}\left(r, \infty; F^{(k)} - (1+1/b)\right) = \overline{N}(r, \infty; f).$$

Using Lemma 2.1 and the same argument as used in the case when $b = -1$ we can get a contradiction.

Subcase 2.2. Let $b \neq 0$ and $a = b$.

If $b = -1$, then from (3.14) we have

$$F^{(k)} G^{(k)} \equiv 1,$$

that is

$$[f^n(af+b)]^{(k)}[g^n(ag+b)]^{(k)} \equiv 1,$$

which is impossible by Lemma 2.10 for $k = 1$. If $b \neq -1$, from (3.14) we have

$$\frac{1}{F^{(k)}} = \frac{bG^{(k)}}{(1+b)G^{(k)} - 1}.$$

Hence from Lemma 2.3 we have

$$\begin{aligned} \overline{N}\left(r, 1/(1+b); G^{(k)}\right) &= \overline{N}\left(r, 0; F^{(k)}\right) \\ &\leq N_{k+1}(r, 0; F) + k\overline{N}(r, \infty; f). \end{aligned}$$

From Lemma 2.1 we have

$$\begin{aligned} &(n+m)T(r,g) + O(1) = T(r,G) \\ &\leq \overline{N}(r, \infty; G) + N_{k+1}(r, 0; G) + \overline{N}\left(r, \frac{1}{b+1}; G^{(k)}\right) + S(r,G) \\ &\leq k\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + N_{k+1}(r, 0; F) + N_{k+1}(r, 0; G) + S(r,G) \\ &\leq (2k+m+1)T(r,f) + (k+m+2)T(r,g) + S(r,g), \end{aligned}$$

which is a contradiction for $n > 3k + m + 8$ for $r \in I$.

Subcase 2.3. Let $b = 0$.

From (3.14) we obtain

$$F^{(k)} = \frac{G^{(k)} + a - 1}{a}. \tag{3.15}$$

If $a - 1 \neq 0$ then From (3.15) we obtain

$$\overline{N}\left(r, 1-a; G^{(k)}\right) = \overline{N}\left(r, 0; F^{(k)}\right).$$

We can similarly deduce a contradiction as in Subcase 2.2. Therefore $a = 1$ and from (3.15) we obtain

$$F = G + p(z), \tag{3.16}$$

where $p(z)$ is a polynomial of degree at most $k-1$. We claim that $p(z) \equiv 0$. Otherwise noting that f is transcendental when $k \geq 2$, in view of Lemma 2.9 we have

$$\begin{aligned}(n+m)T(r,f) &= T(r,F) + O(1) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;f) + \overline{N}(r,p;F) + S(r,F) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;f) + \overline{N}(r,0;G) + S(r,F) \\ &\leq 3T(r,f) + 2T(r,g) + S(r,f).\end{aligned} \tag{3.17}$$

Also from (3.16) we get

$$T(r,f) = T(r,g) + S(r,f),$$

which together with (3.17) implies a contradiction. Hence

$$F \equiv G.$$

So from Lemma 2.11 we get the conclusion of the theorem. □

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